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The semistar operations on certain Prüfer domain

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Abstract

Let D be a 1-dimensional Prüfer domain with exactly two maximal ideals. We determine the semistar operations on D.

Let D be an integral domain, let K be its quotient field, let $\mathrm{F}(D)$ be the set of nonzero fractional ideals of D, and let $\bar{\mathrm{F}}(D)$ be the set of nonzero D-submodules of K. A mapping $I \longmapsto I^\star$ from $\bar{\mathrm{F}}(D)$ to $\bar{\mathrm{F}}(D)$ is called a semistar operaion on D, if it satisfies the following conditions: (1) $(xI)^\star = xI^\star$ for each $x \in K \setminus \{0\}$ and $I \in \bar{\mathrm{F}}(D)$; (2) $I \subset I^\star$ for each $I \in \bar{\mathrm{F}}(D)$; (3) $(I^\star)^\star = I^\star$ for each $I \in \bar{\mathrm{F}}(D)$; (4) $I \subset J$ implies $I^\star \subset J^\star$ for each $I, J \in \bar{\mathrm{F}}(D)$. Let $\Sigma'(D)$ (resp. $\Sigma(D)$) be the set of semistar operations (resp. star operations) on D. We have the following,

Theorem ([H]). Let V be a valuation domain with maximal ideal M.

(1) If M is principal, then $|\Sigma(V)| = 1$.

(2) If M is not principal, then $|\Sigma(V)| = 2$.

Theorem ([M1]). Let V be an n-dimensional valuation domain, let v be a valuation belonging to V, and let Γ be the value group of v. Let $M = P_n \supsetneqq P_{n-1} \supsetneqq \cdots \supsetneqq P_1 \supsetneqq (0)$ be the prime ideals of V, and let $H_n = \{0\} \subsetneqq H_{n-1} \subsetneqq \cdots \subsetneqq H_1 \subsetneqq \Gamma$ be the convex subgroups of Γ. Let m be an integer with $n+1 \leq m \leq 2n+1$. Then the following conditions are equivalent.

(1) $|\Sigma'(V)| = m$.

(2) The maximal ideal of V_{P_i} is principal for exactly $2n+1-m$ of i in $\{1, \cdots, n\}$.

(3) Γ/H_i has a minimal positive element for exactly $2n+1-m$ of i in $\{1, \cdots, n\}$.

We want to know $|\Sigma'(D)|$ for Prüfer domains D. Let Γ be a totally ordered set. If each nonempty subset S of Γ which is bounded below has its infimum inf (S) in Γ, then Γ is called complete. In this paper we determine $|\Sigma'(D)|$ for a 1-dimensional Prüfer domain D with two maximal ideals. Our result is the following,

Theorem. Let D be a 1-dimensional Prüfer domain with exactly two maximal ideals M and N, and let Γ (resp. Γ') be the value group of D_M (resp. D_N).

(1) If both M and N are principal, then $|\Sigma(D)| = 1$ and $|\Sigma'(D)| = 7$.

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(2) If M is principal, and Γ' is not complete, then $|\Sigma(D)| = 2$ and $|\Sigma'(D)| = 14$.
(3) If Γ is not complete, and N is principal, then $|\Sigma(D)| = 2$ and $|\Sigma'(D)| = 14$.
(4) If neither Γ nor Γ' is complete, then $|\Sigma(D)| = 9$ and $|\Sigma'(D)| = 54$.

In §2 we will prove (1). In §3 we will prove (2). In §4 we will prove (4). The proof for (3) is similar to that of (2). §1 is the general case.

Throughout the paper, let D be a 1-dimensional Prüfer domain with exactly two maximal ideals M and N, let $V = D_M$ (resp. $W = D_N$), let v (resp. w) be a valuation belonging to V (resp. W), let Γ (resp. Γ') be the value group of v (resp. w), and let K be the quotient field of D. Let A (resp. B) be the D-submodule of K generated by the subset $\{\frac{1}{p} \mid p \in M \setminus N\}$ (resp. $\{\frac{1}{q} \mid q \in N \setminus M\}$) of K.

This is a continuation of [M2].

§1 The general case

Throughout the paper, p denotes an element of $M \setminus N$, and q denotes an element of $N \setminus M$.

(1.1) (1) Each element $x \in D \setminus \{0\}$ can be expressed as 1 or p or q or pq uniquely up to associates, where $p \in M \setminus N$ and $q \in N \setminus M$.

(2) Each element $x \in K \setminus \{0\}$ can be expressed as 1 or p or q or pq or $\frac{1}{p}$ or $\frac{1}{q}$ or $\frac{q}{p}$ or $\frac{p}{q}$ or $\frac{1}{pq}$ uniquely up to associates.

Proof. (1) Let $0 \neq x \in M \cap N$. Choose an element $p \in M \setminus N$. Since Γ has rank 1, there exists a positive integer n such that $v(p^n) > v(x)$. Let $p_1 = p^n + x$. Then we have $v(p_1) = v(x)$ and $p_1 \in M \setminus N$. Similarly, there exists an element $q_1 \in N \setminus M$ such that $w(q_1) = w(x)$. Then we have $x = p_1 q_1 u$ for a unit u of D.

We note that $V = B$ and $W = A$.

A fractional ideal $A_0 \in \mathrm{F}(D)$ is called of type α, if there exists a subset $\{p_\lambda \mid \Lambda\}$ of $M \setminus N$ so that A_0 is generated by the set $\{\frac{1}{p_\lambda} \mid \Lambda\}, v(A_0)$ is bounded below, and there does not exist inf $v(A_0)$. A fractional ideal $B_0 \in \mathrm{F}(D)$ is called of type β, if there exists a subset $\{q_\sigma \mid \Sigma\}$ of $N \setminus M$ so that B_0 is generated by the set $\{\frac{1}{q_\sigma} \mid \Sigma\}, w(B_0)$ is bounded below, and there does not exist inf $w(B_0)$.

