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Note on g-monoids

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Abstract

We study almost pseudo-valuation semigroups S, especially will study semistar operations on S, and will determine the complete integral closure of S. We will study various cancellation properties of semistar operations on g-monoids. Also, we will study Kronecker function rings of any semistar operations on g-monoids.

A. Badawi and E. Houston [BH] introduced an almost pseudo-valuation domain. An integral domain D with quotient field K is called an almost pseudo-valuation domain (or, an APVD) if every prime ideal P of D is strongly primary, that is, if, for elements $x, y \in K$, $xy \in P$ and $x \notin P$ implies $y^n \in P$ for some positive integer n. In this paper we will introduce an almost pseudo-valuation semigroup (or, an APVS), and will study it, especially will study semistar operations on an APVS, and will determine the complete integral closure of an APVS. Let G be a torsion-free abelian additive group. A subsemigroup S of G which contains 0 is called a grading monoid (or, a g-monoid). We may confer [M3] for g-monoids. Also, we will study various cancellation properties of semistar operations on g-monoids. Moreover, we will study Kronecker function rings of any semistar operations on g-monoids. The paper consists of seven sections. In §1, we will introduce an APVS, and will show that [BH] holds for g-monoids. In §2, we will show a semigroup version of [KMOS], and will determine the complete integral closure of the APVS. In §3, we will give conditions for an APVS to have only a finite number of semistar operations. In §4, we will study conditions for an APVD to have only a finite number of semistar operations. In §5, we will introduce various cancellation properties of semistar operations on a g-monoid, and will show various implications of the cancellation properties. In §6, we will study results for Kronecker function rings of e.a.b. semistar operations for any semistar operations on g-monoids. §7 is an appendix. Many parts in every §1 $\sim$ §4 are restatements of [M7]. Since it seems that [M7] has not appeared about six years, and we refered [M7] in

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other papers, we will state them for the convenience.

§1 Almost pseudo-valuation semigroups

In this section we will show a semigroup version of [BH]. Almost all proofs of the semigroup version are easy and simple modification of those of [BH]. However, for convenience sake, we will note the definitions and the results.

Throughout the section, S will denote a g-monoid. The group $\mathrm{q}(S) = \{x - y \mid x, y \in S\}$ is called the quotient group of S. A non-empty subset I of S is called an ideal if $S + I \subset I$. We define an ideal I of S to be powerful if, whenever $x + y \in I$ for elements $x, y \in \mathrm{q}(S)$, we have $x \in S$ or $y \in S$.

(1.1). An ideal I of S is powerful if and only if $-x+I \subset S$ for every $x \in \mathrm{q}(S)-S$.

A proper ideal P of S is called a prime ideal if $x + y \in P$ for $x, y \in S$, then $x \in P$ or $y \in P$. A prime ideal of S is called a strongly prime ideal if $x + y \in P$ for $x, y \in \mathrm{q}(S)$, then $x \in P$ or $y \in P$.

(1.2). A prime ideal of S is strongly prime if and only if it is powerful.

(1.3). If $J \subset I$ are ideals of S with I powerful, then J is also powerful.

Let I be an ideal of S, and n be a positive integer. Then nI denotes the ideal generated by $\{x_1 + \cdots + x_n \mid \text{every } x_i \in I\}$. For subsets A, B of a torsion-free abelian group, the subset $\{x \in B \mid nx \in A \text{ for some } n > 0\}$ of B is denoted by $\mathrm{Rad}_B(A)$.

(1.4) Theorem. Let I be a powerful ideal of S.
(1) If J is an ideal of S, then either $J \subset I$ or $2I \subset J$.
(2) If J is a prime ideal of S, then I and J are comparable.
(3) The prime ideals of S contained in $\mathrm{Rad}_S(I)$ are linearly ordered.

An element a of S is called a unit of S if $-a \in S$. If $\mathrm{q}(S) = S$, then S is a unigue ideal of S. If $\mathrm{q}(S) \supsetneqq S$, then S has a unique maximal ideal. If every prime ideal of S is a strongly prime ideal, then S is called a pseudo-valuation semigroup (or a PVS).

(1.5). S is a PVS if and only if the maximal ideal of S is powerful.

(1.6). If S contains a powerful ideal, then S contains a unique largest powerful ideal.

(1.7). If I is a proper powerful ideal of S, and if $P = \cap_{k=0}^{\infty} kI$ is non-empty, then P is a strongly prime ideal.

(1.8). Let I be a powerful ideal of S. If $x, y \in \mathrm{q}(S)$ and $x + y \in \mathrm{Rad}_S(I)$, then there is a positive integer m such that either $mx \in I$ or $my \in I$. In particular, if I is a proper powerful ideal, then $\mathrm{Rad}_S(I)$ is a prime ideal.

(1.9). Let I be a powerful ideal of S. If $x \in \mathrm{q}(S)$ and $nx \in I$ for some $n > 0$, then $(n+k)x \in S$ for every $k \geq 0$.

An ideal J of S is called a radical ideal of S if $\mathrm{Rad}_S(I) = I$. A radical ideal J of S is called strongly radical if $x \in \mathrm{q}(S)$ and $nx \in J$ for some $n > 0$ implies $x \in J$. S is called a seminormal semigroup if $x \in S$ whenever $x \in \mathrm{q}(S)$ and $nx \in S$ for all sufficiently large n.

(1.10). Let I be a proper powerful ideal of S. Then $\mathrm{Rad}_S(I)$ is powerful (and therefore strongly prime) if and only if $\mathrm{Rad}_S(I)$ is strongly radical. In particular, if S is seminormal, then $\mathrm{Rad}_S(I)$ is strongly prime.

An intermediate g-monoid between S and $\mathrm{q}(S)$ is called an oversemigroup of S. Let G be a torsion-free abelian group, let Γ be a totally ordered abelian group, and let v be a mapping of G onto Γ. If $v(a+b) = v(a) + v(b)$ for all $a, b \in G$, then v is called a valuation on G, and the subsemigroup $\{x \in G \mid v(x) \geq 0\}$ of G is called the valuation semigroup belonging to v. v is said to belong to V, Γ is called the value group of v, and Γ is also called the value group of V.

(1.11). Let I be a powerful ideal of S, and let T be an oversemigroup of S. Then $I + T$ is a powerful ideal of T. In particular, if $I + T = T$, then T is a valuation semigroup.

(1.12). Let I be a powerful ideal of S, and suppose that $P \subset I$ is a finitely generated prime ideal of S. Then S is a PVS with maximal ideal P.

The oversemigroup $\mathrm{Rad}_{\mathrm{q}(S)}(S)$ of S is called the integral closure of S, and is denoted by $\bar{S}$.

(1.13) Theorem. Suppose that S admits a powerful ideal I and that $M = \mathrm{Rad}_S(I)$ is a maximal ideal of S. Then :

(1) $I + \bar{S} \subset M$, and therefore $I + \bar{S}$ is an ideal of S.

(2) $\bar{S}$ is a PVS with maximal ideal $N = \mathrm{Rad}_{\bar{S}}(I + \bar{S})$, and hence $(N : N) = \{x \in \mathrm{q}(S) \mid x + N \subset N\}$ is a valuation oversemigroup of S with maximal ideal N.

If P is a prime ideal of S, then the oversemigroup $\{x - y \mid x \in S \text{ and } y \in S - P\}$ is denoted by S_P.

(1.14). Let I be a powerful ideal of S, and let $P = \mathrm{Rad}_S(I)$. Then $T = \overline{S_P}$ is a PVS with maximal ideal $N = \mathrm{Rad}_T(I + T)$. It follows that $(N : N)$ is a valuation oversemigroup of S with maximal ideal N.

(1.15). Let I be a powerful ideal of S, and let $T \neq \mathrm{q}(S)$ be an oversemigroup of S with maximal ideal N. Then S and T share an ideal which is powerful in both S and T. In fact:

(1) If $I+T=T$, then $P=N\cap S$ is a common ideal which is powerful in both semigroups.

(2) If $I+T\neq T$, then $2I+T$ is a common ideal, and $3I+T$ is powerful in both semigroups.

(1.16). Suppose that T is an oversemigroup of S, and that S and T share the ideal J. If J is powerful in T, then $3J$ is a powerful ideal of S.

A proper ideal I of S is called a primary ideal of S if $x+y\in I$ and $x\notin I$, then $ny\in I$ for some $n>0$.

(1.17). A primary ideal of a valuation semigroup is strongly primary.

For a subset A of S, we define $\mathrm{E}(A)$ by $\mathrm{E}(A)=\{x\in \mathrm{q}(S)\mid nx\notin A$ for every $n\geq 1\}$.

(1.18). An ideal I of S is strongly primary if and only if $-x+I\subset I$ for every $x\in \mathrm{E}(I)$.

(1.19) Theorem. Let S be a seminormal semigroup. If I is a proper stongly primary ideal of S, then I is powerful, and $\mathrm{Rad}_S(I)$ is strongly prime. In particular, a prime ideal of S is strongly prime if and only if it is strongly primary.

(1.20). Let I be a proper strongly primary ideal of S, and let T be an oversemigroup of S. Then either $I+T=T$ or $I+T=I$.

(1.21). If I is a proper strongly primary ideal of S, then $I+\bar{S}=I$. Moreover, $3I$ is powerful in both S and $\bar{S}$.

(1.22). If I is a proper strongly primary ideal of S, and if $\cap_{n=1}^{\infty}nI$ is non-empty, then $\cap_{n=1}^{\infty}nI$ is a strongly prime ideal of S.

(1.23) Theorem. If I is a strongly primary ideal of S, then I is comparable to every radical ideal of S. Moreover, the prime ideals of S which are properly contained in I are strongly prime and linearly ordered.

(1.24). If P is a prime ideal of S which is strongly primary but not strongly prime, then P is the only prime with this property.

(1.25) Theorem. Let I be a strongly primary ideal of S, and let $T\neq \mathrm{q}(S)$ be an oversemigroup of S. Then S and T share a strongly primary ideal. In fact:

(1) If $I+T\neq T$, then $I+T=I$ is a common strongly primary ideal;

(2) If $I+T=T$, then T is strongly primary, and, for the maximal ideal N of T, $N\cap S$ is a common strongly prime ideal of S and T.

(1.26) Theorem. Let I be a proper ideal of S. Then the following statements are equivalent.

(1) I is a strongly primary ideal of S.

(2) I is a primary ideal in some valuation oversemigroup of S.

(3) $V = (I : I)$ is a valuation semigroup, and I is (an ideal of V which is) primary to the maximal ideal of V.

(1.27). If S admits a proper principal strongly primary ideal, then S is a valuation semigroup.

(1.28). Let I be a strongly primary ideal of S. Then,

(1) $I \subset x + S$ for every $x \in S - \mathrm{Rad}_S(I)$, and

(2) If I is finitely generated, then S has maximal ideal $\mathrm{Rad}_S(I)$.

(1.29). Let P be a strongly primary prime ideal of S, and let I be an ideal of S with $\mathrm{Rad}_S(I) = P$. Then $P + I$ is strongly primary. In particular, nP is strongly primary for every $n \geq 1$.

