Math. J. Ibaraki Univ., 42 (2010), 3–16

On polynomial ascent and descent semistar operations on an integral domain

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INTRODUCTION

In 1994, A. Okabe and R. Matsuda introduced the notion of a semistar operation in [OM] as a generalization of the notion of a star operation which was introduced in 1936 by W. Krull and was developed in [G] by R. Gilmer. In 2000, M. Fontana and J.A. Huckaba investigated the relation between semistar operations and localizing systems and they associated the semistar operation $*_{\mathcal{F}}$ for each localizing system $\mathcal{F}$ on D and the localizing system $\mathcal{F}^*$ for each semistar operation $*$ on D. Using these correspondences, they established a very natural bridge between semistar operations and localizing systems which has been proven to be a very important and essential tool in the study of semistar operation theory.

Let D be an integral domain with quotient field K and let $D[X]$ be the ring of polynomials over D in indeterminate X. We shall denote the set of all semistar operations on D (resp. $D[X]$) by $\mathbf{SS}(D)$ (resp. $\mathbf{SS}(D[X])$) as in [O5]. We have much interest in considering the relation between $\mathbf{SS}(D)$ and $\mathbf{SS}(D[X])$. First, in [OM], a correspondence $* \mapsto *'$ from $\mathbf{SS}(D[X])$ into $\mathbf{SS}(D)$ was given by setting $E^{*'} = (ED[X])^* \bigcap K$ for each nonzero D-submodule E of K. In this paper, this semistar operation $*'$ is called *the polynomial descent semistar operation associated to* $*$ and is denoted by $*^\delta$. Next, in [P3], G. Picozza defined a reverse correspondence $* \mapsto *'$ from $\mathbf{SS}(D)$ into $\mathbf{SS}(D[X])$ by setting $*' = *_{\mathcal{F}^*[X]}$ for each $* \in \mathbf{SS}(D)$. In this paper, this semistar operation $*'$ is called *the polynomial ascent semistar operation associated to* $*$ and is denoted by $*^\alpha$. Thus we have two correspondences between $\mathbf{SS}(D)$ and $\mathbf{SS}(D[X])$. The purpose of this paper is to investigate the relation between $\mathbf{SS}(D)$ and $\mathbf{SS}(D[X])$ using these two semistar operations $*^\alpha$ and $*^\delta$.

In Section 1, we first recall some well-known results on semistar operations and localizing systems on an integral domain D which will be used in sequel and we shall show some new results concerning semistar operations $[*]$ and $*_a$ which were introduced in [FL1].

In Section 2, we shall prove some important properties of semistar operations $*^\delta$ and $*^\alpha$. In Theorem 27, we show that $(*^\alpha)^\delta = \bar{*}$ for each semistar operation $*$ on D

Received 12 February, 2009; revised 24 August, 2009.
2000 *Mathematics Subject Classification.* 13A15.
Key Words and Phrases. semistar operation, polynomial ascent, polynomial descent.
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and $\mid \overline{\mathbf{SS}}(D) \mid \leq \mid \overline{\mathbf{SS}}(D[X]) \mid$ and in Theorem 28, we show that $(*^{\alpha})^{\delta} = \tilde{*}$ for each semistar operation $*$ of finite type on D and $\mid \widetilde{\mathbf{SS}}(D) \mid \leq \mid \widetilde{\mathbf{SS}}(D[X]) \mid$, where $\overline{\mathbf{SS}}(D)$ (resp. $\widetilde{\mathbf{SS}}(D)$) denotes the set of all stable semistar operations (resp. all stable semistar operations of finite type) on D. Moreover, we shall obtain two necessary and sufficient conditions for a semistar operation $*$ to be stable in Theorem 27 and Corollary 40. Lastly, in Corollary 46, we shall give a semistar operation theoretic characterization of a Prüfer domain.

Throughout this paper, D will denote an integral domain with quotient field K. An integral domain which lies between D and K is called an *overring* of D. We denote the set of prime ideals of D by $\mathrm{Spec}(D)$ and denote the cardinality of a set X by $\mid X \mid$. The *integral closure* of an integral domain D is denoted by $\bar{D}$.

1. BACKGROUND ON SEMISTAR OPERATIONS AND LOCALIZING SYSTEMS

In this paper, we shall denote the set of all nonzero D-submodules of K by $\mathbf{K}(D)$ and we shall call each element of $\mathbf{K}(D)$ a *K-fractional ideal* of D as in [O1]. Let $\mathbf{F}(D)$ be the set of all nonzero *fractional ideals* of D, that is, all elements $E \in \mathbf{K}(D)$ such that there exists a nonzero element $d \in D$ with $dE \subseteq D$. The set of finitely generated K- fractional ideals of D is denoted by $\mathbf{f}(D)$. Evidently $\mathbf{f}(D) \subseteq \mathbf{F}(D) \subseteq \mathbf{K}(D)$. The set of all nonzero integral ideals of D is denoted by $\mathbf{I}(D)$.

A map $E \mapsto E^*$ of $\mathbf{K}(D)$ into $\mathbf{K}(D)$ is called a *semistar operation* if the following conditions hold for all $a \in K \backslash \{0\}$ and $E, F \in \mathbf{K}(D)$:

(S_1) $(aE)^* = aE^*$;

(S_2) If $E \subseteq F$, then $E^* \subseteq F^*$; and

(S_3) $E \subseteq E^*$ and $(E^*)^* = E^*$.

We shall denote the set of all semistar operations on D by $\mathbf{SS}(D)$ as in [O5].

Proposition 1. Let $*$ be a semistar operation on D. Then, for all $E, F \in \mathbf{K}(D)$ and for every family $\{E_\alpha\}$ of elements in $\mathbf{K}(D)$, we have:

(1) $(EF)^* = (E^*F)^* = (EF^*)^* = (E^*F^*)^*$;

(2) $(E+F)^* = (E^*+F)^* = (E+F^*)^* = (E^*+F^*)^*$;

(3) $(E:F)^* \subseteq E^* : F^* = (E^* : F) = (E^* : F)^*$;

(4) $(\sum_\alpha E_\alpha)^* = (\sum_\alpha E_\alpha^*)^*$;

(5) $\bigcap_\alpha E_\alpha^* = (\bigcap_\alpha E_\alpha^*)^*$, if $\bigcap_\alpha E_\alpha^* \neq \{0\}$.

Example 2. (1) If we set $E^{\bar{d}_D} = E$ for each $E \in \mathbf{K}(D)$, then $\bar{d}_D$ is a semistar operation on D and is called *the identity semistar operation* on D. The semistar operation $\bar{d}_D$ is simply denoted by $\bar{d}$ and is called *the* $\bar{d}$*-operation* on D.

(2) If we set $E^{\bar{e}_D} = K$ for all $E \in \mathbf{K}(D)$, then $\bar{e}_D$ is a semistar operation on D and is called *the trivial semistar operation* on D. The semistar operation $\bar{e}_D$ is simply denoted by $\bar{e}$ and is called *the* $\bar{e}$*-operation* on D.

(3) For every $E, F \in \mathbf{K}(D)$, the set $\{x \in K \mid xE \subseteq F\}$ is denoted by $F :_K E$, or simply by $F : E$. If $F : E \neq \{0\}$, then $F : E$ is also a K-fractional ideal of D. If we set $F :_D E = (F :_K E) \cap D$, then $F :_D E$ is an integral ideal of D.

