Math. J. Ibaraki Univ., 43 (2011), 13–41

Caloric morphisms between different radial metrics on semi-euclidean spaces of same dimension

Katsunori Shimomura*

Dedicated to Professor Yoshihiro Mizuta on the occasion of his sixtieth birthday

Abstract

This paper generalizes and improves the result of [8] to caloric morphisms between manifolds with different radial semi-euclidean metrics. It is based on the similar arguments as were used in [7] and [8] (cf. [4], [5], [6]), but it succeed to remove the technical assumption from the main result of [8].

1. Introduction

In [6], we defined the notion of caloric morphism, the transformation which preserves the solutions of the heat equation, between semi-riemannian manifolds, and obtained a characterization theorem. The Appell transformation is a typical example in euclidean spaces.

Let $n \geqq 2$ and (M, g) be an n-dimensional semi-riemannian manifold. We denote by Δ_g the Laplace-Beltrami operator on (M, g), which is given in a local coordinate $(x_i)_{i=1}^n$ by

$$\Delta_g u = \sum_{i,j=1}^{n} \frac{1}{\sqrt{|\det g|}} \frac{\partial}{\partial x_i}\Big(\sqrt{|\det g|}\, g^{ij} \frac{\partial u}{\partial x_j}\Big),$$

where $\det g = \det(g_{ij})$ and (g^{ij}) denotes the inverse matrix of (g_{ij}).

Definition 1.1. A C^2-function $u(t, x)$ defined on an open set $D \subset \mathbb{R} \times M$ is said to be *caloric* if u satisfies the heat equation

$$\frac{\partial u}{\partial t} - \Delta_g u = 0$$

on D. The operator $H_g := \dfrac{\partial}{\partial t} - \Delta_g$ is called the *heat operator* on $\mathbb{R} \times M$.

Received 22 April 2010; revised 7 October 2010
2000 *Mathematics Subject Classification.* 31B99, 35K99, 35A30
Key Words and Phrases. caloric morphism, riemannian manifold, Appell transformation
* Partially supported by Grant-in-aid for Scientific Research (C) No.17540144, No.19540161, Japan Society for the Promotion of Science.
*Ibaraki University, Mito, Ibaraki 310-8512, Japan. (shimomur@mx.ibaraki.ac.jp)

Definition 1.2. Let M and N be semi-riemannian manifolds, f a C^2-mapping from a domain $D \subset \mathbb{R} \times M$ to $\mathbb{R} \times N$ and φ a strictly positive C^2-function on D. A pair (f, φ) is said to be a *caloric morphism*, if f and φ satisfy the following conditions:

(1) $f(D)$ is a domain in $\mathbb{R} \times N$;

(2) For any caloric function u defined on an open set E in $\mathbb{R} \times N$, the function $\varphi \cdot (u \circ f)$ is caloric on $f^{-1}(E)$.

Let $n \geqq 2$ and $\gamma = (\gamma_{ij})$ be a non-degenerate real symmetric (n, n)-matrix. Assume that γ is not negative definite. Then the set $M_0 = \{x \in \mathbb{R}^n; \gamma(x, x) > 0\}$ is not empty and we consider M_0 as an open set of n-dimensional semi-euclidean space with the inner product

$$\gamma(x, y) = \sum_{i,j} \gamma_{ij} x_i y_j.$$

We write $\langle x \rangle = \sqrt{\gamma(x, x)}$ for $x \in M_0$.

Let ρ be a strictly positive C^∞-function defined on an open interval $J_\rho \subset \mathbb{R}_+ := (0, \infty)$ and let $M = M_0 \cap \{x; \langle x \rangle \in J_\rho\}$. We consider the semi-riemannian manifold (M, g) with the metric of form

$$g(x) = \rho(\langle x \rangle)\gamma.$$

We call the metric of this type *radial metric*.

In our previous paper [8], we considered caloric morphisms with respect to a radial metric such that f has one of the following forms:

$$f(t, x) = (f_0(t), \nu(t)R(t)x) \quad \text{or} \quad f(t, x) = (f_0(t), \langle x \rangle^{-2}\nu(t)R(t)x),$$

where $\nu(t)$ is a strictly positive C^∞-function and $R(t)$ is an $O_\gamma(n)$-valued C^∞-function, where $O_\gamma(n) := \{R; R\gamma^{-1}{}^tR = \gamma^{-1}\}$. In [8], we determined all the caloric morphisms under the assumption that $f(D) \cap D \neq \emptyset$.