We say that a g-monoid S is an almost pseudo-valuation semigroup (or, an APVS) if every prime ideal of S is strongly primary. A prime ideal of S is called divided if it is comparable to every ideal of S. If every prime ideal of S is divided, then S is called a divided semigroup.

(1.30). Let S be an APVS. Then S is a divided semigroup. Moreover, every non-maximal prime ideal of S is strongly prime.

(1.31). The followings are equivalent for a g-monoid S.

(1) Every primary ideal of S is strongly primary.

(2) Either S is a valuation semigroup or S is a PVS with unbranched maximal ideal.

(1.32) Theorem. The following statements are equivalent for a g-monoid S.

(1) S is an APVS.

(2) The maximal ideal of S is strongly primary.

(3) If N is the maximal ideal of S, then $-x+N \subset N$ for every element $x \in \mathrm{E}(N)$.

(4) The maximal ideal M of S is such that $(M : M)$ is a valuation semigroup with M primary to the maximal ideal of $(M : M)$.

(5) There is a valuation oversemigroup in which M is a primary ideal.

(1.33). If S is strongly primary, then S is an APVS (and hence S admits a proper strongly primary ideal).

(1.34). Let S be an APVS with maximal ideal M. If T is an oversemigroup of S with $M + T = T$, then T is also an APVS.

(1.35). If S is an APVS with maximal ideal M, then $\bar{S}$ is a PVS with maximal ideal $N = \mathrm{Rad}_{\bar{S}}(M + \bar{S})$.

(1.36). If every oversemigroup of a g-monoid S is an APVS, then $\bar{S}$ is a valuation semigroup.

Proof. Suppose the contrary. Let P be the maximal ideal of S, $(P : P) = V, M = \mathrm{Rad}_V(P)$, G be the unit group of V, and K be the unit group of $\bar{S}$. $\bar{S}$ is a PVS with maximal ideal M. We may take an element $g \in G - K$. Set $T = \bar{S}[2g] = \bar{S} + \boldsymbol{Z}_0 2g$, and let N be the maximal ideal of T.

Let $l \in \boldsymbol{Z}$. Then $lg \in T$ if and only if $l \in 2\boldsymbol{Z}_0$. It follows that $2g \in N$, $3g \notin N$, and $-ng \notin N$ for every positive integer n.

Since $2g = 3g + (-g)$, we have that N is not strongly primary, hence T is not an APVS; a contradiction.

(1.37). Let S be an APVS with $\bar{S}$ a valuation semigroup, and assume that every integral oversemigroup of S is an APVS. Then every oversemigroup of S is an APVS.

If every oversemigroup of S which is different from $\mathrm{q}(S)$ has a non-empty conductor to S, then S is called a conducive semigroup.

(1.38) Theorem. The following conditions are equivalent for a g-monoid S.
(1) S is a conducive semigroup.
(2) S admits a powerful ideal.
(3) S admits a strongly primary ideal.
(4) S shares an ideal with some conducive oversemigroup.

Assume that there are prime ideals P_i of S such that $P_1 \subsetneqq P_2 \subsetneqq \cdots \subsetneqq P_n$, and that there does not exist prime ideals Q_i of S such that $Q_1 \subsetneqq Q_2 \subsetneqq \cdots \subsetneqq Q_{n+1}$, then n is called the Krull dimension (or, the dimension) of S.

If every ideal of S is finitely generated, then S is called a Noetherian semigroup.

Let X be a non-empty set, and assume that, for every $s \in S$ and $x \in X$, there is defined the element $s + x$ of X. If $0 + x = x$ and, for every $s_1, s_2 \in S$, $(s_1 + s_2) + x = s_1 + (s_2 + x)$, then X is called an S-module.

(1.39) Theorem. A Noetherian semigroup S with $S \neq \mathrm{q}(S)$ is conducive if and only if each of the following conditions holds:
(1) S is of dimension 1.
(2) $\bar{S}$ is a rank one discrete valuation semigroup.
(3) $\bar{S}$ is a finitely generated S-module.

§2 The complete integral closure of an APVS

M. Kanemitsu, R. Matsuda, N. Onoda and T. Sugatani [KMOS] determined the complete integral closure of an APVD. In this section, we will show a semigroup version of [KMOS], and will determine the complete integral closure of an APVS. The proofs of the semigroup version are easy modification of those in [KMOS]. However, for convenience sake, we will note the results. Throughout this section, S will denote a g-monoid.

Let t be an element of an extension semigroup T of S. If there is $s \in S$ such that $s + nx \in S$ for every positive integer n, then x is called almost integral over S. The set of almost integral elements in $\mathrm{q}(S)$ is called the complete integral closure of S, and is denoted by S^c.

(2.1). Let I be an ideal of S such that $(I : I)$ is a valuation semigroup, then I is comparable with any prime ideal of S.

(2.2). If $P \subsetneqq I$ are ideals of S with P prime, then $(I : I) \subset (S : I) \subset (P : P)$.

(2.3). The following two statements are equivalent.

(1) S has the maximal ideal M such that $(M : M)$ is a valuation semigroup.

(2) For any prime ideal P of S, $(P : P)$ is a valuation semigroup.

Furthermore, if S satisfies one of these conditions, then the following statement holds.

(3) The prime ideals of S are linearly ordered.

(2.4). Let P be a prime ideal of S, $V = (P : P)$, and $M = \mathrm{Rad}_V(P)$. Then the following statements are equivalent.

(1) P is a strongly primary ideal of S.

(2) V is a valuation semigroup and M is the maximal ideal of V.

(2.5). Assume that S is of dimension 1. If the maximal ideal P of S is strongly primary, then $(P : P)$ is of dimension 1.

(2.6). Let P be an ideal of S, $V = (P : P)$, and $M = \mathrm{Rad}_V(P)$. Then the following conditions are equivalent.

(1) P is a strongly prime ideal of S.

(2) V is a valuation semigroup, and P is the maximal ideal of V.

(2.7). Let P be a prime ideal of S and $V = (P : P)$. Suppose that P is strongly primary. Then the following statements hold.

(1) $S_P \subset V$ and $P + S_P = P$.

(2) For every $a \in S - P$, we have $P = a + P$. In particular, $P \subset \cap_{n=1}^{\infty}(na + S)$.

(3) For an ideal I of S, we have either $I \subset P$ or $P \subset I$.

(4) If there is a prime ideal Q of S such that $Q \not\subset P$ and $(Q : Q)$ is a valuation semigroup, then $S_P = V$. In particular, P is strongly prime.

(2.8). The following statements hold.

(1) $S^c = \mathrm{q}(S)$ if and only if $\cap_{n=1}^{\infty}(na + S) \neq \emptyset$ for every $a \in S$.

(2) Let P be a prime ideal of S of height 1. Then $\cap_{n=1}^{\infty}(na + S) = \emptyset$ for every element $a \in P$.

(2.9) Theorem. Let S be an APVS. The following statements hold.

(1) If S has no prime ideal with height 1, then $S^c = \mathrm{q}(S)$.

(2) If S has a prime ideal P with height 1, then,
(i) $S^c = (P : P)$.
(ii) If S is of dimension ≥ 2, then $S^c = S_P$.
(iii) S^c is of dimension 1.

§3 Semistar operations on an APVS

Let I be an S-submodule of $\mathrm{q}(S)$ such that $s + I \subset S$ for some $s \in S$. Then I is called a fractional ideal of S. The set of fractional ideals of S is denoted by $\mathrm{F}(S)$.

A mapping $\star$ of $\mathrm{F}(S)$ to $\mathrm{F}(S)$ is called a star operation on S if $\star$ satisfies the following conditions: For all $a \in \mathrm{q}(S)$ and $I, J \in \mathrm{F}(S)$, $(a)^\star = (a), (a+I)^\star = a + I^\star, I \subset I^\star, I \subset J$ implies $I^\star \subset J^\star$, and $(I^\star)^\star = I^\star$. The set of star operations on S is denoted by $\mathrm{Star}(S)$.

Let $\bar{\mathrm{F}}(S)$ be the set of S-submodules of $\mathrm{q}(S)$. A mapping $\star$ of $\bar{\mathrm{F}}(S)$ to $\bar{\mathrm{F}}(S)$ is called a semistar operation on S if $\star$ satisfies the following conditions: For all $a \in \mathrm{q}(S)$ and $I, J \in \bar{\mathrm{F}}(S)$, $(a+I)^\star = a+I^\star, I \subset I^\star, I \subset J$ implies $I^\star \subset J^\star$, and $(I^\star)^\star = I^\star$. The set of semistar operations on S is denoted by Sstar (S).

(3.1) ([M2, Theorem 2]). Let V be a valuation semigroup with finite dimension n, and let Γ be its value group. Let $M = P_n \supsetneqq \cdots \supsetneqq P_1$ be the prime ideals of V, and let $\{0\} \subsetneqq H_{n-1} \subsetneqq \cdots \subsetneqq H_1 \subsetneqq \Gamma$ be the convex subgroups of Γ. Let m be a positive integer with $n + 1 \leq m \leq 2n + 1$. The following conditions are equivalent.
(1) $|\mathrm{Sstar}\ (V)\ | = m$.
(2) The maximal ideal of V_{P_i} is principal for exactly $2n + 1 - m$ of i.
(3) The ordered abelian grouop Γ/H_i has a minimal positive element for exactly $2n + 1 - m$ of i.

For a subset I of $\mathrm{q}(S)$, the subset $\{x \in \mathrm{q}(S) \mid x + I \subset S\}$ of $\mathrm{q}(S)$ is denotede by I^{-1} (We set $\emptyset^{-1} = \mathrm{q(S)}$), and $(I^{-1})^{-1}$ is denotede by I^v.

(3.2) (cf., [M5, Theorem 2]). Let S be a PVS which is not a valuation semigroup, let M be the maximal ideal of S, and let $V = M^{-1}$. Let Σ_1' be the set of semistar operations $\star$ on S such that $S^\star \supset V$, and let Σ_2' be the set of semistar operations $\star$ on S such that $S^\star \subsetneqq V$. Let H be the unit group of S, and let G be the unit group of V. Assume that $\mid G/H \mid < \infty$ and that $\dim\ (S) < \infty$. Let $H_1, \cdots, H_l$ be the subgroups H' of G such that $G \supsetneqq H' \supset H$, and let $S_i = H_i \cup M$ for every i.
(1) $\mathrm{Sstar}(S) = \Sigma_1' \cup \Sigma_2'$.
(2) $\mid \mathrm{Sstar}(V) \mid < \infty$, and $\mid \mathrm{Star}(S_i) \mid < \infty$ for every i.
(3) $\mid \Sigma_1' \mid = \mid \mathrm{Sstar}(V) \mid$.
(4) $\mid \Sigma_2' \mid = \sum_1^l \mid \mathrm{Star}(S_i) \mid$.

(3.3). Let S be an APVS, P the maximal ideal of S, and $V = (P : P)$. If $|\mathrm{Sstar}(S)\ | < \infty$, then $\dim(S) < \infty$ and V is a finitely generated S-module.