For each $E \in \mathbf{K}(D)$, we set $E^{-1} = D :_K E = \{x \in K \mid xE \subseteq D\}$ and $E^{\bar{v}} = (E^{-1})^{-1}$. Then $\bar{v}$ is a semistar operation on D and is called *the divisorial semistar operation* (or simply *the $\bar{v}$-operation*) on D. If $E \in \mathbf{K}(D) \setminus \mathbf{F}(D)$, then $E^{-1} = (0)$ and so $E^{\bar{v}} = K$.

(4) Let R be an overring of D. If we set $E^{*_{(R)}} = ER$ for each $E \in \mathbf{K}(D)$, then $*_{(R)}$ is a semistar operation on D and is called *the semistar operation defined by an overring R*. As easily seen, $*_{(D)} = \bar{d}_D$.

(5) Let $\mathcal{D} = \{D_\alpha\}$ be a family of overrings of D. If we set $E^{*_{\mathcal{D}}} = \bigcap\{ED_\alpha \mid D_\alpha \in \mathcal{D}\}$ for each $E \in \mathbf{K}(D)$, then $*_{\mathcal{D}}$ is a semistar operation on D and is called *the semistar operation defined by the family $\mathcal{D}$*.

(6) Let $\mathcal{V}$ be the set of all valuation overrings of D. If we set $E^{\bar{b}} = \bigcap\{EV_\alpha \mid V_\alpha \in \mathcal{V}\}$ for each $E \in \mathbf{K}(D)$, then $\bar{b}$ is a semistar operation on D and is called *the $\bar{b}$-operation* on D. It follows that $D^{\bar{b}} = \bigcap\{V_\alpha \mid V_\alpha \in \mathcal{V}\} = \bar{D}$, the *integral closure* of D.

Let $\mathcal{W}$ be a set of valuation overrings of D. If we set $E^{\bar{w}} = \bigcap\{EV_\alpha \mid V_\alpha \in \mathcal{W}\}$ for each $E \in \mathbf{K}(D)$, then $\bar{w}$ is a semistar operation on D and is called *the $\bar{w}$-operation* on D.

A semistar operation $*$ on D is said to be *of finite type* (or *of finite character*) if $E^* = \bigcup\{F^* \mid F \subseteq E \text{ and } F \in \mathbf{f}(D)\}$ for each $E \in \mathbf{K}(D)$. For each semistar operation $*$ on D and each $E \in \mathbf{K}(D)$, we set $E^{*_f} = \bigcup\{F^* \mid F \subseteq E \text{ and } F \in \mathbf{f}(D)\}$. Then the map $E \mapsto E^{*_f}$ is a semistar operation of finite type on D and is called *the semistar operation of finite type associated to $*$*. It is easy to see that $*$ is of finite type if and only if $* = *_f$. In particular, $*_f$ is of finite type for each semistar operation $*$ on D. The semistar operation $\bar{v}_f$ associated to $\bar{v}$ is denoted by $\bar{t}$ and is called *the $\bar{t}$-operation*. It is easily seen that $E^* = E^{*_f}$ for all $E \in \mathbf{f}(D)$. Note that $*_{(R)}$ is a semistar operation of finite type for all overrings R of D. We shall denote the set of all semistar operations of finite type on D by $\mathbf{SS}_f(D)$.

A map $E \longmapsto E^\star$ of $\mathbf{F}(D)$ into $\mathbf{F}(D)$ is called a *star operation* on D, if the following conditions hold for all $a \in K - \{0\}$ and $E, F \in \mathbf{F}(D)$:

(S_0) $(aD)^\star = aD$;

(S_1) $(aE)^\star = aE^\star$;

(S_2) If $E \subseteq F$, then $E^\star \subseteq F^\star$; and

(S_3) $E \subseteq E^\star$ and $(E^\star)^\star = E^\star$.

If we set $E^d = E$ for all $E \in \mathbf{F}(D)$ then d is a star operation on D and is called *the identity operation* (or simply *the d-operation*). Next, for each $E \in \mathbf{F}(D)$, we set $E^{-1} = D :_K E = \{x \in K \mid xE \subseteq D\}$ and $E^v = (E^{-1})^{-1}$ for each $E \in \mathbf{F}(D)$, then v is a star operation on D and is called *the v-operation*.

We shall denote the set of all star operations on D by $\mathbf{S}(D)$. For any overring R of D, we denote the set $\{* \in \mathbf{SS}(D) \mid D^* = R\}$ by $\mathbf{SS}(D, R)$ as in [P2].

We recall that there exists a canonical method which corresponds each star operation $\star$ on D to a semistar operation $\star^e$ on D:

Proposition 3 ([OM, Proposition 17]). Let $\star$ be a star operation on D. For each

$E \in \mathbf{K}(D)$, we set:

$$E^{\star^e} = \begin{cases} E^\star, & \text{for } E \in \mathbf{F}(D) \\ K, & \text{for } E \in \mathbf{K}(D)\backslash\mathbf{F}(D) \end{cases}$$

Then the map $E \longmapsto E^{\star^e}$ is a semistar operation on D.

This semistar operation $\star^e$ is called *the trivial semistar extension of a star operation* $\star$. It is easily seen that the $\bar{v}$-operation is the trivial semistar extension of the v-operation.

In [O1], a semistar operation $*$ is said to be *weak* if $D^* = D$ and is said to be *strong* if $D^* \neq D$. We denote the set of all weak semistar operations on D by $\mathbf{SS}_w(D)$ or $(\mathbf{S})\mathbf{S}(\mathbf{D})$. Evidently $\star^e$ is a weak semistar operation for all star operations $\star$.

We denote the trivial semistar extension d^e of the d-operation on D by $\bar{f}$. For each overring R of D, we denote the d-operation on R by d_R and denote the trivial semistar extension $(d_R)^e$ of d_R by $\bar{f}_R$.

Remark 4. The map $\star \mapsto \star^e$ is evidently an injective map of $\mathbf{S}(D)$ into $\mathbf{SS}(D)$ and hence $\mid \mathbf{S}(D) \mid \leq \mid \mathbf{SS}(D) \mid$.

Remark 5. An integral domain D is called a *conducive domain* if $D : R = \{x \in K \mid xR \subseteq D\} \neq (0)$ for each overring R of D other than K. It is easy to see that D is a conducive domain if and only if $\bar{d} = \bar{f}$ holds. (see [O5, Proposition 5]).

In [OM] we defined a partial order $\leq$ on $\mathbf{SS}(D)$ by $*_1 \leq *_2$ if and only if $E^{*_1} \subseteq E^{*_2}$ for each $E \in \mathbf{K}(D)$. It is evident that $\bar{d} \leq * \leq \bar{e}$ holds for each semistar operation $*$ on D.

Proposition 6 (cf.[OM, Lemma 16]). For $*_1, *_2 \in \mathbf{SS}(D)$, the following conditions sre equivalent:
(1) $*_1 \leq *_2$;
(2) $(E^{*_2})^{*_1} = E^{*_2}$ for all $E \in \mathbf{K}(D)$;
(3) $(E^{*_1})^{*_2} = E^{*_2}$ for all $E \in \mathbf{K}(D)$.