The aim of this paper is to generalize the results in [8] to caloric morphisms between two different radial metrics on semi-riemannian spaces of same dimension. It is remarkable that this generalization makes it possible to remove the assumption $f(D) \cap D \neq \emptyset$ from the main result of [8].

Let $\gamma = (\gamma_{ij})$ and $\eta = (\eta_{ij})$ be two non-degenerate real symmetric (n, n)-matrices $(n \geqq 2)$, and consider two n-dimensional semi-euclidean spaces with the inner products $\gamma(x, y) = \sum_{i,j} \gamma_{ij} x_i y_j$ and $\eta(x, y) = \sum_{i,j} \eta_{ij} x_i y_j$. Assume that neither γ nor η is negative definite. Then the sets $M_0 = \{x \in \mathbb{R}^n; \gamma(x, x) > 0\}$ and $N_0 = \{y \in \mathbb{R}^n; \eta(y, y) > 0\}$ are not empty. For $x \in M_0$ and $y \in N_0$, we can put

$$\langle x \rangle_\gamma = \sqrt{\gamma(x, x)} \quad \text{and} \quad \langle y \rangle_\eta = \sqrt{\eta(y, y)},$$

respectively. We define the set $O_{\gamma,\eta}(n)$ as

$$O_{\gamma,\eta}(n) = \{R \in GL(n, \mathbb{R}); R\gamma^{-1}{}^tR = \eta^{-1}\}.$$

Let ρ and σ are strictly positive C^∞-functions defined on open intervals $J_\rho, J_\sigma \subset \mathbb{R}_+$, respectively and let $M := \{x \in M_0; \langle x \rangle_\gamma \in J_\rho\}$ and $N := \{y \in N_0; \langle y \rangle_\eta \in J_\sigma\}$. We consider two semi-riemannian manifolds (M, g) and (N, h) with metrics of forms

$$g = \rho(\langle x \rangle_\gamma)\gamma \quad \text{and} \quad h = \sigma(\langle y \rangle_\eta)\eta,$$

respectively.

Let (f, φ) be a caloric morphism from a domain $D \subset \mathbb{R} \times M$ to $\mathbb{R} \times N$ such that $f(t, x)$ has one of the following forms:

$$f(t, x) = (f_0(t), A(t)x) \tag{a}$$

and

$$f(t, x) = (f_0(t), \langle x \rangle_\gamma^{-2} A(t)x), \tag{b}$$

where $A(t) \in GL(n, \mathbb{R})$ is a C^∞-function defined on the open interval $I_0 = \{t \in \mathbb{R}; (\{t\} \times \mathbb{R}^n) \cap D \neq \emptyset\}$.

Our main result is the following

Theorem 1.1. *Let $M = \{x \in M_0; \langle x \rangle_\gamma \in J_\rho\}$ and $N = \{y \in N_0; \langle y \rangle_\eta \in J_\sigma\}$ are semi-riemannian manifolds with metrics $g = \rho(\langle x \rangle_\gamma)\gamma$ and $h = \sigma(\langle y \rangle_\eta)\eta$, respectively. If (f, φ) be a caloric morphism from a domain $D \subset \mathbb{R} \times M$ to $\mathbb{R} \times N$ such that the mapping f has the form* (a) *or* (b) *in the above, then one of the following cases occurs:*

Case 1-a. $n = 2$, $\rho(r) = p_1 r^{-2}$, $\sigma(r) = p_2 r^{-2}$,

$$f(t, x) = (\frac{p_2}{p_1} t + d, c e^{at} R_0 e^{t\gamma^{-1}\left(\begin{smallmatrix} 0 & -b \\ b & 0 \end{smallmatrix}\right)} x),$$

$$\varphi(t, r, \theta) = C r^{\frac{1}{2} a p_1} \exp \frac{p_1}{2} \Big(\frac{b}{\sqrt{|\det \gamma|}} \theta + \frac{1}{2}(a^2 + \frac{b^2}{\det \gamma}) t \Big).$$