Proof. Then V is a finitely generated oversemigroup of S. By [M2, Lemma 8], $\bar{S}$ is a valuation semigroup. Since $\bar{S} \subset V$, and the maximal ideal of $\bar{S}$ is $\mathrm{Rad}_{\mathrm{q}(S)}(P)$, we have $V = \bar{S}$. Therefore V is a finitely generated S-module.

Let T be an oversemigroup of S. Then we have canonical mappings α: Sstar(S) $\longrightarrow$ Sstar(T) and δ: Sstar$(T) \longrightarrow$ Sstar(S). Thus, for every $\star \in$ Sstar$(S), \alpha(\star)$ is the restriction of $\star$ to $\bar{\mathrm{F}}(T)$. And, for every $\star' \in$ Sstar$(T), I^{\delta(\star')} = (I+T)^{\star'}$ for every $I \in \bar{\mathrm{F}}(S)$. $\alpha(\star)$ is called the ascent of $\star$ to T, and $\delta(\star')$ is called the descent of $\star'$ to S.

The semistar operation $I \longmapsto \mathrm{q}(S)$ for every $I \in \bar{\mathrm{F}}(S)$ is called the e-semistar operation on S.

(3.4). Assume that $|\mathrm{Sstar}(S)\,| < \infty$, $\bar{S}$ is a valuation semigroup, the unit group of $\bar{S}$ coincides with the unit group of S, and that $\bar{S}$ is a finitely generated S-module. Then S need not be an APVS.

Example: Let $S = \{0, 2, 4, 5, 6, \cdots\}$. Then $V = \bar{S} = \boldsymbol{Z}_0$ is a valuation semigroup, the unit group of $\bar{S} = \{0\}$, the unit group of $S = \{0\}$, $\bar{S}$ is a finitely generated S-module, S is not an APVS, and $\dim(S) = 1$. We must show that $|\mathrm{Sstar}(S)\,| < \infty$.

Set $\Sigma'_1 = \{\star \in \mathrm{Sstar}(S) \mid S^\star = \boldsymbol{Z}\ \}$, $\Sigma'_2 = \{\star \in \mathrm{Sstar}(S) \mid S^\star = V\}$, $\Sigma'_3 = \{\star \in \mathrm{Sstar}(S) \mid S \underset{\neq}{\subsetneq} S^\star \underset{\neq}{\subsetneq} V\}$, and $\Sigma'_4 = \{\star \in \mathrm{Sstar}(S) \mid S^\star = S\}$. Then we have $\mathrm{Sstar}(S) = \cup_{i=1}^4 \Sigma'_i$. And we have $\Sigma'_1 = \{e\}$, and $|\ \Sigma'_1\ | = 1$. Every $* \in \Sigma'_2$ is induced from a star operation on V. Since $|\mathrm{Star}(V)\,| = 1$, $|\ \Sigma'_2\ | = 1$.

Let $T = \{0, 2, 3, 4, \cdots\}$. Every $\star \in \Sigma'_3$ is induced from a star operation on T. Hence $|\ \Sigma'_3\ | = |\mathrm{Star}(T)\,|$. Similarly, we have $|\ \Sigma'_4\ | = |\mathrm{Star}(S)\,|$.

We will show that $|\mathrm{Star}(S)\,| < \infty$ (The proof of $|\mathrm{Star}(T)\,| < \infty$ is simpler). Set $I_0 = \{0, 2, 3, 4, \cdots\}$, $F_1 = \{S, I_0, V\}$, $F_2 = \{I \in \mathrm{F}(S) \mid S \subset I \subset V - 4\}$. Let $F_2^{F_1}$ be the set of mappings of F_1 to F_2. Then F_2 is a finite set, and $F_2^{F_1}$ is a finite set. Since $V + 4 \subset S$, we have $V^\star \subset S - 4 \subset V - 4$ for every $\star \in \mathrm{Star}(S)$.

For every $\star \in \mathrm{Star}(S)$, there is a canonical mapping $\theta_\star$ of F_1 to F_2: $S \longmapsto S$, $I_0 \longmapsto I_0^\star$, $V \longmapsto V^\star$. There arises a canonical mapping θ of Star(S) to $F_2^{F_1}$: $\star \longmapsto \theta_\star$.

Assume that $\theta(\star_1) = \theta(\star_2)$ for $\star_1, \star_2 \in \mathrm{Star}(S)$. Let $I \in \mathrm{F}(S)$. There is $x \in \boldsymbol{Z}$ such that $x + I \subset \mathrm{F}_1$. Since $\theta_{\star_1} = \theta_{\star_2}$, we have $(x+I)^{\star_1} = (x+I)^{\star_2}$. It follows that $I^{\star_1} = I^{\star_2}$, and hence $\star_1 = \star_2$. That is, θ is an injection, and hence $|\mathrm{Star}(S)\,| < \infty$.

In this section, we will prove the following,

(3.5) Theorem. Let S be an APVS, P be the maximal ideal of S, and let $V = (P : P)$. Then $|\mathrm{Sstar}(S)\,| < \infty$ if and only if $\dim(S) < \infty$ and V is a finitely generated S-module.

(3.1) and (3.2) show that (3.5) holds for any PVS.

Thus in the remainder of this section, S denotes an APVS with dimension $< \infty$ which is not a PVS, P is the maximal ideal of S, $V = (P : P)$ which is a finitely generated S-module, M is the maximal ideal of V, G is the unit group of V, H is the unit group of S, v is the valuation which belongs to V, and Γ is the value group of v.

(3.6). (1) $V = P^{-1}$.

(2) $V = \bar{S}$.

(3) $| G/H | < \infty$.

Proof. (1) Suppose the contrary. There is $x \in P^{-1} - V$. Since $x + P \not\subset P$ and $x + P \subset S$, we have $S = x + P$. It follows that $P = -x + S$, and $V = S$; a contradiction.

(3.7). $\bar{\mathrm{F}}(S) = \mathrm{F}(S) \cup \{\mathrm{q}(S)\}$.

Proof. We note that $V \in \mathrm{F}(S)$. Let $I \in \bar{\mathrm{F}}(S)$.

The case that $v(I)$ is bounded below: There is $x \in \mathrm{q}(S)$ such that $v(x) < v(I)$. Then $-x + I \subset V$. Hence I is a fractional ideal of S.

The case that $v(I)$ is not bounded below: Let $x \in \mathrm{q}(S)$, and let $p \in P$. There is $y \in I$ such that $v(y) < v(x-p)$. Then $x = (x-y-p)+p+y \in V+p+y \subset P+y \subset I$. Hence $I = \mathrm{q}(S)$.

(3.8). Let T be an oversemigroup of S. Then either $T \supset V$ or $T \subsetneqq V$.

Proof. Assume that $T \not\subset V$. There is $t \in T - V$. Let $x \in V$. Then $-t \in M$, and hence $-nt \in P$ for some $n > 0$. Then $x = nt + (x - nt) \in T + P \subset T$, and hence $V \subset T$.

(3.9). (1) There are no $g_i \in G$ such that $V = S[g_1, \cdots, g_l]$.

(2) There are $g_i \in G$ and $x_0 \in M$ such that $V = S[g_1, \cdots, g_l, x_0]$.

(3) In (2), $v(x_0)$ is a minimal positive element of Γ.

(4) $\boldsymbol{Z}v(x_0)$ is the rank 1 convex subgroup of Γ.

(5) Let m be the minimal positive integer k such that $kx_0 \in S$. Then $v(P) = \{\gamma \in \Gamma \mid \gamma \geq mv(x_0)\}$.

Proof. (1) Suppose the contrary. For any $x \in M$, we have $x = s + \sum k_i g_i$. Then $s \in P$, hence $x \in P$. Therefore $P = M$; a contradiction.

(2) There are $g_1, \cdots, g_l \in G$ and $x_1, \cdots, x_m \in M$ such that $V = S[g_1, \cdots, g_l, x_1, \cdots, x_m]$ with $m > 0$. Assume that, for instance, $v(x_2) > v(x_1)$. We have $x_2 - x_1 = s + \sum k_i g_i + \sum k'_i x_i$. Hence $x_2 = s + \sum k_i g_i + (1 + k'_1)x_1 + \sum_{i=2}^{m} k'_i x_i$. It follows that $k'_2 = 0$, and $V = S[g_1, \cdots, g_l, x_1, x_3, x_4, \cdots, x_m]$. Therefore we may assume that $v(x_1) = \cdots = v(x_m)$. It follows that $V = S[g_1, \cdots, g_l, x_1]$.

(3) Suppose the contrary. There is $x \in M$ such that $v(x) < v(x_0)$. Then $x_0 - x_1 = s + \sum k_i g_i + kx_0$. If $k > 0$, then $x \in G$; a contradiction. If $k = 0$, then $s \in P$. Then $x_0 = x_1 + s + \sum k_i g_i \in P$, and hence $V = S[g_1, \cdots, g_l]$; a contradiction.

(4) follows from (3).

(5) Then $mv(x_0) = v(mx_0) \in v(P)$.

Assume that $\gamma > mv(x_0)$. Then $\gamma - mv(x_0) = v(x)$ for some $x \in M$, and $\gamma = v(mx_0 + x) \in v(P)$.

Suppose that $v(p) < mv(x_0)$ for some $p \in P$. By (3), we have $v(p) \leq (m-1)v(x_0)$, and hence $(m-1)x_0 \in S$; a contradiction.

Set $\Sigma' = \{\star \in \mathrm{Sstar}(S) \mid S^\star = S\}$, $\Sigma'_1 = \{\star \in \mathrm{Sstar}(S) \mid S^\star \supset V\}$, and set

$\Sigma_2' = \{\star \in \mathrm{Sstar}(S) \mid S^\star \subsetneqq V\}$.

(3.10). (1) There is a canonical bijection from $\mathrm{Sstar}(V)$ onto Σ_1'.
(2) $\mathrm{Sstar}(S) = \Sigma_1' \cup \Sigma_2'$ (disjoint).

Proof. (1) follows from (3.7).
(2) follows from (3.8).

Let $g_1, \cdots, g_l$ be a complete representatives of G modulo H. Then we have $V = S[g_1, \cdots, g_l, x_0]$ for some $x_0 \in M$. Let m be the minimal positive integer k such that $kx_0 \in S$. Let $C = \{g_i + kx_0 \mid 1 \leq i \leq l, 0 \leq k < m\}$, and $\Pi = \{\sigma \mid \sigma$ is a subset of C which contains some $g_i\}$. We may assume that $v(x_0) = 1$, and that $\boldsymbol{Z}$ is the rank 1 convex subgroup of Γ.

For every $\sigma \in \Pi$, the fractional ideal of S generated by σ is denoted by $\sigma + S$.

(3.11). (1) Let I be a fractional ideal of S with $I \subset V$ which meets with G. Then $I \supset P$, and there is a unique element $\sigma \in \Pi$ such that $I = \sigma + S$.

(2) Let $\star \in \Sigma(S)$ and $\sigma \in \Pi$. Then there is a unique element $\sigma' \in \Pi$ such that $(\sigma + S)^\star = \sigma' + S$.

Proof. (1) Let $g \in G \cap I$ and $p \in P$. Then $p = (p - g) + g \in P + I \subset I$, and hence $P \subset I$.