Definition 7 ([FL1, Definition 4.2]). For each semistar operation $*$ on D, we set:

$$F^{[*]} = \bigcup\{((H^* : H^*)F)^{*_f} \mid H \in \mathbf{f}(D)\} \text{ for each } F \in \mathbf{f}(D),$$

and

$$E^{[*]} = \bigcup\{F^{[*]} \mid F \in \mathbf{f}(D) \text{ and } F \subseteq E\} \text{ for each } E \in \mathbf{K}(D).$$

Definition 8 ([FL1, Definition 4.4]). For each semistar operation $*$ on D, we set:

$$F^{*_a} = \bigcup\{((FH)^* : H^*) \mid H \in \mathbf{f}(D)\} \text{ for each } F \in \mathbf{f}(D),$$

and

$$E^{*_a} = \bigcup\{F^{*_a} \mid F \in \mathbf{f}(D) \text{ and } F \subseteq E\} \text{ for each } E \in \mathbf{K}(D).$$

Now we must recall the definitons of an e.a.b. semistar operation and an a.b. semistar operation in order to state some of the fundamental properties of $[*]$ and $*_a$. A semistar operation $*$ on D is said to be *endlich arithmetisch brauchbar* (for short, e.a.b.) if for all $E, F, G \in \mathbf{f}(D), (EF)^* \subseteq (EG)^*$ implies $F^* \subseteq G^*$ and is said to be *arithmetisch brauchbar* (for short, a.b.) if for all $F, G \in \mathbf{K}(D)$ and for all $E \in \mathbf{f}(D), (EF)^* \subseteq (EG)^*$ implies $F^* \subseteq G^*$.

Proposition 9. Let $*$ be a semistar operation on D. Then
(1) $[*]$ is a semistar operation of finite type;
(2) $D^{[*]}$ is an integrally closed overring of D;
(3) $*_a$ is an e.a.b. semistar operation of finite type;
(4) $*_a = *_f \Longleftrightarrow *_f$ is an e.a.b. semistar operation;
(5) $[*]$ is e.a.b. if and only if $[*] = *_a$.

Proof. The proof of (1) is straightforward. (2) is in [FL1, Propositions 4.3]. (3) and (4) are in [FL1, Proposition 4.5]. (5) Since $[*]$ is of finite type, by (4) of Proposition 9, $[*] = [*]_f$ is e.a.b. if and only if $[*]_a = [*]_f = [*]$. But, by [FL1, Proposition 4.5 (9)], $[*]_a = *_a$ and hence our assertion follows. □

Note. In general, for each semistar operation $*$ on D, we have $[*] \leq *_a$ as shown in [FL1, Proposition 4.5 (3)].

Proposition 10 ([OM, Lemma 45]). Let R be an overring of D. Then

(1) For each $* \in \mathbf{SS}(R)$, if we define $E^{\delta_D(*)} = (ER)^*$ for all $E \in \mathbf{K}(D)$, then $\delta_D(*) \in \mathbf{SS}(D)$.

(2) If we define $\delta_{R/D} : \mathbf{SS}(R) \to \mathbf{SS}(D)$ by $\delta_{R/D}(*) = \delta_D(*)$, then $\delta_{R/D}$ is an injective map and therefore $\mid \mathbf{SS}(R) \mid \leq \mid \mathbf{SS}(D) \mid$.

(3) For each $* \in \mathbf{SS}(D)$, if we define $E^{\alpha_R(*)} = E^*$ for all $E \in \mathbf{K}(R) (\subseteq \mathbf{K}(D))$, then $\alpha_R(*) \in \mathbf{SS}(R)$.

(4) If we define $\alpha_{R/D} : \mathbf{SS}(D) \to \mathbf{SS}(R)$ by $\alpha_{R/D}(*) = \alpha_R(*)$, then $\alpha_{R/D} \circ \delta_{R/D}$ is the identity map of $\mathbf{SS}(R)$.

The map $\delta_{R/D}$ (resp. $\alpha_{R/D}$) is called *the descent map* (resp. *the ascent map*). Here we collect fundamental properties of $\alpha_{R/D}$ and $\delta_{R/D}$ concerning e.a.b. property and a.b. property.

Proposition 11. (1) If $*$ is an e.a.b. (resp. a.b.) semistar operation on D, then $\alpha_{D^*/D}(*)$ is an e.a.b. (resp. a.b.) semistar operation on D^*.

(2) If $*$ is an e.a.b. (resp. a.b.) semistar operation on an overring R of D, then $\delta_{R/D}(*)$ is an e.a.b. (resp. a.b.) semistar operation on D.

Proof. (1) is in [FL1, Proposition 2.8] and (2) is in [FL1, Proposition 2.9]. □

We shall show two fundamental properties of $\delta_{R/D}$ concerning $[*]$ and $*_a$ in the following two lemmas:

Lemma 12. Let R be an overring of D and let $*$ be a semistar operation on R. If we set $\delta = \delta_{R/D}$, then $(\delta(*))_a = \delta(*_a)$.

Proof. First, for each $F \in \mathbf{f}(D), F^{(\delta(*))_a} = \bigcup\{(FH)^{\delta(*)} : H^{\delta(*)} \mid H \in \mathbf{f}(D)\} = \bigcup\{(FRHR)^* : (HR)^* \mid H \in \mathbf{f}(D)\} = \bigcup\{(FRG)^* : G^* \mid G \in \mathbf{f}(R)\} = (FR)^{*_a} = F^{\delta(*_a)}$.

Next, for each $E \in \mathbf{K}(D), E^{(\delta(*))_a} = \bigcup\{F^{(\delta(*))_a} \mid F \in \mathbf{f}(D)$ and $F \subseteq E\} = \bigcup\{F^{\delta(*_a)} \mid F \in \mathbf{f}(D)$ and $F \subseteq E\} = E^{\delta(*_a)}$ and therefore $(\delta(*))_a = \delta(*_a)$. $\square$

Lemma 13. Let R be an overring of D and let $*$ be a semistar operation on R. If we set $\delta = \delta_{R/D}$, then $[\delta(*)] = \delta([*])$.

Proof. For each $F \in \mathbf{f}(D), F^{\delta([*])} = (FR)^{[*]} = \bigcup\{(H^* : H^*)FR)^{*_f} \mid H \in \mathbf{f}(R)\} = \bigcup\{(((GR)^* : (GR)^*)FR)^{*_f} \mid G \in \mathbf{f}(D)\} = \bigcup\{(G^{\delta(*)} : G^{\delta(*)})FR)^{*_f} \mid G \in \mathbf{f}(D)\} = \bigcup\{(((G^{\delta(*)} : G^{\delta(*)})F)R)^{*_f} \mid G \in \mathbf{f}(D)\} = \bigcup\{((G^{\delta(*)} : G^{\delta(*)})F)^{\delta(*_f)} \mid G \in \mathbf{f}(D)\}$. Then, since $\delta(*_f) = (\delta(*))_f$ by [P2, Proposition 3.2 (1)], we have $F^{\delta([*])} = \bigcup\{((G^{\delta(*)} : G^{\delta(*)})F)^{(\delta(*))_f} \mid G \in \mathbf{f}(D)\} = F^{[\delta(*)]}$. Hence, for each $E \in \mathbf{K}(D), E^{\delta([*])} = \bigcup\{F^{\delta([*])} \mid F \in \mathbf{f}(D)$ and $F \subseteq E\} = \bigcup\{F^{[\delta(*)]} \mid F \in \mathbf{f}(D)$ and $F \subseteq E\} = E^{[\delta(*)]}$ which implies that $[\delta(*)] = \delta([*])$. $\square$

Remark 14. (1) $[\bar{e}] = (\bar{e})_a = \bar{e}$. In fact, since $\bar{e}$ is of finite type, $\bar{e} = (\bar{e})_f \leq [\bar{e}] \leq (\bar{e})_a$ by [FL1, Proposition 4.5 (3)]. Hence $[\bar{e}] = (\bar{e})_a = \bar{e}$.