Case 1-b. $n = 2$, $\rho(r) = p_1 r^{-2}$, $\sigma(r) = p_2 r^{-2}$,

$$f(t, x) = (\frac{p_2}{p_1} t + d, c e^{at} \langle x \rangle_\gamma^{-2} R_0 e^{t\gamma^{-1}\left(\begin{smallmatrix} 0 & -b \\ b & 0 \end{smallmatrix}\right)} x),$$

$$\varphi(t, r, \theta) = C r^{-\frac{1}{2} a p_1} \exp \frac{p_1}{2} \Big(\frac{b}{\sqrt{|\det \gamma|}} \theta + \frac{1}{2}(a^2 + \frac{b^2}{\det \gamma}) t \Big).$$

In the cases 1-a and 1-b, $a, b, d \in \mathbb{R}$, $c, C, p_1, p_2 \in \mathbb{R}_+$, $R_0 \in O_{\gamma,\eta}(2)$, and (r, θ) is the polar coordinate of $\mathbb{R}^2$ with respect to γ (see §4 below).

Case 2-a. $n \geqq 2$, $\rho(r) = p_1 r^{-2}$, $\sigma(r) = p_2 r^{-2}$,

$$f(t, x) = (\frac{p_2}{p_1} t + d, c e^{at} R_0 x), \quad \varphi(t, x) = C \langle x \rangle_\gamma^{\frac{1}{2} a p_1} \exp \Big(\frac{p_1}{4} a^2 t \Big).$$

Case 2-b. $n \geqq 2$, $\rho(r) = p_1 r^{-2}$, $\sigma(r) = p_2 r^{-2}$,

$$f(t, x) = (\frac{p_2}{p_1} t + d, c e^{at} \langle x \rangle_\gamma^{-2} R_0 x), \quad \varphi(t, x) = C \langle x \rangle_\gamma^{-\frac{1}{2} a p_1} \exp \Big(\frac{p_1}{4} a^2 t \Big).$$

In the cases 2-a and 2-b, $a, d \in \mathbb{R}$, $c, C, p_1, p_2 \in \mathbb{R}_+$ and $R_0 \in O_{\gamma,\eta}(n)$.

Case 3-a. $n \geqq 2$, $\rho(r) = p_1 r^q$, $\sigma(r) = p_2 r^q$,

$$f(t, x) = (\frac{p_2}{p_1} \frac{ct + d}{at + b}, |at + b|^{-2/(q+2)} R_0 x),$$

$$\varphi(t, x) = \frac{C}{|at + b|^{n/2}} \exp \Big[- \frac{p_1 a \langle x \rangle_\gamma^{q+2}}{(q + 2)^2 (at + b)} \Big].$$

Case 3-b. $n \geqq 2$, $\rho(r) = p_1 r^q$, $\sigma(r) = p_2 r^{-q-4}$,

$$f(t,x) = (\frac{p_2}{p_1}\frac{ct+d}{at+b}, |at+b|^{2/(q+2)}\langle x\rangle_\gamma^{-2} R_0 x),$$

$$\varphi(t,x) = \frac{C}{|at+b|^{n/2}} \exp\Big[-\frac{p_1 a \langle x\rangle_\gamma^{q+2}}{(q+2)^2(at+b)}\Big].$$

In the cases 3-a and 3-b, $a,b,c,d,q \in \mathbb{R}$ $(bc-ad=1,\ q \neq -2)$, $C, p_1, p_2 \in \mathbb{R}_+$ and $R_0 \in O_{\gamma,\eta}(n)$.

Case 4-a. $n \geqq 2$, $\sigma(\nu r) = \dfrac{\lambda}{\nu^2}\rho(r)$ *holds for all r with some positive constants ν and λ,*

$$f(t,x) = (\lambda t + d, \nu R_0 x), \quad \varphi(t,x) = C,$$

where $C \in \mathbb{R}_+$, $d \in \mathbb{R}$ and $R_0 \in O_{\gamma,\eta}(n)$.

Case 4-b. $n \geqq 2$, $\sigma(\dfrac{\nu}{r}) = \dfrac{\lambda r^4}{\nu^2}\rho(r)$ *holds for all r with some positive constants ν and λ,*

$$f(t,x) = (\lambda t + d, \nu\langle x\rangle_\gamma^{-2} R_0 x), \quad \varphi(t,x) = C,$$

where $C \in \mathbb{R}_+$, $d \in \mathbb{R}$ and $R_0 \in O_{\gamma,\eta}(n)$.