Let σ be the set of elements of I which are contained in C. Then $\sigma \in \Pi$, and $P \subset \sigma + S \subset I$. Let $x \in I - P$. Then $v(x) = v(rx_0)$ for some $0 \leq r < m$.

Then $x = rx_0 + g_i + h$, and then $rx_0 + g_i \in I \cap C$, and hence $x \in \sigma + S$. Therefore $I = \sigma + S$.

Assume that $I = \sigma_1 + S = \sigma_2 + S$ for $\sigma_1, \sigma_2 \in \Pi$. Let $g_i + kx_0 \in \sigma_1$. Then $g_i + kx_0 = g_j + k'x_0 + s$ with $g_j + k'x_0 \in \sigma_2$. Clearly $k' \leq k$. If $k' < k$, then $s = g_i - g_j + (k - k')x_0$, and $0 < v(s) < m$; a contradiction. Hence $k = k'$, $g_i = g_j + s$, $g_i = g_j$, and $s = 0$. Therefore $\sigma_1 \subset \sigma_2$. Similarly $\sigma_2 \subset \sigma$.

(2) We have $(\sigma + S)^\star \subset V^\star \subset V^v$. Then $(\sigma + S)^\star \subset V$ by (3.6). The proof is complete by (1).

In (3.11)(2), set $\sigma' = f_\star(\sigma)$, and set $F(\sigma) = f_\star$. Then $f_\star$ is a mapping from Π to Π, and F is a mapping from $\Sigma_2'(S)$ to Π^Π, where Π^Π denotes the set of mappings from Π to Π. Π^Π is a finite set.

(3.12). The set of non-maximal prime ideals of S coincides with the set of non-maximal prime ideals of V.

(3.13). If $\dim(S) = 1$, then F is injecticve. In particular, $\mid \Sigma_2'(S) \mid < \infty$.

Proof. We have $\Gamma = \boldsymbol{Z}$. Assume that $F(\star) = F(\star')$ for $\star, \star' \in \Sigma_2'(S)$. We must show that $\star = \star'$.

Let $I \in \mathrm{F}(S)$. We have $\min\,(v(x + I)) = 0$ for some $x \in \mathrm{q}(S)$. There is $\sigma \in \Pi$ such that $x + I = \sigma + S$. Hence $x + I^\star = (\sigma + S)^\star$ and $x + I^{\star'} = (\sigma + S)^{\star'}$. Since

$f_\star = f_{\star'}$, we have $(\sigma + S)^\star = (\sigma + S)^{\star'}$. It follows that $I^\star = I^{\star'}$, and hence $\star = \star'$.

(3.14). Assume that $\dim(S) \geq 2$. Then F is injective. In particular, $\mid \Sigma'_2(S) \mid < \infty$.

Proof. Assume that $F(\star) = F(\star')$ for $\star, \star' \in \mathrm{Star}(S)$. We must show that $\star = \star'$. Let Q be the prime ideal of V which correspond to the convex subgroup $\boldsymbol{Z}$ of Γ. Q is a prime ideal of S, and $Q = \{x \in V \mid v(x) \notin \boldsymbol{Z}\}$.

We note that $\Gamma/\boldsymbol{Z}$ is a totally ordered abelian group. For every element γ (resp., subset A) of Γ, $\gamma + \boldsymbol{Z}$ (resp., $\{a + \boldsymbol{Z} \mid a \in A\}$) is denoted by $\bar{\gamma}$ (resp., $\bar{A}$).

Let $I \in \mathrm{F}(S)$. We have three cases: (1) $v(I)$ has a minimal element. (2) $v(I)$ does not have a minimal element, and $\overline{v(I)}$ does not have a minimal element. (3) $v(I)$ does not have a mnimal element, and $\overline{v(I)}$ has a minimal element.

Case (1): Let $v(x)$ be the minimal element of $v(I)$ with $x \in I$. There is $\sigma \in \Pi$ such that $I - x = \sigma + S$. Since $f_\star = f_{\star'}$, we have $(\sigma + S)^\star = (\sigma + S)^{\star'}$. Then $I^\star = x + (\sigma + S)^\star = x + (\sigma + S)^{\star'} = I^{\star'}$.

Case (2): Assume that $x \notin I$. For any $i \in I$, we have $\overline{v(x)} < \overline{v(i)}$. Hence also $\overline{v(x+1)} < \overline{v(i)}$. It follows that $i \in x+1+S$, and that $I \subset x+1+S$. Since $x \notin x+1+S$, we have $I = I^v$. Hence $I^\star = I^{\star'}$.

Case (3): Let $\overline{v(x)}$ be the minimal element of $\overline{v(I)}$ with $x \in I$. Then we have $v(I - x) = \{\gamma \in \Gamma \mid l < \gamma \text{ for some integer } l\}$. Let $\{x \mid v(x) < l \text{ for every integer } l\} = \{x_\lambda \mid \Lambda\}$. Then $I - x = \cap_\lambda (x_\lambda + S)$. Hence $I = I^v$, and hence $I^\star = I^{\star'}$.

We have shown that $\star = \star'$.

(3.10) completes the proof of (3.5).

For a general g-monoid S, conditions for $|\mathrm{Sstar}(S)| < \infty$ were studied in [M5].

§4 Semistar operations on an APVD

In this section we study semistar operations on APVD's. For a domain D, the set of non-zero fractional ideals of D is denoted by $\mathrm{F}(D)$, and the set of star operations on D is denoted by $\mathrm{Star}(D)$.

(4.1) ([M2, Theorem 3]). Let V be a valuation domain with finite dimension n, and let Γ be its value group. Let $M = P_n \supsetneqq \cdots \supsetneqq P_1 \supsetneqq (0)$ be the prime ideals of V, and let $\{0\} \subsetneqq H_{n-1} \subsetneqq \cdots \subsetneqq H_1 \subsetneqq \Gamma$ be the convex subgroups of Γ. Let m be a positive integer with $n + 1 \leq m \leq 2n + 1$. The following conditions are equivalent:

(1) $|\mathrm{Sstar}(V)| = m$.

(2) The maximal ideal of V_{P_i} is principal for exactly $2n + 1 - m$ of i.

(3) The ordered group Γ/H_i has a minimal positive element for exactly $2n+1-m$ of i.

(4.2) ([M8, §2, Proposition 3 and Lemma 3]). Let D be a PVD with maximal ideal M, and set $V = M^{-1}$. Assume that $|\mathrm{Sstar}(D) \mid < \infty$, then $\dim(D) < \infty$, and V/M is a simple extension field of D/M with $[V/M : D/M] < \infty$.

(4.3) ([M8, §1, Proposition 1]). In (4.2), the converse does not hold.

Example: Let k be a field of characteristic 0, K be an extension field of k with $[K : k] = 4$, $V = K[[X]]$, M be the maximal ideal of V, and let $D = k + M$.

(4.4) ([M8, §2, Proposition 4]). Let D be a PVD which is not a valuation domain, let M be the maximal ideal of D, and let $V = M^{-1}$. Assume that dim $(D) < \infty$, and that V/M is a simple extension field of D/M with $[V/M : D/M] < \infty$. Let $\{D_1, \cdots, D_l\}$ be the set of overrings T of D such that $T \subsetneqq V$. Let Σ'_1 be the semistar operations $\star$ on D such that $D^\star \supset V$, and let Σ'_2 be the set of semistar operations $\star$ on D such that $D^\star \subsetneqq V$.

(1) $\mathrm{Sstar}(D) = \Sigma'_1 \cup \Sigma'_2$.
(2) $|\mathrm{Sstar}(V)| < \infty$.
(3) $|\Sigma'_1| = |\mathrm{Sstar}(V)|$.
(4) $\cup_1^l \mathrm{Star}(D_i)$ is a disjoint union.
(5) There is a canonical bijection from Σ'_2 onto $\cup_1^l$ $\mathrm{Star}(D_i)$.

(4.5). (1) Assume that $\bar{D} = D$ and $|\mathrm{Sstar}(D)| < \infty$, then D need not be an APVD (cf., [M5, Remark 1]).

(2) If $|\mathrm{Sstar}(D)| < \infty$, and if $\bar{D}$ is quasi-local, then $\bar{D}$ is a valuation domain (cf., [M4, Theorem 3]).

In the remainder of this section, let D be an APVD with maximal ideal P, $(P : P) = V$, M be the maximal ideal of V, v be the valuation which belongs to V, and Γ be the value group of v.

(4.6). (1) If D is not a valuation domain, then $V = P^{-1}$.
(2) $\bar{\mathrm{F}}(D) = \mathrm{F}(D) \cup \{\mathrm{q}(D)\}$.

Proof. (1) Suppose that $P^{-1} \supsetneqq V$. There is $x \in P^{-1} - V$. Then $xP \subset D$ and $xP \not\subset P$. Hence $xP = D$ and $P = x^{-1}D$. Hence $D = V$; a contradiction.

(2) Let $I \in \bar{\mathrm{F}}(D)$. If $v(I)$ is bounded below, then $I \in \mathrm{F}(D)$. If $v(I)$ is not bounded below, then $I = \mathrm{q}(D)$.

(4.7). Let T be an overring of D. Then $T \supset V$ or $T \subsetneqq V$.

The proof is a ring version of that of (3.8).

(4.8). Assume that $|\mathrm{Sstar}(D)| < \infty$.
(1) dim $(D) < \infty$.
(2) V is a finitely generated D-module.
(3) $V = \bar{D}$.
(4) V/M is a simple extension field of D/P with $[V/M : D/P] < \infty$.

Proof. V is a finitely generated overring of D, and $\bar{D} \subset V$. $\bar{D}$ is a quasi-local ring with maximal ideal $M = \mathrm{Rad}_V(P)$. Since $|\mathrm{Sstar}(\bar{D})| < \infty$, $\bar{D}$ is a valuation ring.

Hence $V = \bar{D}$. Therefore V is a finitely generated D-module.

(4.9) Proposition. Let D be an APVD which is not a PVD, let P be the maximal ideal of D, and let $V = (P : P)$. Assume that dim $(D) < \infty$, and V is a finitely generated D-module. Let $\{D_\lambda \mid \lambda \in \Lambda\}$ be the set of overrings T of D such that $T \subsetneqq V$. Let Σ_1' be the semistar operations $\star$ on D such that $D^\star \supset V$, and let Σ_2' be the set of semistar operations $\star$ on D such that $D^\star \subsetneqq V$.

(1) $\mathrm{Sstar}(D) = \Sigma_1' \cup \Sigma_2'$.
(2) $|\mathrm{Sstar}(V)\,| < \infty$.
(3) $|\ \Sigma_1'\ | = |\mathrm{Sstar}(V)\,|$.
(4) $\cup_\lambda$ $\mathrm{Star}(D_\lambda)$ is a disjoint union.
(5) There is a canonical bijection from Σ_2' onto $\cup_\lambda$ $\mathrm{Star}(D_\lambda)$.

For any APVD D, conditions for $|\mathrm{Sstar}(D)| \ < \infty$ were studied in [M9].