(2) $[\bar{d}] = *_{(\bar{D})}$. If $F \in \mathbf{f}(D)$, then $F^{[\bar{d}]} = \bigcup\{((H^{\bar{d}} : H^{\bar{d}})F)^{\bar{d}} \mid H \in \mathbf{f}(D)\} = \bigcup\{(H : H)F \mid H \in \mathbf{f}(D)\} = \bar{D}F$. Hence $F^{[\bar{d}]} = F^{*_{(\bar{D})}}$ for all $F \in \mathbf{f}(D)$. Then for each $E \in \mathbf{K}(D), E^{[\bar{d}]} = \bigcup\{F^{[\bar{d}]} \mid F \in \mathbf{f}(D)$ and $F \subseteq E\} = \bigcup\{\bar{D}F \mid F \in \mathbf{f}(D)$ and $F \subseteq E\} = \bar{D}E = E^{*_{(\bar{D})}}$. Hence we get $[\bar{d}] = *_{(\bar{D})}$.

In [GA], the notion of a *localizing* (or *topologizing*) *system of ideals* was introduced by Gabriel. A set $\mathcal{F}$ of ideals of D is called a *localizing system of ideals* (for short, *localizing system*) on D if the following conditions are satisfied:

(**LS1**) If $I \in \mathcal{F}$ and J is an ideal of D such that $I \subseteq J$, then $J \in \mathcal{F}$;

(**LS2**) If $I \in \mathcal{F}$ and J is an ideal of D such that $J :_D iD \in \mathcal{F}$ for all $i \in I$, then $J \in \mathcal{F}$.

To avoid uninteresting cases, we assume that every localizing system $\mathcal{F}$ is *non-trivial*, i.e., $(0) \notin \mathcal{F}$ and $\mathcal{F}$ is not empty.

A localizing system $\mathcal{F}$ of D is said to be of *finite type* if for each $I \in \mathcal{F}$, there exists a finitely generated ideal $J \in \mathcal{F}$ such that $J \subseteq I$. If $I, J \in \mathcal{F}$, then $IJ \in \mathcal{F}$ and hence $I \bigcap J \in \mathcal{F}$ by (**LS1**) (see [FHP, Proposition 5.1.1]). Thus every localizing system becomes a *generalized multiplicative system*.

We shall denote the set of localizing systems on D by $\mathcal{LS}(D)$ and the set of localizing systems of finite type on D by $\mathcal{LS}_f(D)$.

Let $\mathcal{F}$ be a localizing system on D. If we set $D_{\mathcal{F}} = \{x \in K \mid D :_D x \in \mathcal{F}\}$, then $D_{\mathcal{F}}$ is a subring of K and is called the *quotient ring of D relative to $\mathcal{F}$* (see [PO, p.778]). It is easy to see that $D_{\mathcal{F}} = \bigcup\{D : I \mid I \in \mathcal{F}\}$.

Let T be a flat overring of D. If we set $\mathcal{F}(T) = \{I \in \mathbf{I}(D) \mid IT = T\}$, then $\mathcal{F}(T)$ is a localizing system on D and $D_{\mathcal{F}(T)} = T$ [PO, Proposition 1.2 (i)]. It is easily seen that $\mathcal{F}(T)$ is a localizing system of finite type. Here we note that $\mathcal{F}(T)$ is denoted by $\mathcal{F}_1(T)$ in [FP] and $\mathcal{F}_0(T)$ in [FHP].

It is easily seen that if $\mathcal{F}_1 \subseteq \mathcal{F}_2$ are localizing systems of D, then $D_{\mathcal{F}_1} \subseteq D_{\mathcal{F}_2}$ and if $T_1 \subseteq T_2$ are overrings of D, then $\mathcal{F}(T_1) \subseteq \mathcal{F}(T_2)$.

Proposition 15 ([FH, Proposition 2.8]). Let $*$ be a semistar operation on D. Then $\mathcal{F}^* = \{I \mid I \in \mathbf{I}(D) \text{ and } I^* = D^*\}$ is a localizing system on D.

$\mathcal{F}^*$ is called *the localizing system associated to* $*$ for each $* \in \mathbf{SS}(D)$. Let $*$ be a semistar operation on D. Then $*$ is said to be *stable* if $(E \cap F)^* = E^* \cap F^*$ for all $E, F \in \mathbf{K}(D)$.

Proposition 16 ([FH, Proposition 2.4]). Let $\mathcal{F}$ be a localizing system on D. For each $E \in \mathbf{K}(D)$, we set $E_{\mathcal{F}} = \bigcup\{E : J \mid J \in \mathcal{F}\}$. Then the map $E \mapsto E_{\mathcal{F}}$ of $\mathbf{K}(D)$ into $\mathbf{K}(D)$ is a stable semistar operation on D.

The semistar operation defined in Proposition 16 is denoted by $*_{\mathcal{F}}$ and is called *the semistar operation associated to* $\mathcal{F}$.

In general, $*_{(D_{\mathcal{F}})} \leq *_{\mathcal{F}}$ and the equality holds if and only if $D_{\mathcal{F}}$ is flat over D and $\mathcal{F} = \{I \in \mathbf{I}(D) \mid ID_{\mathcal{F}} = D_{\mathcal{F}}\}$ (see [FH, Proposition 2.6]).

Proposition 17 ([FH, Proposition 3.2]). Let $\mathcal{F}$ be a localizing system on D and let $*$ be a semistar operation on D. Then

(1) If $\mathcal{F}$ is of finite type, then $*_{\mathcal{F}}$ is of finite type.

(2) If $*$ is of finite type, then $\mathcal{F}^*$ is of finite type.

2. POLYNOMIAL ASCENT AND DESCENT SEMISTAR OPERATIONS

In [P3], Picozza has proved that each localizing system $\mathcal{F}$ on D induces in a canonical way the localizing system $\mathcal{F}[X]$ on the polynomial ring $D[X]$.

Lemma 18 ([P3, Proposition 3.1]). Let $\mathcal{F}$ be a localizing system on D. If we set

$$\mathcal{F}[X] = \{I \text{ is an ideal of } D[X] \mid I \supseteq JD[X] \text{ for some } J \in \mathcal{F}\}$$

then $\mathcal{F}[X]$ is a localizing system on $D[X]$.

Note that it is shown in the proof of [P3, Proposition 3.1] that for an ideal I of $D[X]$, $I \in \mathcal{F}[X]$ if and only if $I \bigcap D \in \mathcal{F}$.

Proposition 19 ([P3, Proposition 3.2]). If $\mathcal{F}$ is a localizing system of finite type on D, then $\mathcal{F}[X]$ is a localizing system of finite type on $D[X]$.

Proposition 20 (cf. [P3, Theorem 3.3]). Let $*$ be a semistar operation on D. If we set $*^\alpha = *_{\mathcal{F}^*[X]}$ then $*^\alpha$ is a semistar operation on $D[X]$.