Case 5. $n \geqq 2$, *ρ and σ are any strictly positive C^∞-functions,*

$$f(t,x) = (t+d, R_0 x), \quad \varphi(t,x) = C,$$

where $C \in \mathbb{R}_+$, $d \in \mathbb{R}$ and $R_0 \in O_{\gamma,\eta}(n)$.

Remark 1. In [8], we treated the case of $M = N$ and proved the same result with the assumption $D \cap f(D) \neq \emptyset$.

2. Preliminaries

In [6], we proved the following characterization theorem.

Theorem A (Characterization). *Let (M,g) and (N,h) be two n-dimensional semi-riemannian manifolds, f a C^2-mapping from a domain $D \subset \mathbb{R}\times M$ to $\mathbb{R}\times N$ such that $f(D)$ is a domain, and φ a strictly positive C^2-function on D. Then the following three statements are equivalent:*

(1) (f,φ) is a caloric morphism;

(2) Take a local coordinate $(y_1,\cdots,y_n)$ of N and write the mapping f as $f = (f_0, f_1, \cdots, f_n)$ by the local coordinate. Then f_0 depends only on t and the functions $f_0, f_1, \ldots, f_n$ and φ satisfy the following equations (E-1)– (E-4)*:*

$$H_g\,\varphi = 0, \tag{E-1}$$

$$H_g f_\alpha = 2\,g(\nabla_g \log\varphi, \nabla_g f_\alpha) + \sum_{\beta,\gamma=1}^{n} g(\nabla_g f_\beta, \nabla_g f_\gamma)\cdot {}^h\Gamma^\alpha_{\beta\,\gamma}\circ f \quad (1 \leqq \alpha \leqq n), \tag{E-2}$$

$$\nabla_g f_0 = 0, \tag{E-3}$$

$$g(\nabla_g f_\alpha, \nabla_g f_\beta) = (h^{\alpha\,\beta}\circ f)\cdot f_0'(t) \quad (1 \leqq \alpha,\beta \leqq n), \tag{E-4}$$

where ∇_g *denotes the gradient operator of* (M,g) *and* ${}^h\Gamma^{\alpha}_{\beta\gamma}$ *denotes the Christoffel symbol of* (N,h)*;*

(3) There exists a continuous function λ *on* D*, depending only on* t*, such that*

$$H_g(\varphi \cdot u \circ f)(t,x) = \lambda(t) \cdot \varphi(t,x) \cdot H_h u \circ f(t,x)$$

for any C^2*-function* u *defined on a subdomain of* $f(D)$*.*

Proposition 2.1. *Let* (M,g) *and* (N,h) *be* n*-dimensional semi-riemannian manifolds. If* (f,φ) *is a caloric morphism from a domain* $D \subset \mathbb{R}\times M$ *to* $\mathbb{R}\times N$*, then* $f_0'(t) \neq 0$ *holds for all* $t \in I_0 = \{t \in \mathbb{R}; (\{t\}\times\mathbb{R}^n)\cap D \neq \emptyset\}$*.*

Proof. Assume that there exists $a \in I_0$ satisfying $f_0'(a) = 0$. Then by (E-4):

$$g(\nabla_g f_\alpha(a,x), \nabla_g f_\beta(a,x)) = 0 \quad (1 \leqq \alpha, \beta \leqq n),$$

we have

$$\nabla_g f_1(a,x) = \cdots = \nabla_g f_n(a,x) = 0$$

for all $(a,x) \in D$, and hence the mapping $x \mapsto (f_0(a), f_1(a,x), \dots, f_n(a,x))$ is (at least locally) constant. Thus the set $(\{f_0(a)\}\times M)\cap D$ is not open, which contradicts the condition (1) in the definition of caloric morphism. Therefore $f_0'(t) \neq 0$ for all $t \in I_0$. □

The composition of two caloric morphisms is also a caloric morphism. Let M, N and L be semi-riemannian manifolds. Let D, E be domains in $\mathbb{R}\times M$, $\mathbb{R}\times N$, respectively. If (f,φ) is a caloric morphism from D to $\mathbb{R}\times N$ and (h,ψ) is a caloric morphism from E to $\mathbb{R}\times L$ such that $f(D)\subset E$, then $(F,\Phi) := (h\circ f, \varphi\cdot(\psi\circ f))$ is a caloric morphism from D to $\mathbb{R}\times L$.