(1.2) Let I be a nonzero fractional ideal of D. Then there may arise the following 9-cases:
(1) There exists min $v(I)$, and there exists min $w(I)$;
(2) There exists min $v(I)$, and there exists inf $w(I)$ with inf $w(I) \notin w(I)$;
(3) There exists min $v(I)$, and there does not exist inf $w(I)$;
(4) There exists inf $v(I)$ with inf $v(I) \notin v(I)$, and there exists min $w(I)$;
(5) There exists inf $v(I)$ with inf $v(I) \notin v(I)$, and there exists inf $w(I)$ with

$\inf w(I) \notin w(I)$;

(6) There exists $\inf v(I)$ with $\inf v(I) \notin v(I)$, and there does not exist $\inf w(I)$;
(7) There does not exist $\inf v(I)$, and there exists $\min w(I)$;
(8) There does not exist $\inf v(I)$, and there exists $\inf w(I)$ with $\inf w(I) \notin w(I)$;
(9) There does not exist $\inf v(I)$, and there does not exist $\inf w(I)$.

Case (1): There exist elements $a, b \in I$ such that $v(a) = \min v(I)$ and $w(b) = \min w(I)$. Let $(a, b) = (c)$ for an element $c \in I$. Then we have $I = (c)$.

Case (2): Assume that $D \subsetneqq I$ and $\min v(I) = 0$. Let $w(\frac{1}{q_0}) = \inf w(I)$ for an element $q_0 \in N \setminus M$. Then we have $I = \frac{1}{q_0}N$ and $I^v = (\frac{1}{q_0})$.

Let J be a fractional ideal of D such that $I \subset J \subset I^v$. Then J is either I or I^v.

Case (3): Assume that $D \subsetneqq I$ and $\min v(I) = 0$. Let $\{q \in N \setminus M \mid \frac{1}{q} \in I\} = \{q_\lambda \mid \Lambda\}$. Then the fractional ideal $(\frac{1}{q_\lambda} \mid \Lambda)$ of D is of type β, and $I = (\frac{1}{q_\lambda} \mid \Lambda)$. Let $\{q_\sigma \mid \Sigma\}$ be a subset of $N \setminus M$ so that $\{w(\frac{1}{q_\sigma}) \mid \Sigma\}$ is the lower bounds of $w(I)$. Then we have $I = \cap_\sigma(\frac{1}{q_\sigma})$, hence $I^v = I$.

Case (5): Assume that $D \subsetneqq I$. Let $v(\frac{1}{p_0}) = \inf v(I)$ for an element $p_0 \in M \setminus N$, and let $w(\frac{1}{q_0}) = \inf w(I)$ for an element $q_0 \in N \setminus M$. Then we have $I = \frac{1}{p_0 q_0}MN, I^v = (\frac{1}{p_0 q_0})$ and $I^v \setminus I = \{\frac{u}{p_0 q_0}, \frac{q}{p_0}, \frac{q}{p_0 q_0}, \frac{p}{q_0}, \frac{p}{p_0 q_0} \mid u \text{ is a unit of } D, p \in M \setminus N, q \in N \setminus M\}$.

Let J be a fractional ideal of D such that $I \subset J \subset I^v$. Then J is either I or $(I, \frac{1}{p_0})$ or $(I, \frac{1}{q_0})$ or I^v.

Case (6): Assume that $D \subsetneqq I$. Let $v(\frac{1}{p_0}) = \inf v(I)$ for an element $p_0 \in M \setminus N$, and let $\{q \in N \setminus M \mid \frac{1}{q} \in I\} = \{q_\lambda \mid \Lambda\}$. Then we have $I = \frac{1}{p_0}M(\frac{1}{q_\lambda} \mid \Lambda)$, and the fractional ideal $(\frac{1}{q_\lambda} \mid \Lambda)$ of D is of type β. Let $\{q_\sigma \mid \Sigma\}$ be a subset of $N \setminus M$ so that $\{w(\frac{1}{q_\sigma}) \mid \Sigma\}$ is the lower bounds of $w(I)$. Then we have $I^v = \cap_\sigma(\frac{1}{p_0 q_\sigma}) = \frac{1}{p_0}(\frac{1}{q_\lambda} \mid \Lambda)$ and $I^v \setminus I = \{\frac{u}{p_0}, \frac{q}{p_0}, \frac{u}{p_0 q_\lambda}, \frac{q}{p_0 q_\lambda} \mid q \in N \setminus M, \lambda \in \Lambda, u \text{ is a unit of } D\}$.

Let J be a fractional ideal of D such that $I \subset J \subset I^v$. Then J is either I or I^v.

Case (9): Assume that $D \subsetneqq I$. Let $\{p \in M \setminus N \mid \frac{1}{p} \in I\} = \{p_\lambda \mid \Lambda\}$, and let $\{q \in N \setminus M \mid \frac{1}{q} \in I\} = \{q_\sigma \mid \Sigma\}$. Then the fractional ideal $(\frac{1}{p_\lambda} \mid \Lambda)$ of D is of type α, the fractional ideal $(\frac{1}{q_\sigma} \mid \Sigma)$ of D is of type β, and $I = (\frac{1}{p_\lambda} \mid \Lambda)(\frac{1}{q_\sigma} \mid \Sigma)$. Let $\{p_{\lambda'} \mid \Lambda'\}$ be a subset of $M \setminus N$ so that $\{v(\frac{1}{p_{\lambda'}}) \mid \Lambda'\}$ is the lower bounds of $v(I)$, and let $\{q_{\sigma'} \mid \Sigma'\}$ be a subset of $N \setminus M$ so that $\{w(\frac{1}{q_{\sigma'}}) \mid \Sigma'\}$ is the lower bounds of $w(I)$. Then we have $I = \cap_{\lambda',\sigma'}(\frac{1}{p_{\lambda'}q_{\sigma'}})$, hence $I^v = I$.