Proof. If $*$ is a semistar operation on D, then $\mathcal{F}^*$ is a localizing system on D and then $\mathcal{F}^*[X]$ is a localizing system on $D[X]$. Hence $*^\alpha$ is a semistar operation on $D[X]$ by Proposition 16.

Note. Let $*$ be a semistar operation on D. Then it follows from the definition that $E^{*^\alpha} = \bigcup\{E : J \mid J \in \mathcal{F}^*[X]\} = \bigcup\{E : J \mid J \in \mathbf{I}(D[X])$ and $J \supseteq I[X]$ for some $I \in \mathcal{F}^*\} = \bigcup\{E : J \mid J \supseteq I[X]$ and $I^* = D^*\} = \bigcup\{E : I[X] \mid I \in \mathbf{I}(D)$ and $I^* = D^*\}$ for each $E \in \mathbf{K}(D[X])$.

Proposition 21. Let $*$ be a semistar operation on D. Then

(1) $*^\alpha$ is a stable semistar operation on $D[X]$.

(2) If $*$ is of finite type, then $*^\alpha$ is also of finite type.

(3) If $*_1 \leq *_2$ in $\mathbf{SS}(D)$, then $(*_1)^\alpha \leq (*_2)^\alpha$ in $\mathbf{SS}(D[X])$.

Proof. (1) follows from Proposition 16 and (2) follows from Propositions 17 and 19.

(3) As easily seen, $*_1 \leq *_2$ implies $\mathcal{F}^{*_1} \subseteq \mathcal{F}^{*_2}$ and then $\mathcal{F}^{*_1}[X] \subseteq \mathcal{F}^{*_2}[X]$. Hence $(*_1)^\alpha = *_{\mathcal{F}^{*_1}[X]} \leq *_{\mathcal{F}^{*_2}[X]} = (*_2)^\alpha$. $\square$

The semistar operation $*^\alpha$ is called *the polynomial ascent semistar operation associated to a semistar operation* $*$.

Example 22 (cf.[P3, Remark (1)]). If $\bar{d}$ is the identity semistar operation on D, then $(\bar{d})^\alpha = \bar{d}_{D[X]}$, i.e., the identity semistar operation on $D[X]$. By definition, $\mathcal{F}^{\bar{d}} = \{I$ is an ideal of $D \mid I^{\bar{d}} = D^{\bar{d}}\} = \{D\}$. Then it is easily seen that $\mathcal{F}^{\bar{d}}[X] = \{D[X]\}$ and then $E^{(\bar{d})^\alpha} = E^{*_{\mathcal{F}^{\bar{d}}[X]}} = \{E : J \mid J \in \mathcal{F}^{\bar{d}}[X]\} = E : D[X] = E$ for all $E \in \mathbf{K}(D[X])$. Therefore we have $(\bar{d})^\alpha = \bar{d}_{D[X]}$.

Proposition 23. Let $*$ be a semistar operation on $D[X]$. We set $E^{*^\delta} = (ED[X])^* \bigcap K$ for all $E \in \mathbf{K}(D)$. Then

(1) $*^\delta$ is a semistar operation on D.

(2) $(ED[X])^* = (E^{*^\delta} D[X])^*$ for all $E \in \mathbf{K}(D)$.

(3) If $*$ is a stable semistar operation on $D[X]$, then $*^\delta$ is a stable semistar operation on D.

(4) If $*$ is of finite type, then $*^\delta$ is of finite type.

(5) If $*_1 \leq *_2$ in $\mathbf{SS}(D[X])$, then $(*_1)^\delta \leq (*_2)^\delta$ in $\mathbf{SS}(D)$.

Proof. (1) and (2) are consequences of [OM, Proposition 35].

(3) If $*$ is stable, then $(E \bigcap F)^{*^\delta} = ((E \bigcap F)D[X])^* \bigcap K = (ED[X] \bigcap FD[X])^* \bigcap K = ((ED[X])^* \bigcap (FD[X])^*) \bigcap K = ((ED[X])^* \bigcap K) \bigcap ((FD[X])^* \bigcap K) = E^{*^\delta} \bigcap F^{*^\delta}$ for all $E, F \in \mathbf{K}(D)$ and therefore $*^\delta$ is also stable.

(4) Choose an element $E \in \mathbf{K}(D)$. Then $E^{*^\delta} = (ED[X])^* \bigcap K = (\bigcup\{(FD[X])^* \mid$

$F \subseteq E$ and $F \in \mathbf{f}(D)\}) \bigcap K = \bigcup\{(FD[X])^* \bigcap K \mid F \subseteq E$ and $F \in \mathbf{f}(D)\} = \bigcup\{F^{*^\delta} \mid F \subseteq E$ and $F \in \mathbf{f}(D)\}$ which implies that $*^\delta$ is of finite type.

(5) This is straightforward from the definition.□

The semistar operation $*^\delta$ defined in Proposition 23 is called *the polynomial descent semistar operation associated to a semistar operation* $*$.

As in [FH], we set $\bar{*} = *_{\mathcal{F}^*}$ and $\tilde{*} = (*)_{(\mathcal{F}^*)_f} = *_{\mathcal{F}^{*f}}$ for each $* \in \mathbf{SS}(D)$. Here we collect some fundamental properties of these semistar operations which were proved in [FH].

Proposition 24. Let $*$ be a semistar operation on D. Then
(1) $(\bar{*})_f \leq *_f \leq *$ and $(\bar{*})_f \leq \bar{*} \leq *$;
(2) $\widetilde{(*_f)} = \overline{(*_f)} = \tilde{*}$;
(3) $*$ is a stable semistar operation if and only if $* = \bar{*}$;
(4) $\bar{*} = \tilde{*} \Longleftrightarrow \bar{*}$ is of finite type;
(5) $* = \tilde{*} \Longleftrightarrow *$ is stable and of finite type;
(6) $\bar{\tilde{*}} = \tilde{\tilde{*}} = \tilde{*}$.
(7) $\bar{\bar{*}} = \bar{*}, \tilde{\bar{*}} = \tilde{*}$ and $\tilde{*} \leq \bar{*}$.

In [FL2], an ideal $I \in \mathbf{I}(D)$ is called a *quasi-$*$-ideal* of D if $I^* \bigcap D = I$. A maximal element in the set of all proper quasi-$*$-ideals of D is called a *quasi-$*$-maximal ideal* of D and the set of all quasi-$*$-maximal ideals of D is denoted by $\mathrm{QMax}^*(D)$. For the sake of simplicity, when $*$ is of finite type, $\mathrm{QMax}^*(D)$ is simply denoted by $\mathcal{M}(*)$ (see [FL2, p.4782]).

If $*$ is a semistar operation of finite type on D, then $\mathrm{QMax}^*(D)$ is not an empty set and each element P of $\mathrm{QMax}^*(D)$ is a prime ideal of D [FL2, Lemma 2.3]. Furthermore, it was proved in [FL2, Corollary 2.7 (1)] that $\tilde{*} = *_{\mathcal{M}(*_f)}$ for each semistar operation $*$ on D.

Proposition 25. Let $*$ be a semistar operation of finite type on D. Then $*^\alpha = (\tilde{*})^\alpha$.