(4.10) ([M6, (2.5) Proposition]). Let D be a quasi-local domain with maximal ideal P, and assume that $\bar{D} = V$ is a valuation ring with maximal ideal M, v be a valuation belonging to V with value group Γ. Asume that $D \supset M^3$. Then,

(1) D is either a PVD or, we may assume that $\boldsymbol{Z}$ is the rank one convex subgroup of Γ.
(2) If $D/P = V/M$, then D is an APVD.

§5 Cancellation properties of semistar operations

In this section, we will study G. Picozza [P], and will introduce some cancellation properties of semistar operations on a g-monoid.

Let S be a g-monoid with quotient group $\mathrm{q}(S)$. A star operation $\star$ on S is called a.b. if, for every $F \in \mathrm{f}(S)$ and every $G_1, G_2 \in \mathrm{F}(S)$, $(F + G_1)^\star = (F + G_2)^\star$ implies $G_1^\star = G_2^\star$, and $\star$ is called e.a.b. if the same holds for every $F, F_1, F_2 \in \mathrm{f}(S)$. A semistar operation $\star$ on S is called a.b. if, for every $F \in \mathrm{f}(S)$ and every $H_1, H_2 \in \bar{\mathrm{F}}(S)$, $(F + H_1)^\star = (F + H_2)^\star$ implies $H_1^\star = H_2^\star$, and $\star$ is called e.a.b. if the same holds for every $F, F_1, F_2 \in \mathrm{f}(S)$. The mapping $\bar{\mathrm{F}}(S) \longrightarrow \bar{\mathrm{F}}(S), H \longmapsto H^e = \mathrm{q}(S)$ is a semistar operation on S, and is called the e-semistar operation on S as defined in §3.

A semistar operation $\star$ on S is called cancellative if, for every $H, H_1, H_2 \in \bar{\mathrm{F}}(S)$, $(H + H_1)^\star = (H + H_2)^\star$ implies $H_1^\star = H_2^\star$.

(5.1). Let $\star$ be a semistar operation on S. Then $\star$ is cancellative if and only if $\star = e$.

Proof. The necessity: Let $H \in \bar{\mathrm{F}}(D)$. Since $\mathrm{q}(S)+H = \mathrm{q}(S)$, we have $(\mathrm{q}(S)+H)^\star = (\mathrm{q}(S) + \mathrm{q}(S))^\star$, and hence $H^\star = \mathrm{q}(S)^\star$. Since $\mathrm{q}(S)^\star = \mathrm{q}(S)$, we have $H^\star = \mathrm{q}(S)$, and hence $\star = e$.

Let S be a g-monoid, and let $\star$ be a semistar operation on S. Set $(\mathrm{f}(S))^\star = \{E^\star \mid E \in \mathrm{f}(S)\}$.

(5.2). Let S be a g-monoid, let T be an oversemigroup of S, let $\star$ be a semistar operation on S, and let $\alpha(\star)$ be the ascent of $\star$ to T.

(1) If $\star$ is cancellative, then $\alpha(\star)$ is cancellative.

(2) If $\star$ is a.b., then $\alpha(\star)$ is a.b.

(3) Assume that $T^\star \in (\mathrm{f}(S))^\star$. If $\star$ is e.a.b., then $\alpha(\star)$ is e.a.b.

(5.3). Let S be a g-monoid, let T be an oversemigroup of S, let $\star$ be a semistar operation on T, and let $\delta(\star)$ be the descent of $\star$ to S.

(1) If $\star$ is cancellative, then $\delta(\star)$ is cancellative.

(2) If $\star$ is a.b., then $\delta(\star)$ is a.b.

(3) If $\star$ is e.a.b., then $\delta(\star)$ is e.a.b.

(5.4) Proposition. Let S be a g-monoid, and let $\mathcal{T} = \{T_\lambda \mid \lambda \in \Lambda\}$ be the set of oversemigroups of S.

(1) There is a canonical bijection between the set of cancellative semistar operations on S and the set $\cup_\lambda \{\star \mid \star$ is a cancellative semistar operation on T_λ with $T_\lambda^\star = T_\lambda\}$.

(2) There is a canonical bijection between the set of a.b. semistar operations on S and the set $\cup_\lambda \{\star \mid \star$ is an a.b. semistar operation on T_λ with $T_\lambda^\star = T_\lambda\}$.

(3) There is a canonical bijection between the set of e.a.b. semistar operations on S and the set $\cup_\lambda \{\star \mid \star$ is an e.a.b. semistar operation on T_λ with $T_\lambda^\star = T_\lambda\}$.

Some of the sets in (5.4) may be empty sets. For instanse, the set $\{\star \mid \star$ is a cancellative semistar operation on T_λ with $T_\lambda^\star = T_\lambda\}$ is an empty set unless $T_\lambda = \mathrm{q}(S)$.

The notion of a cancellative semistar operation derives an s.a.b. semistar operation. Thus, we will say that a semistar operation $\star$ on S is s.a.b. (or, strongly arithmetisch brauchbar) if, for every $G \in \mathrm{F}(S)$, and $H_1, H_2 \in \bar{\mathrm{F}}(S)$, $(G+H_1)^\star = (G+H_2)^\star$ implies $H_1^\star = H_2^\star$. Clearly, the e-semistar operation is an s.a.b. semistar operation, and an s.a.b. semistar operation is an a.b. semistar operation.

An s.a.b. semistar operation need not be the e-semistar operation. For example, let S be a principal ideal semigroup which is not a group, and let $\star$ be a semistar operation on S with $\star \neq e$. Then $\star$ is s.a.b.

We note that, for domains, A.Okabe [O] calles an s.a.b. semistar operation a cancellative semistar operation, and gives its characterization ([O, Theorem 28]).

The identity mapping d on $\bar{\mathrm{F}}(S)$ is a semistar operation on S, and is called the d-semistar operation on S.

(5.5). An a.b. semistar operation need not be an s.a.b. semistar operation.

For example, let $S = V$ be a valuation semigroup which is not a group, let M be the maximal ideal with $M = 2M$, and let $\star = d$. Then $\star$ is a.b., and $\star$ is not s.a.b., in fact, $(M+M)^\star = (M+S)^\star$ and $M^\star \neq S^\star$.

Let S be a g-monoid, and let $\star$ be a semistar operation on S. Set $(\mathrm{F}(S))^\star = \{G^\star \mid G \in \mathrm{F}(S)\}$.

(5.6) Proposition. (1) Let S be a g-monoid, let $\star$ be a semistar operation on

S, let T be an oversemigroup of S with $(\mathrm{F}(T))^\star \subset (\mathrm{F}(S))^\star$, and let $\alpha(\star)$ be the ascent of $\star$ to T. If $\star$ is s.a.b., then $\alpha(\star)$ is s.a.b.

(2) Let S be a g-monoid, let T be an oversemigroup of S, let $\star$ be a semistar operation on T, and let $\delta(\star)$ be the descent of $\star$ to S. If $\star$ is s.a.b., then $\delta(\star)$ is s.a.b.

(3) Let S be a g-monoid, and let $\mathcal{T} = \{T_\lambda \mid \lambda \in \Lambda\}$ be the set of oversemigroup T of S with $T \in \mathrm{F}(S)$. Then there is a canonical bijection between the set $A = \{\star \mid \star$ is an s.a.b. semistar operations $\star$ on S with $S^\star \in \mathrm{F}(S)\}$ and the set $B = \cup_\lambda \{\star \mid \star$ is an s.a.b. semistar operation on T_λ with $T_\lambda^\star = T_\lambda\}$.

Proof. (1) Let $(G+H_1)^{\alpha(\star)} = (G+H_2)^{\alpha(\star)}$, where $G \in \mathrm{F}(T)$ and $H_1, H_2 \in \bar{\mathrm{F}}(T)$. Then we have $G \in \mathrm{F}(S), H_1, H_2 \in \bar{\mathrm{F}}(S)$, and $(G + H_1)^\star = (G + H_2)^\star$. It follows that $H_1^\star = H_2^\star$, and hence $H_1^{\alpha(\star)} = H_2^{\alpha(\star)}$.

(2) Let $(g + h_1)^{\delta(\star)} = (g + h_2)^{\delta(\star)}$, where $g \in \mathrm{F}(S)$ and $h_1, h_2 \in \bar{\mathrm{F}}(S)$. Then we have $g + T \in \mathrm{F}(T), h_1 + T, h_2 + T \in \bar{\mathrm{F}}(T)$, and $(g + T + h_1 + T)^\star = (g + T + h_2 + T)^\star$. Since $\star$ is s.a.b., we have $(h_1 + T)^\star = (h_2 + T)^\star$, and hence $h_1^{\delta(\star)} = h_2^{\delta(\star)}$. Hence $\delta(\star)$ is s.a.b.

(3) Let $\star \in A$, and let $\alpha(\star)$ be the ascent of $\star$ to $S^\star$. Then $\alpha(\star) \in B$ by (1). For every $h \in \bar{\mathrm{F}}(S)$, we have $h^\star = (h + S^\star)^{\alpha(\star)}$. Assume that $\alpha(\star_1) = \alpha(\star_2)$, where $\star_1, \star_2 \in A$. Sine $\alpha(\star_1)$ (resp., $\alpha(\star_2)$) is a semistar operation on $S^{\star_1}$ (resp., $S^{\star_2}$), we have $S^{\star_1} = S^{\star_2}$. Then we have $h^{\star_1} = (h + S^{\star_1})^{\alpha(\star_1)}$ and $h^{\star_2} = (h + S^{\star_2})^{\alpha(\star_2)}$. Hence we have $\star_1 = \star_2$. Assume that $\star \in B$, and let $\delta(\star)$ be the descent of $\star$ to S. Then we have $\delta(\star) \in A$ by (2), and $\alpha(\delta(\star)) = \star$.

We may canonically introduce five more cancellation properties of semistar operations. Thus, let S be a g-monoid, and let $\star$ be a semistar operation on S.

Then $\star$ is called h.g. if, for every $H \in \bar{\mathrm{F}}(S)$ and $G_1, G_2 \in \mathrm{F}(S)$, $(H + G_1)^\star = (H + G_2)^\star$ implies $G_1^\star = G_2^\star$;

$\star$ is called g.g. if, for every $G, G_1, G_2 \in \mathrm{F}(D)$, $(G + G_1)^\star = (G + G_2)^\star$ implies $G_1^\star = G_2^\star$;

$\star$ is called f.g. if, for every $F \in \mathrm{f}(S)$ and $G_1, G_2 \in \mathrm{F}(S)$, $(F + G_1)^\star = (F + G_2)^\star$ implies $G_1^\star = G_2^\star$;

$\star$ is called h.f. if, for every $H \in \bar{\mathrm{F}}(S)$ and $F_1, F_2 \in \mathrm{f}(S), (H + F_1)^\star = (H + F_2)^\star$ implies $F_1^\star = F_2^\star$;

$\star$ is called g.f. if, for every $G \in \mathrm{F}(S)$ and $F_1, F_2 \in \mathrm{f}(S), (G + F_1)^\star = (G + F_2)^\star$ implies $F_1^\star = F_2^\star$.

If $\star$ is cancellative (resp., s.a.b., a.b., e.a.b.), then we may call $\star$ to be h.h. (resp., g.h., f.h., f.f.).