From here, we return to the case of semi-riemannian manifolds with radial metrics. Hereafter, we use the following notations: for an (n,n)-matrix $A = (A_{ij})$,

$$A(x,y) = \sum_{i,j=1}^{n} A_{ij}x_i y_j, \quad (Ax)_i = \sum_{j=1}^{n} A_{ij}x_j, \ (i = 1,\dots,n).$$

In this notation, we have

$$\frac{\partial\langle x\rangle_\gamma}{\partial x_j} = \frac{1}{2\sqrt{\gamma(x,x)}}\frac{\partial\gamma(x,x)}{\partial x_j} = \frac{(\gamma x)_j}{\langle x\rangle_\gamma}, \quad \frac{\partial\rho(\langle x\rangle_\gamma)}{\partial x_j} = \rho'(\langle x\rangle_\gamma)\frac{(\gamma x)_j}{\langle x\rangle_\gamma}.$$

We also have

$$\det g = \rho(\langle x\rangle_\gamma)^n \det\gamma, \quad \sqrt{|\det g|} = \rho(\langle x\rangle_\gamma)^{n/2}\sqrt{|\det\gamma|} \quad \text{and} \quad g^{ij} = \frac{1}{\rho(\langle x\rangle_\gamma)}\gamma^{ij},$$

where (γ^{ij}) denotes the inverse matrix of (γ_{ij}). We can choose the usual cartesian coordinate system as a local coordinate of M. Then the Laplacian of a function u is given by

$$\Delta_g u = \frac{1}{\rho(\langle x\rangle_\gamma)}\sum_{i,j=1}^{n}\gamma^{ij}\frac{\partial^2 u}{\partial x_i\partial x_j} + \frac{n-2}{2}\frac{\rho'(\langle x\rangle_\gamma)}{\rho(\langle x\rangle_\gamma)^2}\sum_{j=1}^{n}\frac{x_j}{\langle x\rangle_\gamma}\frac{\partial u}{\partial x_j}. \tag{2.1}$$

The gradient of a function u is given by

$$\nabla_g u = \frac{1}{\rho(\langle x\rangle_\gamma)} \sum_{i,j=1}^{n} \gamma^{ij} \frac{\partial u}{\partial x_i} \frac{\partial}{\partial x_j},$$

and hence the inner product of the gradients of two functions u and v is given by

$$g(\nabla_g u, \nabla_g v) = \frac{1}{\rho(\langle x\rangle_\gamma)} \sum_{i,j=1}^{n} \gamma^{ij} \frac{\partial u}{\partial x_i} \frac{\partial v}{\partial x_j}. \tag{2.2}$$

Let $D \subset M$ be a domain, $f : D \to N$ a C^∞-mapping and (f, φ) a caloric morphism. Then f is expressed as

$$f(t,x) = (f_0(t), f_1(t,x), \dots, f_n(t,x)).$$

Because of equation (E-4): $g(\nabla_g f_j, \nabla_g f_k) = f_0'(t)(h^{jk} \circ f)$, $(\alpha, \beta = 1, \dots, n)$, the second term of the right hand side of (E-2) equals to $\sum_{j,k=1}^{n} f_0'(t)(h^{jk} \cdot {}^h\Gamma^i_{jk}) \circ f$. On the other hand,

