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A lower bound of nilpotency class of the group of self-homotopy classes

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We work in the category of path-connected topological spaces with nondegenerate base points. If G is a topological group (or a group-like space) whose unit is the base point and X is a space, then the set $[X, G]$ of homotopy classes of based maps from X to G inherits a group structure from G. We have been interested in the group $\mathcal{H}(G) = [G, G]$ [**3**, **4**]. Let $\mathrm{nil}(\Gamma)$ be the nilpotency class of a group Γ and $X^{\times n}$ the product space $X \times \cdots \times X$ (n factors) for an integer $n \geq 1$. The purpose of this note is to prove

Proposition. If G is not homotopy nilpotent, then

$$\mathrm{nil}\big(\mathcal{H}(G^{\times n})\big) \geq \max\big\{n,\, \mathrm{nil}\big(\mathcal{H}(G)\big)\big\}. \tag{1}$$

It follows from [**5**, **6**, **8**] that a connected Lie group is not homotopy nilpotent if and only if it has torsion in homology. Hence, for example, we have $\mathrm{nil}\big(\mathcal{H}(G_2^{\times n})\big) \geq n$ from (1), where G_2 is the exceptional Lie group of rank 2. Notice that $G_2^{\times n}$ is simply connected and not simple when $n \geq 2$. On the other hand, the main theorem of [**1**] says that $\mathrm{nil}\big(\mathcal{H}(\mathrm{PU}(p))\big) \geq n$ provided p is a prime number $\geq n + 2$. Here $\mathrm{PU}(p)$ is the projective unitary group of order p and so it is simple and not simply connected.

Let $T^n(X)$ be the subspace of X^n consisting of all points, at least one of whose coordinates is the base point. Let $j_n : T^n(X) \to X^{\times n}$ be the inclusion map, $d_{n,X} : X \to X^{\times n}$ the diagonal map, and $\mathrm{cat}(X)$ the Lusternik-Schnirelmann category of X defined by Whitehead [**2**]. Then $\mathrm{cat}(X) < n$ if and only if there is a map $\varphi : X \to T^n(X)$ such that $j_n \circ \varphi$ is homotopic to $d_{n,X}$. Let $c_n : G^{\times n} \to G$ be the iterated commutator map, that is, $c_1 = \mathrm{id}_G$ (the identity map), $c_2(x, y) = [x, y] = xyx^{-1}y^{-1}$, and $c_n = c_2 \circ (c_{n-1} \times \mathrm{id}_G)$ for $n \geq 3$. We define $\mathrm{hpnil}(G)$ to be the least integer n such that c_{n+1} is nullhomotopic. If there is no such integer, we define $\mathrm{hpnil}(G) = \infty$. Then G is homotopy nilpotent if and only if $\mathrm{hpnil}(G)$ is finite.

Given maps $f_1, f_2, \ldots, f_n : X \to G$, we have

$$[[\ldots[[f_1, f_2], f_3], \ldots], f_n] = c_n \circ (f_1 \times \cdots \times f_n) \circ d_{n,X}. \tag{2}$$

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Hence we readily obtain the following well-known inequality [**7**].

$$\mathrm{nil}\big([X,G]\big) \le \min\big\{\mathrm{cat}(X),\ \mathrm{hpnil}(G)\big\}. \tag{3}$$

Since $[X, G^{\times n}] \cong [X,G] \oplus \cdots \oplus [X,G]$ (n factors), we have

$$\mathrm{nil}\big([X,G^{\times n}]\big) = \mathrm{nil}\big([X,G]\big). \tag{4}$$

Since $\mathrm{cat}(X \times Y) \le \mathrm{cat}(X) + \mathrm{cat}(Y)$ [**2**], it follows that $\mathrm{cat}(G^{\times n}) \le n \cdot \mathrm{cat}(G)$. Let $\mathrm{pr}_i : G^{\times n} \to G$ be the projection to the i-th factor $(1 \le i \le n)$. Since ${\mathrm{pr}_i}^* : \mathcal{H}(G) \to [G^{\times n}, G]$ is a monomorphism, we have $\mathrm{nil}\big(\mathcal{H}(G)\big) \le \mathrm{nil}\big([G^{\times n}, G]\big)$. Hence we have

$$\mathrm{nil}\big(\mathcal{H}(G)\big) \le \mathrm{nil}\big(\mathcal{H}(G^{\times n})\big) \le \min\{n \cdot \mathrm{cat}(G), \mathrm{hpnil}(G)\}. \tag{5}$$

Proof of Proposition. Suppose that G is not homotopy nilpotent. It suffices to prove $\mathrm{nil}\big([G^{\times n}, G]\big) \ge n$ because of (4) and (5). By the assumption, c_n is essential for every $n \ge 1$. When $n = 1$, it means that $\mathcal{H}(G)$ is not trivial, that is, $\mathrm{nil}\big(\mathcal{H}(G)\big) \ge 1$. If $n \ge 2$, then $[[\ldots[[\mathrm{pr}_1, \mathrm{pr}_2], \mathrm{pr}_3], \ldots, \mathrm{pr}_{n-1}], \mathrm{pr}_n] = c_n$ by (2) and so $\mathrm{nil}\big([G^{\times n}, G]\big) \ge n$. □

Remark. Since $\mathrm{cat}\big(\mathrm{S}^{3^{\times 3}}\big) = \mathrm{hpnil}(\mathrm{S}^3) = \mathrm{nil}\big(\mathcal{H}(\mathrm{S}^{3^{\times n}})\big) = 3$ for all $n \ge 3$ by [**4**, §4], it follows that the assumption "not homotopy nilpotent" is essential in Proposition and that (3) is best possible.

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