Let A_0 (resp. B_0) be a fractional ideal of D with type α (resp. type β). Then we have $A_0 = A_0^v, B_0 = B_0^v$ and $A_0B_0 = (A_0B_0)^v$ by (1.2).

(1.3) Let A_0 (resp. B_0) be a fractional ideal of D with type α (resp. type β), and let $x \in K \setminus \{0\}$. If either $xM = M$ or $xN = N$ or $xMN = MN$ or $xA_0N = A_0N$ or $xB_0M = B_0M$, then $(x) = D$.

(1.4) Example. Let A_1 (resp. A_2) be the additive group of all integers $\mathbf{Z}$ (resp. all rational numbers $\mathbf{Q}$), and introduce on each of them the canonical order. Let $A_1 \oplus A_2$ be their direct sum with the lexicographic order: Let $x = (a_1, a_2), y = (b_1, b_2)$ be elements of $A_1 \oplus A_2$ with $a_2 < b_2$, then let $x < y$. Let v_0 (resp. w_0) be the projection mapping of $A_1 \oplus A_2$ to the ordered group A_1 (resp. A_2). Let k be a field, and let K be the quotient field of the semigroup ring $k[X; A_1 \oplus A_2]$. Let v (resp. w) be the canonical extension of the valuation v_0 (resp. w_0) on $A_1 \oplus A_2$ to a valuation on K, and let V (resp. W) be the valuation ring on K belonging to v (resp. w), and let M' (resp. N') be the maximal ideal of V (resp. W). Let $D = V \cap W$, and let $M = M' \cap D$ and $N = N' \cap D$. Then we have $V = D_M, M' = MV, W = D_N$ and $N' = NW$. We have $\Gamma = \mathbf{Z}$, and $\Gamma' = \mathbf{Q}$ is not complete. Let r be a real number with $\mathbf{Q} \not\ni r < 0$, and le $\{q \in N \setminus M \mid w(\frac{1}{q}) > r\} = \{q_\sigma \mid \Sigma\}$. Then the fractional ideal $(\frac{1}{q_\sigma} \mid \Sigma)$ of D has type β. Let $I = M(\frac{1}{q_\sigma}) \mid \Sigma)$. Then there exists inf $v(I)$ with inf $v(I) \notin v(I)$, and there does not exist inf $w(I)$.

(1.5) Let $\star$ be a star operation on D. Then we have that $M^\star$ is either M or D, $N^\star$ is either N or D, and $(MN)^\star$ is either MN or M or N or D. Either $(A_0N)^\star = A_0N$ for each fractional ideal A_0 of D with type α, or $(A_0N)^\star = A_0$ for each fractional ideal A_0 of D with type α. Either $(B_0M)^\star = B_0M$ for each fractional ideal B_0 of D with type β, or $(B_0M)^\star = B_0$ for each fractional ideal B_0 of D with type β.

Proof. Suppose that $(A_0^1N)^\star = A_0^1N$ and $(A_0^2N)^\star = A_0^2$ for some fractional ideals A_0^1, A_0^2 of D with type α. There exists an element $p \in M \setminus N$ such that $A_0^2 \subset \frac{1}{p}A_0^1$. It follows that $A_0^2 \subset \frac{1}{p}A_0^1N$; a contradiction.

Let $\star$ be a star operation on D. If $(A_0N)^\star = A_0N$ for each fractional ideal A_0 of D with type α, and $(B_0M)^\star = B_0M$ for each fractional ideal B_0 of D with type β, then we say that $\star$ is of type (α, β). If $(A_0N)^\star = A_0N$ for each fractional ideal A_0 of D with type α, and $(B_0M)^\star = B_0$ for each fractional ideal B_0 of D with type β, then we say that $\star$ is of type (α, β'). If $(A_0N)^\star = A_0$ for each fractional ideal A_0 of D with type α, and $(B_0M)^\star = B_0M$ for each fractional ideal B_0 of D with type β, then we say that $\star$ is of type (α', β). If $(A_0N)^\star = A_0$ for each fractional ideal A_0 of D with type α, and $(B_0M)^\star = B_0$ for each fractional ideal B_0 of D with type β, then we say that $\star$ is of type (α', β').

(1.6) We have $|\Sigma(D)| \leq 64$.

Proof. (1.2) implies that $\{I \in \mathrm{F}(D) \mid I \subsetneqq I^v\} \subset \{xM, xN, xMN, xA_0N, xB_0M \mid x \in K \setminus \{0\}, A_0$ (resp. B_0) is a fractional ideal of D with type α (resp. type β)$\}$. Let $\star$ be a star operation on D. Then we have that $M^\star$ is either M or D, $N^\star$ is either N or D, and $(MN)^\star$ is either MN or M or N or D. $\star$ has type (α, β) or type (α, β') or type (α', β) or type (α', β').

(1.7) Let $I \in \bar{\mathrm{F}}(D) \setminus \mathrm{F}(D)$. There may arise the following 7-cases:

(1) There exists min $v(I)$, and $w(I) = \Gamma'$;

(2) $v(I)$ is bounded below, there exists inf $v(I)$ with inf $v(I) \notin v(I)$, and $w(I) = \Gamma'$;

(3) $v(I)$ is bounded below, there does not exist inf $v(I)$, and $w(I) = \Gamma'$;

(4) $v(I) = \Gamma$, and there exists min $w(I)$;

(5) $v(I) = \Gamma$, and there exists inf $w(I)$ with inf $w(I) \notin w(I)$;

(6) $v(I) = \Gamma$, $w(I)$ is bounded below, and there does not exist inf $w(I)$;

(7) $v(I) = \Gamma$, and $w(I) = \Gamma'$.