Proof. Let E be an element of $\mathbf{K}(D[X])$. Then $E^{(\tilde{*})^\alpha} = \bigcup\{E : I[X] \mid I^{\tilde{*}} = D^{\tilde{*}}\} = \bigcup\{E : I[X] \mid I \not\subseteq M$ for all $M \in \mathrm{QMax}^{\tilde{*}}(D)\} = \bigcup\{E : I[X] \mid I \not\subseteq M$ for all $M \in \mathcal{M}(\tilde{*})\}$.

But, by [FL2, Corollary 3.5 (2)], $\mathcal{M}(\tilde{*}) = \mathcal{M}(*_f) = \mathcal{M}(*)$. Hence $E^{(\tilde{*})^\alpha} = \bigcup\{E : I[X] \mid I \not\subseteq M$ for all $M \in \mathcal{M}(*)\} = \bigcup\{E : I[X] \mid I^* = D^*\} = E^{*^\alpha}$ which implies that $*^\alpha = (\tilde{*})^\alpha$. □

Remark 26. Let $\mathcal{F}$ be a localizing system on D. Then $\mathcal{F} = \mathcal{F}^{*_\mathcal{F}}$ by [FH, Theorem 2.10 (A)]. Using this result, we can give another proof of Proposition 25. First, by definition $\bar{*} = *_{\mathcal{F}^*}$ for each semistar operation $*$ on D and so $\mathcal{F}^{\bar{*}} = \mathcal{F}^{*_{\mathcal{F}^*}} = \mathcal{F}^*$ by [FH, Theorem 2.10 (A)]. Hence $(\bar{*})^\alpha = *_{\mathcal{F}^{\bar{*}}[X]} = *_{\mathcal{F}^*[X]} = *^\alpha$. In particular, if $*$ is of finite type, then $\bar{*} = \tilde{*}$ and so $*^\alpha = (\tilde{*})^\alpha$.

We shall denote the set of all stable semistar operations on D (resp. all stable semistar operations of finite type on D) by $\overline{\mathbf{SS}}(D)$ (resp. $\widetilde{\mathbf{SS}}(D)$). It easily follows from Proposition 24 that $\overline{\mathbf{SS}}(D) = \{\bar{*} \mid * \in \mathbf{SS}(D)\}$ and $\widetilde{\mathbf{SS}}(D) = \{\tilde{*} \mid * \in \mathbf{SS}_f(D)\}$.

Theorem 27. Let $*$ be a semistar operation on D. Then
(1) $(*^\alpha)^\delta = \bar{*}$.
(2) $*$ is stable if and only if $(*^\alpha)^\delta = *$.
(3) $\mid \overline{\mathbf{SS}}(D) \mid \leq \mid \overline{\mathbf{SS}}(D[X]) \mid$.

Proof. (1) For each $E \in \mathbf{K}(D)$, $E^{(*^\alpha)^\delta} = (ED[X])^{*^\alpha} \bigcap K = (ED[X])^{*_{\mathcal{F}^*[X]}} \bigcap K = \bigcup\{ED[X] : JD[X] \mid J \in \mathcal{F}^*\} \bigcap K = \bigcup\{(ED[X] : JD[X]) \bigcap K \mid J \in \mathcal{F}^*\} = \bigcup\{E : J \mid J \in \mathcal{F}^*\} = E^{*_{\mathcal{F}^*}} = E^{\bar{*}}$ which implies $(*^\alpha)^\delta = \bar{*}$.

(2) This follows from Theorem 27 (1) and [FH, Proposition 3.7 (1)].

(3) First, for each $* \in \mathbf{SS}(D)$, we have $*^\alpha \in \overline{\mathbf{SS}}(D[X])$ by Proposition 21 (1). Next, if $*$ is stable, then $(*^\alpha)^\delta = \bar{*} = *$ by Proposition 24 (3) and Theorem 27 (1). Hence the map $* \mapsto *^\alpha$ of $\overline{\mathbf{SS}}(D)$ into $\overline{\mathbf{SS}}(D[X])$ is injective and therefore $\mid \overline{\mathbf{SS}}(D) \mid \leq \mid \overline{\mathbf{SS}}(D[X]) \mid$. □

Theorem 28. Let $*$ be a semistar operation on D. Then
(1) $(*^\alpha)^\delta = \tilde{*}$ for each $* \in \mathbf{SS}_f(D)$.
(2) $\mid \widetilde{\mathbf{SS}}(D) \mid \leq \mid \widetilde{\mathbf{SS}}(D[X]) \mid$.

Proof. (1) This follows from Proposition 24 (2) and Theorem 27 (1).

(2) Let $* \in \widetilde{\mathbf{SS}}(D)$. Then, by Proposition 21 (1) and (2), $*^\alpha \in \widetilde{\mathbf{SS}}(D[X])$. Moreover, $(*^\alpha)^\delta = \tilde{*}$ by Theorem 28 (1). Therefore the map $* \mapsto *^\alpha$ of $\widetilde{\mathbf{SS}}(D)$ into $\widetilde{\mathbf{SS}}(D[X])$ is injective and so $\mid \widetilde{\mathbf{SS}}(D) \mid \leq \mid \widetilde{\mathbf{SS}}(D[X]) \mid$. □

Let Δ be a subset of Spec(D). Then $\mathcal{F}(\Delta) = \bigcap\{\mathcal{F}(P) \mid P \in \Delta\}$, where $\mathcal{F}(P) = \{I \in \mathbf{I}(D) \mid I \not\subseteq P\}$ is a localizing system on D. A localizing system $\mathcal{F}$ on D is called *spectral* if there exists a subset Δ of Spec(D) such that $\mathcal{F} = \mathcal{F}(\Delta)$. For each subset Δ of Spec(D), if we set $E^{*_\Delta} = \bigcap\{ED_P \mid P \in \Delta\}$ for each $E \in \mathbf{K}(D)$, then $*_\Delta$ is a stable semistar operation on D by [OM, Theorem 20 (1)]. A semistar operation $*$ on D is called *spectral* if $* = *_\Delta$ for some subset Δ of Spec(D).

In this paper, we denote the set $\{P[X] \mid P \in \Delta\}$ by $\Delta[X]$ for each subset Δ of Spec(D).

Lemma 29. If Δ is a subset of Spec(D), then $\mathcal{F}(\Delta)[X] \subseteq \mathcal{F}(\Delta[X])$.

Proof. Let $J \in \mathcal{F}(\Delta)[X]$. Then $J \supseteq I[X]$ for some $I \in \mathcal{F}(\Delta)$. Since $I \not\subseteq P$ for all $P \in \Delta, I[X] \not\subseteq P[X]$ for all $P[X] \in \Delta[X]$ which implies that $I[X] \in \mathcal{F}(\Delta[X])$ and therefore $J \in \mathcal{F}(\Delta[X])$. □

Proposition 30. If $\Delta = \{P_\alpha\}$ is a subset of Spec(D), then $(*_{\Delta[X]})^\delta = *_\Delta$.

Proof. For each $E \in \mathbf{K}(D)$, we have $E^{(*_{\Delta[X]})^\delta} = (ED[X])^{*_{\Delta[X]}} \bigcap K = (\bigcap\{E[X]_{P_\alpha[X]} \mid$

$P_\alpha \in \Delta\})\bigcap K = \bigcap\{E[X]_{P_\alpha[X]}\bigcap K \mid P_\alpha \in \Delta\} = \bigcap\{E_{P_\alpha} \mid P_\alpha \in \Delta\} = E^{*_\Delta}$ and hence $(*_{\Delta[X]})^\delta = *_\Delta$. $\square$

Proposition 31 ([FH, Lemma 4.2]). Let Δ be a subset of $\mathrm{Spec}(D)$. Then $*_\Delta = *_{\mathcal{F}(\Delta)}$ and $\mathcal{F}^{*_\Delta} = \mathcal{F}(\Delta)$.