$$\begin{aligned}
\sum_{j,k=1}^{n} (h^{jk} \cdot {}^h\Gamma^i_{jk})(y) &= \sum_{j,k=1}^{n} h^{jk}(y) \sum_{l=1}^{n} \frac{1}{2} h^{il}(y) \Big(\frac{\partial h_{kl}}{\partial y_j}(y) + \frac{\partial h_{jl}}{\partial y_k}(y) - \frac{\partial h_{jk}}{\partial y_l}(y) \Big) \\
&= \sum_{j,k,l=1}^{n} \frac{\eta^{jk}\eta^{il}}{2\sigma(\langle y\rangle_\eta)^2} \Big(\eta_{kl} \frac{\partial \sigma(\langle y\rangle_\eta)}{\partial y_j} + \eta_{jl} \frac{\partial \sigma(\langle y\rangle_\eta)}{\partial y_k} - \eta_{jk} \frac{\partial \sigma(\langle y\rangle_\eta)}{\partial y_l} \Big) \\
&= \frac{1}{2\sigma(\langle y\rangle_\eta)^2} \sigma'(\langle y\rangle_\eta) \Big(\sum_{j=1}^{n} \eta^{ij} \frac{(\eta y)_j}{\langle y\rangle_\eta} + \sum_{k=1}^{n} \eta^{ik} \frac{(\eta y)_k}{\langle y\rangle_\eta} - \sum_{l=1}^{n} n\eta^{il} \frac{(\eta y)_l}{\langle y\rangle_\eta} \Big) \\
&= \frac{\sigma'(\langle y\rangle_\eta)}{2\sigma(\langle y\rangle_\eta)^2} \frac{y_i + y_i - n y_i}{\langle y\rangle_\eta} = \frac{\sigma'(\langle y\rangle_\eta)}{2\sigma(\langle y\rangle_\eta)^2} \frac{(2-n)y_i}{\langle y\rangle_\eta}.
\end{aligned}$$

Thus we have

$$\sum_{j,k=1}^{n} g(\nabla_g f_j, \nabla_g f_k) \cdot {}^h\Gamma^i_{jk} \circ f = f_0' \frac{2-n}{2} \frac{\sigma'(\langle f\rangle_\eta)}{\sigma(\langle f\rangle_\eta)^2} \frac{f_i}{\langle f\rangle_\eta} \quad (1 \leqq i \leqq n). \tag{2.3}$$

Now let (f, φ) be a caloric morphism such that f is of form (a) or (b). Recall that

$$O_{\gamma,\eta}(n) = \{R \in GL(n,\mathbb{R}); R\gamma^{-1}{}^tR = \eta^{-1}\}.$$

The equation $R\gamma^{-1}{}^tR = \eta^{-1}$ is equivalent to ${}^tR\eta R = \gamma$. Therefore, $R \in O_{\gamma,\eta}(n)$ if and only if

$$\langle Rx\rangle_\eta = \langle x\rangle_\gamma$$

holds for all $x \in \mathbb{R}^n$.

Proposition 2.2. *Let $(M, \rho(\langle x\rangle_\gamma)\gamma)$ and $(N, \sigma(\langle y\rangle_\eta)\eta)$ be the same as in Theorem* 1.1.

(1) Assume that there exists a caloric morphism (f, φ) such that the mapping f has the form (a)*:*

$$f(t,x) = (f_0(t), A(t)x)$$

defined on a domain $D \subset \mathbb{R} \times M$. Then $f'(t) > 0$ holds for each $t \in I_0$ and there exist a strictly positive C^∞-function $\nu(t)$ defined on I_0 and an $O_{\gamma,\eta}(n)$-valued C^∞-function $R(t)$ on I_0 such that $A(t) = \nu(t)R(t)$ holds for each $t \in I_0$. Moreover, the functions ρ, σ, f_0 and ν satisfy the equation

$$\sigma(\nu(t)r) = \frac{f_0'(t)}{\nu(t)^2}\rho(r) \tag{2.4}$$

for all $(t,r) \in E_0 := \{(t, \langle x\rangle_\gamma) \in \mathbb{R} \times \mathbb{R}_+; (t,x) \in D\}$.

(2) Assume that there exists a caloric morphism (f, φ) such that the mapping f has the form (b)*:*

$$f(t,x) = (f_0(t), \langle x\rangle_\gamma^{-2} A(t)x)$$

defined on a domain $D \subset \mathbb{R} \times M$. Then $f'(t) > 0$ holds for each $t \in I_0$ and there exist a strictly positive C^∞-function $\nu(t)$ defined on I_0 and an $O_{\gamma,\eta}(n)$-valued C^∞-function $R(t)$ on I_0 such that $A(t) = \nu(t)R(t)$ holds for each $t \in I_0$. Moreover, the functions ρ, σ, f_0 and ν satisfy

$$\sigma(\frac{\nu(t)}{r}) = \frac{f_0'(t)r^4}{\nu(t)^2}\rho(r) \tag{2.5}$$

for all $(t,r) \in E_0 := \{(t, \langle x\rangle_\gamma) \in \mathbb{R} \times \mathbb{R}_+; (t,x) \in D\}$.