Case (1): Assume that min $v(I) = 0$. Then we have $I = B$.

Case (2): Assume that $v(\frac{1}{p_0}) = \inf v(I)$ for an element $p_0 \in M \setminus N$. Let $\{p \in M \setminus N \mid \frac{1}{p} \in I\} = \{p_\lambda \mid \Lambda\}$. Then we have $(\frac{1}{p_\lambda} \mid \Lambda) = \frac{1}{p_0}M, I = \frac{1}{p_0}MB$, and $M^v = D$.

Case (3): Assume that $I \supsetneqq D$. Let $\{p \in M \setminus N \mid \frac{1}{p} \in I\} = \{p_\lambda \mid \Lambda\}$, and let $A_0 = (\frac{1}{p_\lambda} \mid \Lambda)$. Then A_0 is a fractional ideal of D with type α, and $I = A_0B$.

Case (7): $I = K$.

Let $I^{d'} = I$ (resp. $I^{v'} = (I^{-1})^{-1}, I^e = K$) for each $I \in \bar{\mathrm{F}}(D)$. Then d$'$ (resp. v$'$, e) is a semistar operation on D, and is called the d$'$-operation (resp. v$'$-operation, e-operation) on D.

Let $I \subset J$ be an inclusion relation of elements of $\bar{\mathrm{F}}(D)$. A mapping $\star$ from $\bar{\mathrm{F}}(D)$ to $\bar{\mathrm{F}}(D)$ is called monotone for $I \subset J$, if $I^\star \subset J^\star$.

(1.8) Let $\star$ be a semistar operation on D.

(1) $A^\star$ is either A or K. $B^\star$ is either B or K.

(2) $(AN)^\star$ is either AN or A or K. $(BM)^\star$ is either BM or B or K.

(3) If $A^\star = A$, then $(AB_0)^\star = AB_0$ for each fractional ideal B_0 of D with type β. If $B^\star = B$, then $(A_0B)^\star = A_0B$ for each fractional ideal A_0 of D with type α.

Proof. (1) Assume that $A \subsetneqq A^\star$. There exists an element $q \in N \setminus M$ such that $\frac{1}{q} \in A^\star$. Then we have $\frac{1}{q}A \subset A^\star$, and $A^\star = \frac{1}{q}A^\star$. Hence $A^\star = K$.

(2) We may assume that $A^\star = A$. We have $(AN)^\star \subset A$. Let $AN \subsetneqq J \subset A$ be a D-submodule of K. There exists an element $x \in J$ with $w(x) = 0$. It follows that $\frac{1}{p_0} \in J$ for some $p_0 \in M \setminus N$. Then we have $D \subset J$ and $\frac{1}{p} \in J$ for each $p \in M \setminus N$. Hence $J = A$.

(3) Suppose that $AB_0 \subsetneqq (AB_0)^\star$. There exists an element $q_1 \in N \setminus M$ such that $AB_0 \not\ni \frac{1}{q_1} \in (AB_0)^\star$. Then $w(\frac{1}{q_1})$ is a lower bound of $w(B_0)$, and $AB_0 \subset A\frac{1}{q_1} \subset (AB_0)^\star$. There exists an element $q_2 \in N \setminus M$ with $w(\frac{1}{q_1}) < w(\frac{1}{q_2})$ so that $w(\frac{1}{q_2})$ is a lower bound of $w(B_0)$. Then we have $AB_0 \subset A\frac{1}{q_2} \subsetneqq A\frac{1}{q_1} \subset (AB_0)^\star$. Hence $(AB_0)^\star \subset A\frac{1}{q_2} \subsetneqq A\frac{1}{q_1} \subset (AB_0)^\star$; a contradiction.

(1.9) We have $|\Sigma'(D)| < \infty$.

Proof. Let Σ'_0 be the set of extensions of star operations on D to semistar operations on D. It is sufficient to prove that $|\Sigma'_0| < \infty$.

Let $\star$ be an element of Σ'_0. Then the restriction $\star|_{F(D)}$ belongs to the finite set $\Sigma(D)$. $A^\star$ is either$A$ or K, $B^\star$ is either B or K, $(AN)^\star$ is either AN or A or K, and $(BM)^\star$ is either BM or B or K.

Let $\star'$ be an element of Σ'_0 satisfying the following conditions: $\star|_{F(D)} = \star'|_{F(D)}, A^\star = A^{\star'}, B^\star = B^{\star'}$, $(AN)^\star = (AN)^{\star'}$ and $(BM)^\star = (BM)^{\star'}$. To prove that $|\Sigma'_0| < \infty$, it is sufficient to prove that $\star = \star'$.

Let $I \in \bar{\mathrm{F}}(D)$. We must show that $I^\star = I^{\star'}$.

If $I \in \mathrm{F}(D)$, we have $I^\star = I^{\star'}$, since $\star|_{F(D)} = \star'|_{F(D)}$.

Assume that there exists min $v(I)$ and that $w(I) = \Gamma'$. Then $I = xB$ for some element $x \in K$. Then we have $I^\star = xB^\star = xB^{\star'} = I^{\star'}$.

Assume that there exists inf $v(I)$ with inf $v(I) \notin v(I)$ and that $w(I) = \Gamma'$. Then we have $I = xBM$ for some element $x \in K$. Hence $I^\star = I^{\star'}$.