Note. It follows from Proposition 16 that $*_\Delta$ is stable for each subset Δ of $\mathrm{Spec}(D)$, because, by Proposition 31, $*_\Delta = *_{\mathcal{F}(\Delta)}$ and $\mathcal{F}(\Delta)$ is a localizing system on D. Thus every spectral semistar operation is a stable semistar operation.

Lemma 32. Let $\mathcal{F}$ be a localizing system on $D[X]$. If we set $\mathcal{F}_D = \{I \in \mathbf{I}(D) \mid I[X] \in \mathcal{F}\} = \{J\bigcap D \mid J \in \mathbf{I}(D[X])$ such that $J\bigcap D \neq (0)$ and $(J\bigcap D)[X] \in \mathcal{F}\}$, then $\mathcal{F}_D$ is a localizing system on D.

Proof. We need only to show that $\mathcal{F}_D$ satisfies (**LS1**) and (**LS2**).

(**LS1**) Let $I \in \mathcal{F}_D$ and let J be an ideal of D containing I. Then $I[X] \in \mathcal{F}$ and $I[X] \subseteq J[X]$ and hence $J[X] \in \mathcal{F}$ which implies $J \in \mathcal{F}_D$.

(**LS2**) Suppose that $I \in \mathcal{F}_D$ and J is an ideal of D such that $J :_D iD \in \mathcal{F}_D$ for all $i \in I$. Then $(J :_D iD)[X] \in \mathcal{F}$ and $(J :_D iD)[X]iD[X] \subseteq J[X]$. Hence $(J :_D iD)[X] \subseteq J[X] :_{D[X]} iD[X]$ and so $J[X] :_{D[X]} iD[X] \in \mathcal{F}$, because $(J :_D iD)[X] \in \mathcal{F}$. Thus $J[X] :_{D[X]} iD[X] \in \mathcal{F}$ for all $i \in I$. Choose $f \in I[X]$ and set $f = a_0+a_1X+\cdots+a_nX^n$. Then $J[X] :_{D[X]} a_iD[X] \in \mathcal{F}$ for all $i = 0, 1, \cdots, n$ and so $\bigcap_{i=0}^n(J[X] :_{D[X]} a_iD[X]) \in \mathcal{F}$. If we choose $g(X) \in \bigcap_{i=0}^n(J[X] :_{D[X]} a_iD[X])$, then $f(X)g(X) = (a_0 + a_1X + \cdots + a_nX^n)g(X) = a_0g(X) + a_1g(X)X + \cdots + a_ng(X)X^n \in J[X]$ and so $g(X) \in J[X] :_{D[X]} f(X)D[X]$. Thus $\bigcap_{i=0}^n(J[X] :_{D[X]} a_iD[X]) \subseteq J[X] :_{D[X]} f(X)D[X]$ and then $J[X] :_{D[X]} f(X)D[X] \in \mathcal{F}$. Hence $J[X] :_{D[X]} f(X)D[X] \in \mathcal{F}$ for all $f \in I[X]$ which implies that $J[X] \in \mathcal{F}$ and so $J \in \mathcal{F}_D$. $\square$

Proposition 33. If $\mathcal{F}$ is a localizing system of finite type on $D[X]$, then $\mathcal{F}_D$ is also a localizing system of finite type on D.

Proof. Choose $I \in \mathcal{F}_D$. Then $I[X] \in \mathcal{F}$. By hypothesis, $\mathcal{F}$ is of finite type and so there exists a finitely generated ideal $J \in \mathcal{F}$ such that $J \subseteq I[X]$. Set $J = (f_1, f_2, \cdots, f_n)$. Then $c(f_i) \subseteq I$ for all $i = 1, 2, \cdots, n$. Then $I_0 = \sum_{i=1}^n c(f_i)$ is finitely generated and $I_0 \subseteq I$. Since $J \subseteq I_0[X]$, we obtain $I_0[X] \in \mathcal{F}$. Thus $I_0 \in \mathcal{F}_D$ and $I_0 \subseteq I$. Therefore $\mathcal{F}_D$ is of finite type. $\square$

Proposition 34. Let $\mathcal{F}$ be a localizing system on D. Then $\mathcal{F}[X]_D = \mathcal{F}$.

Proof. ($\supseteq$) If $I \in \mathcal{F}$, then $I[X] \in \mathcal{F}[X]$ and therefore $I \in \mathcal{F}[X]_D$.

($\subseteq$) Suppose that $I \in \mathcal{F}[X]_D$. Then $I[X] \in \mathcal{F}[X]$ and then $I[X] \supseteq J[X]$ for some $J \in \mathcal{F}$. Then, since $I \supseteq J$ and $J \in \mathcal{F}$, we have $I \in \mathcal{F}$. $\square$

Proposition 35. Let $\mathcal{F}$ be a localizing system on $D[X]$. Then

(1) $\mathcal{F}_D[X] \subseteq \mathcal{F}$.

(2) $\mathcal{F} = \mathcal{F}_D[X] \Longleftrightarrow J\bigcap D \neq (0)$ and $(J\bigcap D)[X] \in \mathcal{F}$ for all $J \in \mathcal{F}$.

Proof. (1) Suppose that $J \in \mathcal{F}_D[X]$. Then $J \supseteq I[X]$ for some $I \in \mathcal{F}_D$. Then $I[X] \in \mathcal{F}$ and so we get $J \in \mathcal{F}$.

(2) ($\Longleftarrow$) Choose $J \in \mathcal{F}$. Then $J \bigcap D \neq (0)$ and $(J \bigcap D)[X] \in \mathcal{F}$. Then, since $(J \bigcap D)[X] \in \mathcal{F}_D[X]$, evidently $J \in \mathcal{F}_D[X]$. Thus $\mathcal{F} \subseteq \mathcal{F}_D[X]$ and therefore $\mathcal{F} = \mathcal{F}_D[X]$.

($\Longrightarrow$) Let $J \in \mathcal{F} = \mathcal{F}_D[X]$. Then $J \supseteq I[X]$ with $I \in \mathcal{F}_D$. Hence $J \bigcap D \supseteq I$ with $I \in \mathcal{F}_D$ and therefore $J \bigcap D \neq (0)$ and $(J \bigcap D)[X] \in \mathcal{F}_D[X] = \mathcal{F}$. □

Proposition 36. Let $*$ be a semistar operation on $D[X]$. Then $(*^{\delta})^{\alpha} \leq \bar{*} \leq *$.