What the property names as above stand for: Let $\star$ be a semistar operation on S. Set $\mathrm{f}(S) = X_1$, $\mathrm{F}(S) = X_2$, and set $\bar{\mathrm{F}}(S) = X_3$. If, for every $A \in X_i$ and $B, C \in X_j$, $(A + B)^\star = (A + C)^\star$ implies $B^\star = C^\star$, then $\star$ is called $\mathrm{f}_i.\mathrm{f}_j$. Since an element of $\mathrm{f}(S)$ is called finitely generated, we set also $\mathrm{f}_1 = \mathrm{f}$. And considering the alphabetical order, we set also $\mathrm{f}_2 = \mathrm{g}$ and $\mathrm{f}_3 = \mathrm{h}$.

(5.7). Let $\star$ be a semistar operation on a g-monoid S. Then $\star$ is h.g. if and ony if $\star$ is h.f. if and only if $\star = e$.

Proof. Assume that $\star$ is h.f. Let $F_1, F_2 \in \mathrm{f}(S)$. Since $(\mathrm{q}(S)+F_1)^\star = (\mathrm{q}(S)+F_2)^\star$, we have $F_1^\star = F_2^\star$. Hence there is $H \in \bar{\mathrm{F}}(S)$ such that $H = F^\star$ for every $F \in \mathrm{f}(S)$. Let $a \in \mathrm{q}(S) - \{0\}$. Since $a \in (S+a)^\star = H$, we have $H = \mathrm{q}(S)$. Hence $\star = e$.

(5.8). (1) s.a.b. implies g.g.
(2) g.g. implies g.f.
(3) a.b. implies f.g.
(4) f.g. implies e.a.b.
(5) g.g. implies f.g.
(6) g.f. implies e.a.b.

The proofs of the following (5.9), (5.10) and (5.11) are similar to that of (5.6).

(5.9) Proposition. (1) Let S be a g-monoid, let $\star$ be a semistar operation on S, let T be an oversemigroup of S with $(\mathrm{F}(T))^\star \subset (\mathrm{F}(S))^\star$, and let $\alpha(\star)$ be the ascent of $\star$ to T. If $\star$ is g.g., then $\alpha(\star)$ is g.g.

(2) Let S be a g-monoid, let T be an oversemigroup of S, let $\star$ be a semistar operation on T, and let $\delta(\star)$ be the descent of $\star$ to S. If $\star$ is g.g., then $\delta(\star)$ is g.g.

(3) Let S be a g-monoid, and let $\mathcal{T} = \{T_\lambda \mid \lambda \in \Lambda\}$ be the set of oversemigroups T of D with $T \in \mathrm{F}(S)$. Then there is a canonical bijection between the set $A = \{\star \mid \star$ is a g.g. semistar operation on S with $S^\star \in \mathrm{F}(S)\}$ and the set $B = \cup_\lambda \{\star \mid \star$ is a g.g. semistar operation on T_λ with $T_\lambda^\star = T_\lambda\}$.

(5.10) Proposition. (1) Let S be a g-monoid, let T be an oversemigroup of S with $T \in \mathrm{F}(S)$, let $\star$ be a semistar operation on S, and let $\alpha(\star)$ be the ascent of $\star$ to T. If $\star$ is f.g., then $\alpha(\star)$ is f.g.

(2) Let S be a g-monoid, let T be an oversemigroup of S, let $\star$ be a semistar operation on T, and let $\delta(\star)$ be the descent of $\star$ to S. If $\star$ is f.g., then $\delta(\star)$ is f.g.

(3) Let S be a g-monoid, and let $\mathcal{T} = \{T_\lambda \mid \lambda \in \Lambda\}$ be the set of oversemigroups T of S with $T \in \mathrm{F}(S)$. Then there is a canonical bijection between the set $A = \{\star \mid \star$ is a f.g. semistar operation on S with $S^\star \in \mathrm{F}(S)\}$ and the set $B = \cup_\lambda \{\star \mid \star$ is a f.g. semistar operation on T_λ with $T_\lambda^\star = T_\lambda\}$.

(5.11) Proposition. (1) Let S be a g-monoid, let T be an oversemigroup of S with $T \in \mathrm{F}(S)$, let $\star$ be a semistar operation on S, and let $\alpha(\star)$ be the ascent of $\star$ to T. If $\star$ is g.f., then $\alpha(\star)$ is g.f.

(2) Let S be a g-monoid, let T be an oversemigroup of S, let $\star$ be a semistar operation on T, and let $\delta(\star)$ be the descent of $\star$ to S. If $\star$ is g.f., then $\delta(\star)$ is g.f.

(3) Let S be a g-monoid, and let $\mathcal{T} = \{T_\lambda \mid \lambda \in \Lambda\}$ be the set of oversemigroups T of S with $T \in \mathrm{F}(S)$. Then there is a canonical bijection between the set $A = \{\star \mid \star$ is an g.f. semistar operation on S with $S^\star \in \mathrm{F}(S)\}$ and the set $B = \cup_\lambda \{\star \mid \star$ is a g.f. semistar operation on T_λ with $T_\lambda^\star = T_\lambda\}$.

We must check some more implications of the cancellation properties of semistar operations.

(5.12). Let $\star$ be a semistar operation on S.

$\star$ is g.h. if and only if, for every $G \in \mathrm{F}(S)$ and $H \in \bar{\mathrm{F}}(S)$, $G \subset (G+H)^\star$ implies $0 \in H^\star$.

A similar characterization holds for every g.g., g.f., f.h., f.g., f.f. semistar operation $\star$.

For instance, $\star$ is f.g. if and only if, for every $F \in \mathrm{f}(S)$ and $G \in \mathrm{F}(S)$, $F \subset (F+G)^\star$ implies $0 \in G^\star$.

(5.13). (1) a.b. need not imply g.f.

(2) g.f. need not imply g.g.

Proof. (1) Let $S = V$ be a 2-dimensional valuation semigroup, let $M \supsetneqq P$ be the prime ideals of V, and let $\star = d$. Let $x \in M - P$, and set $F_1 = (x)$ and $F_2 = S$. Then we have $(P + F_1)^\star = (P + F_2)^\star$ and $F_1^\star \neq F_2^\star$. It follows that $\star$ is a.b., and that $\star$ is not g.f.

(2) Let $\boldsymbol{R}$ be the set of real numbers. Let $S = V$ be an $\boldsymbol{R}$-valued valuation semigroup, and let v be the valuation belnging to V with value group $\boldsymbol{R}$, and let $\star = d$. We have $(M + M)^\star = (M + S)^\star$ and $M^\star \neq S^\star$, and hence $\star$ is not g.g.

Let $(G + F_1)^\star = (G + F_2)^\star$, where $G \in \mathrm{F}(S)$ and $F_1, F_2 \in \mathrm{f}(S)$. Let $F_1 = V + a$ and $F_2 = V + b$ with $a, b \in \mathrm{q}(S)$, and set $\inf v(G) = v(x)$ with $x \in \mathrm{q}(S)$. Then $\inf v(G + F_1) = v(x) + v(a)$ and $\inf v(G + F_2) = v(x) + v(b)$. It follows that $v(a) = v(b)$, hence $V + a = V + b$, and hence $F_1 = F_2$. Therefore $\star$ is g.f.

(5.14). (1) (cf., [M3, p.69, Corollary 3]). Let $\star$ be a semistar operation on S. If $S^\star$ is not integrally closed, then $\star$ is not e.a.b.

(2) (cf., [M3, p.76]) Let S be an integrally closed semigroup. Then there is a f.h. semistar operation $\star$ on S such that $S^\star = S$.

Proof. (2) Let $\{V_\lambda \mid \lambda \in \Lambda\}$ be the set of valuation oversemigroups of S. Let $\star$ be the semistar operation $H \longmapsto \cap_\lambda (H + V_\lambda)$.

For every λ_0, we have $H + V_{\lambda_0} = H^\star + V_{\lambda_0}$. For, $H^\star + V_\lambda = (\cap_\lambda H + V_\lambda) + V_{\lambda_0} \subset (H + V_{\lambda_0}) + V_{\lambda_0} = H + V_{\lambda_0}$.

Assume that $(F + H_1)^\star = (F + H_2)^\star$ for $F \in \mathrm{f}(S)$ and $H_1, H_2 \in \bar{\mathrm{F}}(S)$. Then, for every λ, $F + H_1 + V_\lambda = (F + H_1)^\star + V_\lambda = (F + H_2)^\star + V_\lambda = F + H_2 + V_\lambda$. Since $F + V_\lambda$ is a principal ideal of V_λ, we have $H_1 + V_\lambda = H_2 + V_\lambda$. Therefore, $H_1^\star = \cap_\lambda (H_1 + V_\lambda) = \cap_\lambda (H_2 + V_\lambda) = H_2^\star$.

Finally, we will call a star operation $\star$ on S to be $\mathrm{g}_0.\mathrm{g}_0$. if, for every $G, G_1, G_2 \in \mathrm{F}(S)$, $(G + G_1)^\star = (G + G_2)^\star$ implies $G_1^\star = G_2^\star$; and we will call a star operation $\star$ to be $\mathrm{g}_0.\mathrm{f}_0$. if, for every $G \in \mathrm{F}(S)$ and $F_1, F_2 \in \mathrm{f}(S)$, $(G + F_1)^\star = (G + F_2)^\star$ implies $F_1^\star = F_2^\star$.

If $\star$ is an a.b. star operation (resp., e.a.b. star operation), then we may call $\star$ to be $\mathrm{f}_0.\mathrm{g}_0$ star operation (resp., $\mathrm{f}_0.\mathrm{f}_0$. star operation).

(5.15). Let $\star$ be a star operation on S.

(1) (i) $\mathrm{g}_0.\mathrm{g}_0.$ implies a.b.
(ii) $\mathrm{g}_0.\mathrm{g}_0.$ implies $\mathrm{g}_0.\mathrm{f}_0.$
(iii) $\mathrm{g}_0.\mathrm{f}_0.$ implies e.a.b.
(2) (i) a.b. need not imply $\mathrm{g}_0.\mathrm{f}_0.$
(ii) $\mathrm{g}_0.\mathrm{f}_0.$ need not imply $\mathrm{g}_0.\mathrm{g}_0.$

The proof follows from (5.13).

(5.16). The followings are equivalent.
(1) Every e.a.b. semistar operation is an f.g. semistar operation.
(2) Every e.a.b. star operation is an a.b. star operation.

(5.17). Let S be a g-monoid. The followings are equivalent.
(1) The d-semistar operation on S is f.g.
(2) S is a valuation semigroup.

Proof. (1) $\Longrightarrow$ (2): Then every $F \in \mathrm{f}(S)$ is principal by [$\mathrm{MS_i}$, (8.2) Theorem]. Then S is a valuation semigroup by [$\mathrm{MS_k}$, Lemma 13].

(5.18) (cf., [$\mathrm{MS_i}$, (8.3)]). Assume that S is not a group. The followings are equivalent.
(1) The d-semistar operation on S is g.g.
(2) S is a rank 1 discrete valuation semigroup.