Assume that $v(I)$ is bounded, there does not exist inf $v(I)$, and $w(I) = \Gamma'$. There exists an element $x \in K$ and a fractional ideal A_0 with type α such that $xI = A_0B$. If $B^\star = K$, then $I^\star = I^{\star'} = K$. If $B^\star = B$, then $I^\star = I^{\star'} = \frac{1}{x}A_0B$.

Assume that $v(I) = \Gamma$ and $w(I) = \Gamma'$. Then $I = K$. Hence $I^\star = I^{\star'}$.

The proof is complete.

§2 The case where both M and N are principal

This case is just the case of [M2, Proposition 6 (2)]. We will review it for our convenience. Thus, each fractional ideal of D is principal. Hence we have $\Sigma(D) = \{\mathrm{d}\}$.

We have $\bar{\mathrm{F}}(D)\backslash\mathrm{F}(D) = \{xA, xB, K \mid x \in K \setminus \{0\}\}$.

Let $\star$ be a semistar operation on D. Then $A^\star$ is either A or K, and $B^\star$ is either B or K.

(2.1) (1) Set $I_0^\star = I_0$ for each $I_0 \in \mathrm{F}(D)$, and set $A^\star = A$ and $B^\star = B$. Then there is determined a semistar operation $\star$ on D uniquely, and $\star = \mathrm{d}'$.

(2) Set $I_0^\star = I_0$ for each $I_0 \in \mathrm{F}(D)$, and set $A^\star = A$ and $B^\star = K$. Then there is determined a semistar operation $\star$ on D uniquely.

(3) Set $I_0^\star = I_0$ for each $I_0 \in \mathrm{F}(D)$, and set $A^\star = K$ and $B^\star = B$. Then there is determined a semistar operation $\star$ on D uniquely.

(4) Set $I_0^\star = I_0$ for each $I_0 \in \mathrm{F}(D)$, and set $A^\star = K$ and $B^\star = K$. Then there is determined a semistar operation $\star$ on D uniquely, and $\star = \mathrm{v}'$.

(2.2) We have $|\Sigma(D)| = 1$ and $|\Sigma'(D)| = 7$.

§3 The case where M is principal and Γ' is not complete

Then N is not principal.

(3.1) We have $\{I \in \mathrm{F}(D) \mid I \subsetneqq I^v\} = \{xN \mid x \in K \setminus \{0\}\}$.

If $xN \subset N$ for an element $x \in K \setminus \{0\}$, then $x \in D$.

(3.2) (1) Set $N^{\star_1} = N$. Then there is determined a star operation $\star_1$ on D uniquely, and then $\star_1 = \mathrm{d}$.

(2) Set $N^{\star_2} = D$. Then there is determined a star operation $\star_2$ on D uniquely, and then $\star_2 = \mathrm{v}$.

(3.3) We have $|\Sigma(D)| = 2$ and $\Sigma(D) = \{\mathrm{d}, \mathrm{v}\ \}$.

(3.4) We have $\bar{\mathrm{F}}(D)\setminus \mathrm{F}(D) = \{xA, xB, xAN, xAB_0, K \mid x \in K \setminus \{0\}, B_0$ is a fractional ideal of D with type $\beta\}$.

(3.5) Let $\star$ be a semistar operation on D. Then we have that $A^\star$ is either A or K, $B^\star$ is either B or K, and $(AN)^\star$ is either AN or A or K. If $A^\star = A$, then $(AB_0)^\star = AB_0$ for each fractional ideal B_0 of D with type β.

(3.6) Let $\star_i$ be a star operation on D.

(1) Set $I_0^{\star_i^1} = I_0^{\star_i}$ for each $I_0 \in \mathrm{F}(D)$, and set $A^{\star_i^1} = A, B^{\star_i^1} = B$ and $(AN)^{\star_i^1} = AN$. Then there is determined a unique mapping $\star_i^1$ from $\bar{\mathrm{F}}(D)$ to $\bar{\mathrm{F}}(D)$.

(2) Set $I_0^{\star_i^2} = I_0^{\star_i}$ for each $I_0 \in \mathrm{F}(D)$, and set $A^{\star_i^2} = A, B^{\star_i^2} = B$ and $(AN)^{\star_i^2} = A$. Then there is determined a unique mapping $\star_i^2$ from $\bar{\mathrm{F}}(D)$ to $\bar{\mathrm{F}}(D)$.

(3) Set $I_0^{\star_i^3} = I_0^{\star_i}$ for each $I_0 \in \mathrm{F}(D)$, and set $A^{\star_i^3} = A, B^{\star_i^3} = K$ and $(AN)^{\star_i^3} = AN$. Then there is determined a unique mapping $\star_i^3$ from $\bar{\mathrm{F}}(D)$ to $\bar{\mathrm{F}}(D)$.

(4) Set $I_0^{\star_i^4} = I_0^{\star_i}$ for each $I_0 \in \mathrm{F}(D)$, and set $A^{\star_i^4} = A, B^{\star_i^4} = K$ and $(AN)^{\star_i^4} = A$. Then there is determined a unique mappping $\star_i^4$ from $\bar{\mathrm{F}}(D)$ to $\bar{\mathrm{F}}(D)$.

(5) Set $I_0^{\star_i^5} = I_0^{\star_i}$ for each $I_0 \in \mathrm{F}(D)$, and set $A^{\star_i^5} = K$ and $B^{\star_i^5} = B$. Then there is determined a unique mapping $\star_i^5$ from $\bar{\mathrm{F}}(D)$ to $\bar{\mathrm{F}}(D)$.

(6) Set $I_0^{\star_i^6} = I_0^{\star_i}$ for each $I_0 \in \mathrm{F}(D)$, and set $A^{\star_i^6} = K$ and $B^{\star_i^6} = K$. Then there

is determined a unique mapping $\star_i^6$ from $\bar{\mathrm{F}}(D)$ to $\bar{\mathrm{F}}(D)$.