Proof. Let E be an element of $\mathbf{K}(D[X])$. Then $(E^*)^{(*^{\delta})^{\alpha}} = (E^*)^{*_{\mathcal{F}^{*^{\delta}}[X]}} = \bigcup\{E^* : J \mid J \in \mathcal{F}^{*^{\delta}}[X]\} = \bigcup\{E^* : I[X] \mid I \in \mathcal{F}^{*^{\delta}}\} = \bigcup\{E^* : I[X]^* \mid I \in \mathcal{F}^{*^{\delta}}\}$. But, by definition, $I \in \mathcal{F}^{*^{\delta}}$ if and only if $I^{*^{\delta}} = D^{*^{\delta}}$. Then, by [OM, Proposition 35], $(I[X])^* = (I^{*^{\delta}}[X])^* = (D^{*^{\delta}}[X])^* = D[X]^*$ for each $I \in \mathcal{F}^{*^{\delta}}$. Hence $(E^*)^{(*^{\delta})^{\alpha}} = \bigcup\{E^* : I[X]^* \mid I \in \mathcal{F}^{*^{\delta}}\} = E^* : D[X]^* = E^* : D[X] = E^*$ for all $E \in \mathbf{K}(D[X])$ and therefore $(*^{\delta})^{\alpha} \leq *$ by [OM, Lemma 16]. Then, by Proposition 21 (1), $(*^{\delta})^{\alpha}$ is stable and so $(*^{\delta})^{\alpha} \leq \bar{*} \leq *$ by [FH, Proposition 3.6 (a) and Proposition 3.7 (1)]. □

Theorem 37. Let $*$ be a semistar operation on $D[X]$. If we set $*_{[X]} = *_{(\mathcal{F}^*)_D}$, then $*_{[X]}$ is a semistar operation on D and $*_{[X]} \leq *^{\delta}$.

Proof. By definition, $E^{*_{[X]}} = E^{*_{(\mathcal{F}^*)_D}} = \bigcup\{E : J \mid J \in (\mathcal{F}^*)_D\}$ for each $E \in \mathbf{K}(D)$ and $J \in (\mathcal{F}^*)_D \Longleftrightarrow J[X] \in \mathcal{F}^* \Longleftrightarrow J[X]^* = D[X]^*$. Hence if $x \in E : J$, then $xJ \subseteq E$ and then $xJ[X] \subseteq E[X]$ and so $x(J[X])^* \subseteq E[X]^*$. But then, since $J[X]^* = D[X]^*$ for each $J \in (\mathcal{F}^*)_D$, we have $xD[X]^* \subseteq E[X]^*$ and so $x \in E[X]^*$. Thus $E : J \subseteq E[X]^* \bigcap K = E^{*^{\delta}}$ for all $E \in \mathbf{K}(D)$ and all $J \in (\mathcal{F}^*)_D$ which implies that $E^{*_{[X]}} \subseteq E^{*^{\delta}}$ for all $E \in \mathbf{K}(D)$ and therefore $*_{[X]} \leq *^{\delta}$. □

Lemma 38. Let $*$ be a semistar operation on D. Then $(\mathcal{F}^{*^{\alpha}})_D = \mathcal{F}^*$.

Proof. By definition, $*^{\alpha} = *_{\mathcal{F}^*[X]}$ and hence $\mathcal{F}^{*^{\alpha}} = \mathcal{F}^{*_{\mathcal{F}^*[X]}} = \mathcal{F}^*[X]$ by [FH, Theorem 2.10 (A)]. Therefore $(\mathcal{F}^{*^{\alpha}})_D = \mathcal{F}^*[X]_D = \mathcal{F}^*$ by Proposition 34.□

Theorem 39. $(*^{\alpha})_{[X]} = (*^{\alpha})^{\delta}$ for each semistar operation $*$ on D.

Proof. By Lemma 38, $(*^{\alpha})_{[X]} = *_{(\mathcal{F}^{*^{\alpha}})_D} = *_{\mathcal{F}^*} = \bar{*}$. But $(*^{\alpha})^{\delta} = \bar{*}$ also holds by Theorem 27. Therefore we get $(*^{\alpha})_{[X]} = (*^{\alpha})^{\delta}$. □

Corollary 40. Let $*$ be a semistar operation on D. Then $*$ is stable if and only if $* = (*^{\alpha})_{[X]}$.

Proof. This follows immediately from Theorems 27 and 39.□

Example 41. As easily seen, $\mathcal{F}^{\bar{d}_D} = \{D\}$ for each integral domain D and therefore $\mathcal{F}^{\bar{d}_{D[X]}} = \{D[X]\}$. Then $(\mathcal{F}^{\bar{d}_{D[X]}})_D = \{D\}$ and so $(\bar{d}_{D[X]})_{[X]} = *_{(\mathcal{F}^{\bar{d}_{D[X]}})_D} = \bar{d}_D$. On the other hand, $E^{(\bar{d}_{D[X]})^\delta} = E[X]^{\bar{d}_{D[X]}} \bigcap K = E[X] \bigcap K = E = E^{\bar{d}_D}$ for each $E \in \mathbf{K}(D)$ and so $(\bar{d}_{D[X]})^\delta = \bar{d}_D$. Thus $(\bar{d}_{D[X]})^\delta = (\bar{d}_{D[X]})_{[X]} = \bar{d}_D$.

Remark 42. The above result shown in Example 41 follows directly from Theorem 39. In fact, $(\bar{d}_D)^\alpha = \bar{d}_{D[X]}$ by Example 22 and therefore $(\bar{d}_{D[X]})^\delta = ((\bar{d}_D)^\alpha)^\delta = ((\bar{d}_D)^\alpha)_{[X]} = (\bar{d}_{D[X]})_{[X]}$.

Proposition 43. Let R be an overring of D. Then $(*_{(R[X])})^\delta = *_{(R)}$.

Proof. $E^{(*_{(R[X])})^\delta} = (ED[X])^{*_{(R[X])}} \bigcap K = ER[X] \bigcap K = ER = E^{*_{(R)}}$ for each $E \in \mathbf{K}(D)$ and hence $(*_{(R[X])})^\delta = *_{(R)}$. □

Proposition 44. Let R be an overring of D. Then $(*_{(R[X])})_{[X]} = \widetilde{*_{(R)}}$.

Proof. $E^{(*_{(R[X])})_{[X]}} = \bigcup\{E : J \mid J \in (\mathcal{F}^{*_{(R[X])}})_D\} = \bigcup\{E : J \mid J[X]^{*_{(R[X])}} = D[X]^{*_{(R[X])}}\} = \bigcup\{E : J \mid JR[X] = R[X]\} = \bigcup\{E : J \mid JR = R\} = \bigcup\{E : J \mid J^{*_{(R)}} = D^{*_{(R)}}\} = E^{\widetilde{*_{(R)}}} = E^{\widetilde{*_{(R)}}}$ for each $E \in \mathbf{K}(D)$, since $*_{(R)}$ is of finite type. Hence we have $(*_{(R[X])})_{[X]} = \widetilde{*_{(R)}}$. □

Corollary 45. Let R be an overring of D. Then $(*_{(R[X])})^\delta = (*_{(R[X])})_{[X]}$ if and only if $*_{(R)} = \widetilde{*_{(R)}}$,i.e., $*_{(R)}$ is a stable semistar operation on D.

Proof. This follows immediately from Propositions 43 and 44. □

Corollary 46. (1) An overring R of D is a flat overring of D if and only if $(*_{(R[X])})^\delta = (*_{(R[X])})_{[X]}$.

(2) D is a Prüfer domain if and only if $(*_{(R[X])})^\delta = (*_{(R[X])})_{[X]}$ for each overring R of D.

Proof. (1) By [O3, Remark 37 (1)], R is a flat overring of D if and only if $*_{(R)}$ is a stable semistar operation on D and hence our assertion follows from Corollary 45.

(2) Since D is a Prüfer domain if and only if each overring R of D is a flat overring of D, our assertion also follows from Corollary 45. □

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