§6 Kronecker function rings of semistar operations

Let S be a g-monoid, let D be a domain, and let $D[X;\mathrm{q}(S)]$ be the group ring of $\mathrm{q}(S)$ over D. For an element $f = \sum_1^n a_i X^{s_i}$ with every $a_i \neq 0$ and $s_i \neq s_j$ for every $i \neq j$, the fractional ideal $(s_1, \cdots, s_n)$ of S is denoted by $e_S(f)$ (or, by $e(f)$), and the subset $\{s_1, \cdots, s_n\}$ of S is denoted by $\mathrm{Exp}(f)$. The additive group $\{(a) \mid a \in \mathrm{q}(S)\}$ is called the group of divisibility of S. Let $\star$ be a semistar operation on S. We set $U^\star = \{f \in D[X; S^\star] - \{0\} \mid e_S(f)^\star = S^\star\}$.

If the set $\{I^\star \mid I \in \mathrm{f}(S)\}$ is a group under the mapping $(I_1^\star, I_2^\star) \longmapsto (I_1^\star + I_2^\star)^\star$, then S is called a Prüfer $\star$-multiplication semigroup.

Let $\star$ be an e.a.b. semistar operation on S. Then there is defined the Kronecker function ring $\mathrm{Kr}(S, \star, D) = \{\frac{f}{g} \mid f, g \in D[X; S] - \{0\}$ such that $e(f)^\star \subset e(g)^\star\} \cup \{0\}$ (cf., [M1, Proposition 4]). $\mathrm{Kr}(S, \star, D)$ is also denoted simply, by $\mathrm{Kr}(S, \star)$, or by $S_\star^D$, or by $S_\star$.

(6.1) (cf., [M1, Proposition 4]). Let $\star$ be an e.a.b. semistar operation on S.
(1) $S_\star$ is a Bezout domain.
(2) If $I \in \mathrm{f}(S)$, then $IS_\star \cap \mathrm{q}(S) = I^\star$ and $IS_\star = I^\star S_\star$.

(6.2) (cf., [G1, Theorem 2.5], [A, Theorem 4], [AB, Theorem 3], [$\mathrm{MS_k}$, Theorem 25], [M1, Theorem 23]). Let $\star$ be an e.a.b. semistar operation on S. Then the following conditions are equivalent:

(1) S is a Prüfer $\star$-multiplication semigroup.

(2) $D[X;S^\star]_{U^\star} = S_\star$.

(3) $D[X;S^\star]_{U^\star}$ is a Prüfer domain.

(4) $S_\star$ is a quotient ring of $D[X;S^\star]$.

(5) Every prime ideal of $D[X;S^\star]_{U^\star}$ is the contraction of a prime ideal of $S_\star$.

(6) Every non-zero prime ideal of $D[X;S^\star]_{U^\star}$ is the extension of a prime ideal of $S^\star$.

(7) Every proper valuation overring of $S_\star$ is of the form of $D[X;S^\star]_{QD[X;S^\star]}$, where Q is a prime ideal of $S^\star$ such that $(S^\star)_Q$ is a valuation oversemigroup of $S^\star$.

(8) $S_\star$ is a flat $D[X;S^\star]$-module.

(6.3) (cf., [AB, Corollary 4], [$\mathrm{MS_k}$, Theorem 30], [M1, Theorem 26]). Let $\star$ be an e.a.b. semistar operation on S, and let v_1 be the v-star operation on $S^\star$. Assume that v_1 is e.a.b. and $S_\star$ is a flat $D[X;S^\star]$-module. Then $S_\star = (S^\star)_{v_1}$.

(6.4) (cf., [G2, (34.11) Theorem], [M1, Remark 29]). Let $\star$ be an e.a.b. semistar operation on S. If S is a Prüfer $\star$-multiplication semigroup, then the group $\{I^\star \mid I \in \mathrm{f}(S)\}$ is canonically isomorphic with the group of divisibility of $S_\star$.

In this section, we will study (6.2) $\sim$ (6.4) for any semistar operation on S.

Let $\{V_\lambda \mid \lambda \in \Lambda\}$ be a set of valuation oversemigroups of S. Then the mapping $I \longmapsto \cap_\lambda (I + V_\lambda)$ from $\bar{\mathrm{F}}(S)$ to $\bar{\mathrm{F}}(S)$ is a semistar operation, and is called a w-semistar operation induced by the set $\{V_\lambda \mid \lambda \in \Lambda\}$.

Let v be a valuation on $\mathrm{q}(S)$. Let $f = \sum_i a_i X^{s_i} \in D[X;S] - \{0\}$ with every $a_i \neq 0$ and $s_i \neq s_j$ for every $i \neq j$. If we set $w(f) = \min_i \{v(s_i)\}$, then there is a valuation w on $\mathrm{q}(D[X;S])$, and w is called the canonical extension of v to $\mathrm{q}(D[X;S])$.

(6.5) (cf., [M1, Proposition 9]). Let $\{V_\lambda \mid \lambda \in \Lambda\}$ be a set of valuation oversemigroups of S, let w be the w-semistar operation induced by the set $\{V_\lambda \mid \lambda \in \Lambda\}$, and let W_λ be the canonical extension of V_λ to $\mathrm{q}(D[X;S])$. Then w is an a.b. semistar operation on S, and $S_w = \cap_\lambda W_\lambda$.

(6.6) (Dedekind-Mertens Lemma for semigroups) (cf., [GP, 6.2. Proposition]). Let $f, g \in D[X;S] - \{0\}$. Then there is a positive integer m such that $e(g)^{m+1} + e(f) = e(g)^m + e(fg)$.

(6.7) ([OM, Lemma (4.2)]). Let $\star$ be a semistar operation on S. Let $f, g, f', g' \in D[X;S] - \{0\}$ with $\frac{f}{g} = \frac{f'}{g'}$ such that $(e(f) + e(h))^\star \subset (e(g) + e(h))^\star$ for some element $h \in D[X;S] - \{0\}$. Then there is an element $h' \in D[X;S] - \{0\}$ such that $(e(f') + e(h'))^\star \subset (e(g') + e(h'))^\star$.

Set $\mathrm{Kr}(S, \star, D) = \{\frac{f}{g} \mid f, g \in D[X;S] - \{0\}$ such that $(e(f) + e(h))^\star \subset (e(g) + e(h))^\star$ for some element $h \in D[X;S] - \{0\}\} \cup \{0\}$. (6.7) shows that $\mathrm{Kr}(S, \star, D)$ is a well-defined subset of $\mathrm{q}(D[X;S])$. $\mathrm{Kr}(S, \star, D)$ is also denoted simply, by $\mathrm{Kr}(S, \star)$, or by

$S_\star^D$, or by $S_\star$. If $\star$ is e.a.b., this coincides with the Kronecker function ring of the e.a.b. semistar operation $\star$.

(6.8) ([OM, Proposition (4.4)]). $S_\star$ is a Bezout domain.

(6.9) ([R, Theorem 2]). Let D be a domain, and let R be an overring of D. Then R is a flat D-module if and only if $R_M = D_{D\cap M}$ for every maximal ideal M of R.

(6.10). Let $\star$ be a semistar operation on S, let $T = S^\star$, and let $\alpha(\star)$ be the ascent of $\star$ to T.

(1) We have $\{f \in D[X; S^\star]-\{0\} \mid e_S(f)^\star = S^\star\} = \{f \in D[X;T]-\{0\} \mid e_T(f)^\star = T\}$, that is, $U^\star = U^{\alpha(\star)}$.

(2) S is a Prüfer $\star$-multiplication semigroup if and only if T is a Prüfer $\alpha(\star)$-multiplication semigroup.

(3) We have $S_\star = T_{\alpha(\star)}$.

(4) The set $\{I^\star \mid I \in \mathrm{f}(S)\}$ and its addition $(I_1^\star, I_2^\star) \longmapsto (I_1^\star + I_2^\star)^\star$ is identical to the set $\{J^\star \mid J \in \mathrm{f}(T)\}$ and its addition $(J_1^\star, J_2^\star) \longmapsto (J_1^\star + J_2^\star)^\star$.

Proof. The proof is almost straightforward from the definitions.

(6.11) Proposition. Let $\star$ be a semistar operation on S. In the following conditions we have that (1) $\Longrightarrow$ (5) $\Longrightarrow$ (3) $\Longrightarrow$ (4) $\Longrightarrow$ (2), and (1) $\Longrightarrow$ (6) $\Longrightarrow$ (7).

(1) S is a Prüfer $\star$-multiplication semigroup.

(2) $S_\star$ is a quotient ring of $D[X; S^\star]$.

(3) Every proper valuation overring of $S_\star$ is of the form of $D[X; S^\star]_{QD[X;S^\star]}$, where Q is a prime ideal of $S^\star$ such that $(S^\star)_Q$ is a valuation oversemigroup of $S^\star$.

(4) $S_\star$ is a flat $D[X; S^\star]$-module.

(5) $D[X; S^\star]_{U^\star}$ is a Prüfer domain.

(6) Every prime ideal of $D[X; S^\star]_{U^\star}$ is the contraction of a prime ideal of $S_\star$.

(7) Every non-zero prime ideal of $D[X; S^\star]_{U^\star}$ is the extension of a prime ideal of $S^\star$.

Proof. By (6.10), we may assume that $S^\star = S$.

(1) implies (5): Then $\star$ is an e.a.b. semistar operation. (5) follows from (6.2).

(5) implies (3): Let W be a proper valuation overring of $S_\star$ with maximal ideal N. Set $S_\star \cap N = Q, D[X;S]_{U^\star} \cap Q = Q', D[X;S] \cap Q' = P'$, and $S \cap P' = P$. Let $f = a_1X^{s_1} + \cdots + a_nX^{s_n} \in P' - \{0\}$ with every $a_i \neq 0$ and $s_i \neq s_j$ for every $i \neq j$. Since $\frac{s_i}{f} \in S_\star$ for every i, we have that $s_i \in fS_\star \subset Q, s_i \in P$, and $f \in PD[X;S]$. It follows that $P' = PD[X;S]$, and hence $Q' = PD[X;S]_{U^\star}$. Since $D[X;S]_{U_\star}$ is a Prüfer domain, we have $W = ((D[X;S]_{U^\star})_{Q'} = D[X;S]_{PD[X;S]}$. Then S_P is a valuation oversemigroup of S, because $D[X;S]_{PD[X;S]} \cap \mathrm{q}(S) = S_P$.

(3) implies (4): Let M be a maximal ideal of $S_\star$. Let $W = (S_\star)_M$, and let $N = MW$. Since W is of the form $D[X;S]_{PD[X;S]}$, we have $D[X;S] \cap N = PD[X;S]$, and $D[X;S] \cap M = PD[X;S]$. By (6.9), $S_\star$ is a flat $D[X;S]$-module.