(3.7) Each $\star_i^j$ satisfies the following conditions: $(xI)^\star = xI^\star$ for each $x \in K \setminus \{0\}$ and $I \in \bar{\mathrm{F}}(D)$; $I \subset I^\star$ for each $I \in \bar{\mathrm{F}}(D)$; and $(I^\star)^\star = I^\star$ for each $I \in \bar{\mathrm{F}}(D)$.

(3.8) Let B_0 be a fractional ideal of D with type β, and let $x \in K \setminus \{0\}$.
If $xN \subset A$, then $(x) \subset A$.
If $xN \subset B$, then $(x) \subset B$.
If $xN \subset AB_0$, then $(x) \subset AB_0$.
If $xAN \subset A$, then $xA \subset A$.
If $xAN \subset AB_0$, then $xA \subset AB_0$.

Proof. x is either u or p or q or pq or $\frac{1}{p}$ or $\frac{1}{q}$ or $\frac{q}{p}$ or $\frac{p}{q}$ or $\frac{1}{pq}$, where $p \in M \setminus N, q \in N \setminus M$, and u is a unit of D.

(3.9) $\star_1^1$ is a semistar operation on D, and $\star_1^1 = \mathrm{d}'$.
$\star_1^2$ is a semistar operation on D.
$\star_2^2$ is a semistar operation on D.
$\star_1^3$ is a semistar operation on D.
$\star_1^4$ is a semistar operation on D.
$\star_2^4$ is a semistar operation on D.
$\star_1^5$ is a semistar operation on D.
$\star_2^5$ is a semistar operation on D.
$\star_1^6$ is a semistar operation on D.
$\star_2^6$ is a semistar operation on D, and $\star_2^6 = \mathrm{v}'$.

(3.10) $\star_2^1$ is not monotone for $N \subset AN$. $\star_2^3$ is not monotone for $N \subset AN$.

(3.11) $|\Sigma'(D)| = 14$.

§4 The case where neither Γ nor Γ' is complete

Then neither M nor N is principal.

(4.1) We have $\{I \in \mathrm{F}(D) \mid I \subsetneqq I^v\} = \{xM, xN, xMN, xA_0N, xB_0M \mid x \in K \setminus \{0\}, A_0$ (resp. B_0) is a fractional ideal of D with type α (resp. type β)$\}$.

(4.2) Let $\star$ be a star operation on D. Then we have that $M^\star$ is either M or D, $N^\star$ is either N or D, and $(MN)^\star$ is either MN or M or N or D. $\star$ has the type (α, β) or (α, β') or (α', β) or (α', β').

(4.3) Let A_0 (resp. B_0) be a fractional ideal of D with type α (resp. type β).
(1) Set $M^{\star_1} = M$, $N^{\star_1} = N$, $(A_0N)^{\star_1} = A_0N$ and $(B_0M)^{\star_1} = B_0M$. Then there is determined a unique mapping $\star_1$ from $\mathrm{F}(D)$ to $\mathrm{F}(D)$.
(2) Set $M^{\star_2} = M$, $N^{\star_2} = N$, $(A_0N)^{\star_2} = A_0N$ and $(B_0M)^{\star_2} = B_0$. Then there

is determined a unique maping $\star_2$ from $\mathrm{F}(D)$ to $\mathrm{F}(D)$.

(3) Set $M^{\star_3} = M$, $N^{\star_3} = N$, $(A_0N)^{\star_3} = A_0$ and $(B_0M)^{\star_3} = B_0M$. Then there is determined a unique mapping $\star_3$ from $\mathrm{F}(D)$ to $\mathrm{F}(D)$.

(4) Set $M^{\star_4} = M$, $N^{\star_4} = N$, $(A_0N)^{\star_4} = A_0$ and $(B_0M)^{\star_4} = B_0$. Then there is determined a unique mapping $\star_4$ from $\mathrm{F}(D)$ to $\mathrm{F}(D)$.

(5) Set $M^{\star_5} = M$, $N^{\star_5} = D$ and $(B_0M)^{\star_5} = B_0M$. Then there is determined a unique mapping $\star_5$ from $\mathrm{F}(D)$ to $\mathrm{F}(D)$.

(6) Set $M^{\star_6} = M$, $N^{\star_6} = D$ and $(B_0M)^{\star_6} = B_0$. Then there is determined a unique mapping $\star_6$ from $\mathrm{F}(D)$ to $\mathrm{F}(D)$.

(7) Set $M^{\star_7} = D$, $N^{\star_7} = N$ and $(A_0N)^{\star_7} = A_0N$. Then there is determined a unique mapping $\star_7$ from $\mathrm{F}(D)$ to $\mathrm{F}(D)$.

(8) Set $M^{\star_8} = D$, $N^{\star_8} = N$ and $(A_0N)^{\star_8} = A_0$. Then there is determined a unique mapping $\star_8$ from $\mathrm{F}(D)$ to $\mathrm{F}(D)$.

(9) Set $M^{\star_9} = D$ and $N^{\star_9} = D$. Then there is determined a unique mapping $\star_9$ from $\mathrm{F}(D)$ to $\mathrm{F}(D)$.

(4.4) Let A_0 (resp. B_0) be a fractional ideal of D with type α (resp. type β), and let $x \in K \setminus \{0\}$.

If $xM \subset D$, then $(x) \subset D$.
If $xM \subset N$, then $(x) \subset N$.
If $xM \subset A_0N$, then $(x) \subset A_0N$.
If $xM \subset B_0M$, then $(x) \subset B_0$.
If $xN \subset D$, then $(x) \subset D$.
If $xN \subset M$, then $(x) \subset M$.
If $xN \subset A_0N$, then $(x) \subset A_0$.
If $xN \subset B_0M$, then $(x) \subset B_0M$.
If $xMN \subset D$, then $(x) \subset D$.
If $xMN \subset M$, then $xM \subset M$.
If $xMN \subset M$, then $xN \subset D$.
If $xMN \subset N$, then $xM \subset D$.
If $xMN \subset N$, then $xN \subset N$.
If $xMN \subset A_0N$, then $(x) \subset A_0$.
If $xMN \subset B_0M$, then $(x) \subset B_0$.
If $xA_0N \subset D$, then $xA_0 \subset D$.
If $xA_0N \subset M$, then $xA_0 \subset M$.
If $xA_0N \subset N$, then $xA_0 \subset N$.
If $xA_0N \subset MN$, then $xA_0 \subset MN$.
If $xA_0N \subset A_0N$, then $xA_0 \subset A_0$.
If $xA_0N \subset B_0M$, then $xA_0 \subset B_0M$.
If $xB_0M \subset D$, then $xB_0 \subset D$.
If $xB_0M \subset M$, then $xB_0 \subset M$.
If $xB_0M \subset N$, then $xB_0 \subset N$.
If $xB_0M \subset MN$, then $xB_0 \subset MN$.
If $xB_0M \subset A_0N$, then $xB_0 \subset A_0N$.
If $xB_0M \subset B_0M$, then $xB_0 \subset B_0$.

(4.5) Each $\star_i$ is a star operation on D. We have $\star_1 = \mathrm{d}$ and $\star_9 = \mathrm{v}$.

(4.6) $|\Sigma(D)| = 9$.

(4.7) We have $\bar{\mathrm{F}}(D) \setminus \mathrm{F}(D) = \{xA, xB, xAN, xBM, xAB_0, xA_0B, K \mid x \in K \setminus \{0\}, A_0$ (resp. B_0) is a fractional ideal of D with type α (resp. type β)$\}$.

(4.8) Let $\star$ be a semistar operation on D. Then we have that $A^\star$ is either A or K, $B^\star$ is either B or K, $(AN)^\star$ is either AN or A or K, and $(BM)^\star$ is either BM or B or K. If $A^\star = A$, then $(AB_0)^\star = AB_0$ for each fractional ideal B_0 of D with type β, and if $B^\star = B$, then $(A_0B)^\star = A_0B$ for each fractional ideal A_0 of D with type α.

(4.9) Let $\star_i$ be a star operation on D.

(1) Set $I_0^{\star_i^1} = I_0^{\star_i}$ for each $I_0 \in \mathrm{F}(D)$, and set $A^{\star_i^1} = A, B^{\star_i^1} = B$, $(AN)^{\star_i^1} = AN$ and $(BM)^{\star_i^1} = BM$. Then there is determined a unique mapping $\star_i^1$ from $\bar{\mathrm{F}}(D)$ to $\bar{\mathrm{F}}(D)$.

(2) Set $I_0^{\star_i^2} = I_0^{\star_i}$ for each $I_0 \in \mathrm{F}(D)$, and set $A^{\star_i^2} = A, B^{\star_i^2} = B$, $(AN)^{\star_i^2} = AN$ and $(BM)^{\star_i^2} = B$. Then there is determined a unique mapping $\star_i^2$ from $\bar{\mathrm{F}}(D)$ to $\bar{\mathrm{F}}(D)$.

(3) Set $I_0^{\star_i^3} = I_0^{\star_i}$ for each $I_0 \in \mathrm{F}(D)$, and set $A^{\star_i^3} = A, B^{\star_i^3} = B$, $(AN)^{\star_i^3} = A$ and $(BM)^{\star_i^3} = BM$. Then there is determined a unique mapping $\star_i^3$ from $\bar{\mathrm{F}}(D)$ to $\bar{\mathrm{F}}(D)$.

(4) Set $I_0^{\star_i^4} = I_0^{\star_i}$ for each $I_0 \in \mathrm{F}(D)$, and set $A^{\star_i^4} = A, B^{\star_i^4} = B$, $(AN)^{\star_i^4} = A$ and $(BM)^{\star_i^4} = B$. Then there is determined a unique mappping $\star_i^4$ from $\bar{\mathrm{F}}(D)$ to $\bar{\mathrm{F}}(D)$.

(5) Set $I_0^{\star_i^5} = I_0^{\star_i}$ for each $I_0 \in \mathrm{F}(D)$, and set $A^{\star_i^5} = A, B^{\star_i^5} = K$ and $(AN)^{\star_i^5} = AN$. Then there is determined a unique mapping $\star_i^5$ from $\bar{\mathrm{F}}(D)$ to $\bar{\mathrm{F}}(D)$.

(6) Set $I_0^{\star_i^6} = I_0^{\star_i}$ for each $I_0 \in \mathrm{F}(D)$, and set $A^{\star_i^6} = A, B^{\star_i^6} = K$ and $(AN)^{\star_i^6} = A$. Then there is determined a unique mapping $\star_i^6$ from $\bar{\mathrm{F}}(D)$ to $\bar{\mathrm{F}}(D)$.

(7) Set $I_0^{\star_i^7} = I_0^{\star_i}$ for each $I_0 \in \mathrm{F}(D)$, and set $A^{\star_i^7} = K, B^{\star_i^7} = B$ and $(BM)^{\star_i^7} = BM$. Then there is determined a unique mapping $\star_i^7$ from $\bar{\mathrm{F}}(D)$ to $\bar{\mathrm{F}}(D)$.

(8) Set $I_0^{\star_i^8} = I_0^{\star_i}$ for each $I_0 \in \mathrm{F}(D)$, and set $A^{\star_i^8} = K, B^{\star_i^8} = B$ and $(BM)^{\star_i^8} = B$. Then there is determined a unique mapping $\star_i^8$ from $\bar{\mathrm{F}}(D)$ to $\bar{\mathrm{F}}(D)$.