(4) implies (2): Let $E = \{f \in D[X;S] - \{0\} \mid \frac{1}{f} \in S_\star\}$. Let M be a maximal

ideal of $D[X;S]_E$, and set $M \cap D[X;S] = M_0$. Suppose that $MS_\star = S_\star$. There are elements $f_i \in M$ such that $S_\star = (f_1, \cdots, f_n)S_\star$. Clearly, we may assume that $f_i \in M_0$ for every i. There is an element $s \in S$ so that, for $f = f_1X^s + f_2X^{2s} + \cdots + f_nX^{ns}$, we have $\mathrm{Exp}(f) = \mathrm{Exp}(f_1) \cup \cdots \cup \mathrm{Exp}(f_n)$. Then we have $(f_1, \cdots, f_n)S_\star = fS_\star$; and have a contradiction of $f \in M_0 \cap E$. It follows that $MS_\star \subsetneqq S_\star$. For every maximal ideal M' of $S_\star$ containing $MS_\star$, we have $M' \cap D[X;S]_E = M$. Let $\{M_\lambda \mid \lambda \in \Lambda\}$ be the set of maximal ideals of $D[X;S]_E$, and let M'_λ be a maximal ideal of $S_\star$ such that $M'_\lambda \cap D[X;S]_E = M_\lambda$ for every λ. Then we have $(S_\star)_{M'_\lambda} = (D[X;S]_E)_{M_\lambda}$ by (6.9) for every λ, and have $\cap_\lambda (D[X;S]_E)_{M_\lambda} = D[X;S]_E$ (cf., [G2, (4.10) Theorem]). It follows that $S_\star = D[X;S]_E$.

(1) $\Longrightarrow$ (6) $\Longrightarrow$ (7) are easy.

(6.12) (1) ([$\mathrm{MS_i}$, (8.2) Theorem]). Every invertible ideal of S is principal.

(2)([$\mathrm{MS_k}$, Lemma 13]). If every finitely generated ideal of S is principal, then S is a valuation semigroup.

(6.13). If the semistar operation $\star$ is not e.a.b., the eight conditions in (6.11) need not be equivalent.

Example: Let S be 1-dimensional which is not integrally closed, and let $D = k$ be a field. Assume that $k[X;S]$ is 1-dimensional. For example, $S = \{0, 2, 3, 4, \cdots\}$. Let $\star$ be the d-semistar operation on S. Then $k[X;S]_{U^\star}$ is a 1-dimensional quasi-local domain. Every non-zero prime ideal of $k[X;S]_{U^\star}$ is the extension of a prime ideal of S. But S is not a Prüfer $\star$-multiplication semigroup by (6.12).

(6.14). Let $\star$ be a semistar operation on S, and let w be a valuation on $\mathrm{q}(D[X;S])$ non-negative on $S_\star$ with value group Γ. Then the restriction v of w to $\mathrm{q}(S)$ is a valuation on $\mathrm{q}(S)$ non-negative on S with value group Γ, and the canonical extension of v to $\mathrm{q}(D[X;S])$ is w.

Proof. Let v' be the canonical extension of v to $\mathrm{q}(D[X;S])$. Let $f = a_1X^{s_1} + \cdots + a_nX^{s_n} \in D[X;S] - \{0\}$ with every $a_i \neq 0$ and $s_i \neq s_j$ for every $i \neq j$. If $v(s_k) = \min{}_i v(s_i)$, we have $v'(f) = v(s_k)$, and have $w(f) \geq \min{}_i w(a_iX^{s_i}) = v(s_k)$. Since $\frac{s_k}{f} \in S_\star$, we have $0 \leq w(\frac{s_k}{f}) = v(s_k) - w(f)$. It follows that $w(f) = v(s_k) = v'(f)$, and hence $w = v'$.

(6.15). Let v be the v-semistar operation on S. Let $\{P_\lambda \mid \lambda \in \Lambda\}$ be a set of prime ideals of S such that $V_\lambda = S_{P_\lambda}$ is a valuation overring of S with $S = \cap_\lambda V_\lambda$. Let w be the w-semistar operation induced by the set $\{V_\lambda \mid \lambda \in \Lambda\}$. Then we have $S_w = S_v$.

Proof. Let I be a finitely generated ideal of S. Since $S = \cap_\lambda V_\lambda$, it is obvious that $I^w \subset I^v$. Suppose that $I^w \subsetneqq I^v$, and choose an element $x \in I^v - I^w$. Then we have $x \notin I + V_\lambda$ for some λ. There is an element $a \in S$ such that $I + V_\lambda = a + V_\lambda$, and there is an element $s \in S - P_\lambda$ such that $s + I \subset (a)$. Since $a + V_\lambda \subsetneqq x + V_\lambda$, we have $x \notin (a - s)$. Since $I \subset (a - s)$, we have $x \notin I^v$; a contradiction. Hence we have

$I^w = I^v$. It follows that $S_w = S_v$.

(6.16) Proposition. Let $\star$ be a semistar operation on S, and let v_1 be the v-star operation on $S^\star$. Assume that v_1 is e.a.b., and that $S_\star$ is a flat $D[X;S^\star]$-module. Then we have $S_\star = (S^\star)_{v_1}$.

Proof. By (6.10), we may assume that $S^\star = S$. Let W be a proper valuation overring of $S_\star$ with maximal ideal N. We set $N \cap S_\star = Q, Q \cap D[X;S] = P'$ and $P' \cap S = P$. Then we have $PD[X;S] = P'$. By (6.9), we have $W = (S_\star)_Q = D[X;S]_{P'} = D[X;S]_{PD[X;S]}$. Since $D[X;S]_{PD[X;S]} \cap \mathrm{q}(S) = S_P$, S_P is a valuation oversemigroup of S. Let $\{W_\lambda \mid \lambda \in \Lambda\}$ be the set of proper valuation overrings of $S_\star$, let N_λ be the maximal ideal of W_λ, let $S \cap N_\lambda = P_\lambda$, let $W_\lambda \cap \mathrm{q}(S) = S_{P_\lambda} = V_\lambda$ for every λ, and let w be the w-semistar operation induced by the set $\{V_\lambda \mid \lambda \in \Lambda\}$. We have $\cap_\lambda W_\lambda = S_\star$. Since $I^\star \subset I^{v_1}$ for every $I \in \mathrm{f}(S)$, we have $S_\star \subset S_{v_1}$. Since v_1 is e.a.b., we have $S_{v_1} \cap \mathrm{q}(S) = S$. It follows that $S_\star \cap \mathrm{q}(S) = S$, and that $\cap_\lambda V_\lambda = \cap_\lambda W_\lambda \cap \mathrm{q}(S) = S_\star \cap \mathrm{q}(S) = S$. We have $S_w = S_{v_1}$ by (6.15). Since W_λ is the canonical extension of V_λ by (6.14), we have $S_w = \cap_\lambda W_\lambda$ by (6.5). Therefore $S_\star = S_{v_1}$.

(6.17) Proposition. Let $\star$ be a semistar operation on S. Assume that S is a Prüfer $\star$-multiplication semigroup. Then the group $\{I^\star \mid I \in \mathrm{f}(S)\}$ is canonically isomorphic onto the group of divisibility of $S_\star$.

Proof. Then $\star$ is an e.a.b. semistar operation. The proof follows from (6.4).

§7 Appendix

(7.1). Let S be an APVS with maximal ideal P, and let $M = \mathrm{Rad}_{\mathrm{q}(S)}(P)$. Then there is the smallest oversemigroup T of S such that T is a PVS with maximal ideal M.

Proof. Let H be the unit group of S, and set $T = H \cup M$.

(7.2). Let D be an APVD with maximal ideal P, and let $M = \mathrm{Rad}_{\mathrm{q}(D)}(P)$. Then there is the smallest overring T of D such that T is a PVD with maximal ideal M.

Proof. Set $T = (D + M)_M$.

(7.3). Assume that S is a 1-dimensional g-monoid with maximal ideal P. If $(P : P)$ is a valuation semigroup, then S need not be an APVS.

Example: Let V be a 2-dimensional valuation semigroup, let Q be the height 1 prime ideal, and let $q \in Q$. Set $P = q + V$, and set $S = \{0\} \cup P$. Then P is a prime ideal of S, $V = (P : P)$, and in V, $\mathrm{Rad}_V(P)$ is not the maximal ideal of V. Suppose that $P \supsetneqq I$ be an ideal of S, and choose $x \in P - I$. We have $nx \in I$ for a sufficiently large n. Hence S is 1-dimensional.

(7.4). There is a 1-dimensional quasi-local domain D with maximal ideal P such that $(P : P)$ is a valuation domain and D is not an APVD.

Example: Let $1, e$ be linearly independent over $\boldsymbol{Z}$, and set $\Gamma = \boldsymbol{Z} + \boldsymbol{Z}e$. Introduce the lexicographic order on Γ with $1 < e$. Set $v(X) = 1$ and $v(Y) = e$. Then we have a valuation v on $k(X, Y)$, where k is a field. The valuation domain V of v is 2-dimensional. Let $M \supsetneqq Q \supsetneqq (0)$ be the prime ideals of V. Set $P = YV$, and set $D = k + P$. Then P is a maximal ideal of D, and $(P : P) = V$. In V, we see that $\mathrm{Rad}_V(P)$ is not a maximal ideal, and hence D is not an APVD. Let $I \subsetneqq P$ be a non-zero ideal of D. Assume that $P \supsetneqq I$ be an ideal of D, and choose $x \in P - I$. We have $nx \in I$ for a sufficiently large n. Hence I is not a prime ideal of D. Hence D is 1-dimensional.

Let $\star$ be a semistar operation on D. If $(I \cap J)^\star = I^\star \cap J^\star$ for all $I, J \in \bar{\mathrm{F}}(D)$, then $\star$ is called stable.

M. Fontana and J. Huckaba [FH] gives the following example: Let k be a field, and let $D = k + X^3k[[X]]$. Then D is an APVD. The v-semistar operation on D is not stable.

For let $I = (X^3, X^4)$ and $J = (X^3, X^5)$. Then $(I \cap J)^v \neq I^v \cap J^v$.

If V is a valuation domain, then every semistar operation $\star$ on V is stable.

(7.5). There is a PVD D and a semistar operation $\star$ on D such that $\star$ is not stable.

Example: Let k be a field with characteristic 0, K be an extension field with $[K : k] = 4, K = k + ku + kv + ks$, $U_0 = k + ku$, $W_0 = k + kv + ks$, $V = K[[X]]$, and $D = k + M$, where M is the maximal ideal of V. D is a PVD.

Let I be a non-zero fractional ideal of D. Then there is $x \in \mathrm{q}(D) - \{0\}$ and a k-subspace U of K with $U \supset k$ such that $I = xUD$. For, let $v(x) = \min v(I)$ with $x \in I$, where v is the canonical valuation for V. Set $U = x^{-1}I \cap K$.

Set $D^\star = D, V^\star = V$ and $(U_0D)^\star = V$.

For every 2-dimensional k-subspace U' of K such that $U_0 \neq U' \supset k$, set $(U'D)^\star = U'D$.

For every 3-dimensional k-subspace W of K with $W \supset k$, set $(WD)^\star = V$.

Then there is canonically defined a mapping $\star$ from $\bar{\mathrm{F}}(D)$ to $\bar{\mathrm{F}}(D)$, and $\star$ is a semistar operation on D.

Let $I = U_0D$, and $J = W_0D$. Then $I^\star \cap J^\star = V$. $I \cap J = D$, and $(I \cap J)^\star = D$.

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