(9) Set $I_0^{\star_i^9} = I_0^{\star_i}$ for each $I_0 \in \mathrm{F}(D)$, and set $A^{\star_i^9} = K$ and $B^{\star_i^9} = K$. Then there is determined a unique mapping $\star_i^9$ from $\bar{\mathrm{F}}(D)$ to $\bar{\mathrm{F}}(D)$.

(4.10) Each $\star_i^j$ satisfies the following conditions: $(xI)^\star = xI^\star$ for each $x \in K \setminus \{0\}$ and $I \in \bar{\mathrm{F}}(D)$; $I \subset I^\star$ for each $I \in \bar{\mathrm{F}}(D)$; and $(I^\star)^\star = I^\star$ for each $I \in \bar{\mathrm{F}}(D)$.

(4.11) Let $x \in K \setminus \{0\}$, and let A_0 (resp. B_0) be a fractional ideal of D with type α (resp. type β).

If $xM \subset AN$, then $(x) \subset AN$.
If $xM \subset BM$, then $(x) \subset B$.
If $xM \subset AB_0$, then $(x) \subset AB_0$.
If $xM \subset A_0B$, then $(x) \subset A_0B$.

If $xN \subset AN$, then $(x) \subset A$.
If $xN \subset BM$, then $(x) \subset BM$.
If $xN \subset AB_0$, then $(x) \subset AB_0$.
If $xN \subset A_0B$, then $(x) \subset A_0B$.
If $xMN \subset AN$, then $(x) \subset A$.
If $xMN \subset BM$, then $(x) \subset B$.
If $xMN \subset AB_0$, then $(x) \subset AB_0$.
If $xMN \subset A_0B$, then $(x) \subset A_0B$.
If $xA_0N \subset AN$, then $xA_0 \subset A$.
If $xA_0N \subset BM$, then $xA_0 \subset BM$.
If $xA_0N \subset AB_0$, then $xA_0 \subset AB_0$.
If $xA_0N \subset A_0B$, then $xA_0 \subset A_0B$.
If $xB_0M \subset AN$, then $xB_0 \subset AN$.
If $xB_0M \subset BM$, then $xB_0 \subset B$.
If $xB_0M \subset AB_0$, then $xB_0 \subset AB_0$.
If $xB_0M \subset A_0B$, then $xB_0 \subset A_0B$.

(4.12) Let $I \in \mathrm{F}(D)$ and $J \in \bar{\mathrm{F}}(D)$ such that $I \subset J$. Then each member in $\{\star_1^1, \star_1^2, \star_2^2, \star_7^2, \star_1^3, \star_3^3, \star_5^3, \star_1^4, \star_2^4, \star_3^4, \star_4^4, \star_5^4, \star_6^4, \star_7^4, \star_8^4, \star_9^4, \star_1^5, \star_2^5, \star_7^5, \star_1^6, \star_2^6, \star_3^6, \star_4^6, \star_5^6, \star_6^6, \star_7^6, \star_8^6, \star_9^6, \star_1^7, \star_3^7, \star_5^7, \star_1^8, \star_2^8, \star_3^8, \star_4^8, \star_5^8, \star_6^8, \star_7^8, \star_8^8, \star_9^8, \star_1^9, \star_2^9, \star_3^9, \star_4^9, \star_5^9, \star_6^9, \star_7^9, \star_8^9, \star_9^9\}$ is monotone for $I \subset J$.

(4.13) (1) Each of $\star_2^1, \star_7^1, \star_2^3, \star_4^3, \star_6^3, \star_7^3, \star_8^3, \star_9^3, \star_2^7, \star_4^7, \star_6^7, \star_7^7, \star_8^7, \star_9^7$ is not monotone for $B_0M \subset BM$.

(2) Each of $\star_3^1, \star_4^1, \star_5^1, \star_6^1, \star_8^1, \star_9^1, \star_3^2, \star_4^2, \star_5^2, \star_6^2, \star_8^2, \star_9^2, \star_3^5, \star_4^5, \star_5^5, \star_6^5, \star_8^5, \star_9^5$ is not monotone for $A_0N \subset AN$.

(4.14) Let A_0 (resp. B_0) be a fractional ideal of D with type α (resp. type β), and let $x \in K \setminus \{0\}$.
If $xAN \subset A$, then $xA \subset A$.
If $xAN \subset AB_0$, then $xA \subset AB_0$.
If $xBM \subset B$, then $xB \subset B$.
If $xBM \subset A_0B$, then $xB \subset A_0B$.

(4.15) Each member in $\{\star_1^1, \star_1^2, \star_2^2, \star_7^2, \star_1^3, \star_3^3, \star_5^3, \star_1^4, \star_2^4, \star_3^4, \star_4^4, \star_5^4, \star_6^4, \star_7^4, \star_8^4, \star_9^4, \star_1^5, \star_2^5, \star_7^5, \star_1^6, \star_2^6, \star_3^6, \star_4^6, \star_5^6, \star_6^6, \star_7^6, \star_8^6, \star_9^6, \star_1^7, \star_3^7, \star_5^7, \star_1^8, \star_2^8, \star_3^8, \star_4^8, \star_5^8, \star_6^8, \star_7^8, \star_8^8, \star_9^8, \star_1^9, \star_2^9, \star_3^9, \star_4^9, \star_5^9, \star_6^9, \star_7^9, \star_8^9, \star_9^9\}$ is a semistar operation on D. We have $\star_1^1 = \mathrm{d}'$ and $\star_9^9 = \mathrm{v}'$.

(4.16) We have $\mid \Sigma'(D) \mid = 54$.

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