Proof. (1) The equations (E-4):

$$g(\nabla_g f_\alpha, \nabla_g f_\beta) = f_0'(t)(h^{\alpha\beta} \circ f), \quad (1 \leqq \alpha, \beta \leqq n)$$

yield the matrix equation:

$$A(t)\gamma^{-1}{}^tA(t) = f_0'(t)\frac{\rho(\langle x\rangle_\gamma)}{\sigma(\langle A(t)x\rangle_\eta)}\eta^{-1}, \quad (t,x) \in D, \tag{2.6}$$

which is equivalent to

$${}^tA(t)\eta A(t) = f_0'(t)\frac{\rho(\langle x\rangle_\gamma)}{\sigma(\langle A(t)x\rangle_\eta)}\gamma, \quad (t,x) \in D.$$

Then we have

$$f_0'(t) = \frac{\sigma(\langle A(t)x\rangle_\eta)\eta(A(t)x, A(t)x)}{\rho(\langle x\rangle_\gamma)\gamma(x,x)} > 0 \quad (t,x) \in D,$$

because $\gamma(x,x) > 0$ and $\eta(A(t)x, A(t)x) > 0$ follow from the conditions $(t,x) \in D \subset \mathbb{R} \times M_0$ and $f(t,x) = (f_0(t), A(t)x) \in \mathbb{R} \times N_0$.

Since the left hand side of (2.6) is independent of x, we can define a real variable strictly positive function $\nu(t)$ by

$$\nu(t) = \Big(f_0'(t)\frac{\rho(\langle x\rangle_\gamma)}{\sigma(\langle A(t)x\rangle_\eta)}\Big)^{1/2}, \quad t \in I_0. \tag{2.7}$$

Then ν is a strictly positive C^∞-function on I_0 which satisfies

$$A(t)\gamma^{-1\,t}A(t) = \nu(t)^2\eta^{-1}, \quad t \in I_0. \tag{2.8}$$

Hence the matrix $R(t) := \nu(t)^{-1}A(t)$ belongs to $O_{\gamma,\eta}(n) = \{R \in GL(n,\mathbb{R}); R\gamma^{-1\,t}R = \eta^{-1}\}$ for all $t \in I_0$ and satisfies

$$\langle R(t)x\rangle_\eta = \langle x\rangle_\gamma, \quad (t,x) \in I_0 \times \mathbb{R}^n.$$

Thus the equality

$$\langle A(t)x\rangle_\eta = \nu(t)\langle x\rangle_\gamma, \quad (t,x) \in I_0 \times \mathbb{R}^n \tag{2.9}$$

holds. Substituting (2.7), (2.8) and (2.9) into (2.6), we have

$$\frac{1}{\rho(\langle x\rangle_\gamma)}\nu(t)^2\eta^{-1} = f_0'(t)\frac{1}{\sigma(\nu(t)\langle x\rangle_\gamma)}\eta^{-1},$$

and hence

$$\sigma(\nu(t)\langle x\rangle_\gamma) = \frac{f_0'(t)}{\nu(t)^2}\rho(\langle x\rangle_\gamma), \quad (t,x) \in D.$$

Putting $r = \langle x\rangle_\gamma$, we have (2.4).

Next we consider the caloric morphism (f,φ) such that f has the form

$$f(t,x) = (f_0(t), \langle x\rangle_\gamma^{-2}A(t)x),$$

where $A(t) \in GL(n,\mathbb{R})$. The equations (E-4) yield

$$\frac{1}{\rho(\langle x\rangle_\gamma)}\sum_{i,j=1}^{n}\gamma^{ij}\frac{\partial f_\alpha}{\partial x_i}\frac{\partial f_\beta}{\partial x_j} = f_0'(t)\frac{1}{\sigma(\langle x\rangle_\gamma^{-2}\langle A(t)x\rangle_\eta)}\eta^{\alpha\beta} \quad (1 \leqq \alpha,\beta \leqq n). \tag{2.10}$$

Since

$$\frac{\partial f_\alpha}{\partial x_i} = \frac{A_{\alpha i}(t)}{\langle x\rangle_\gamma^2} - 2\frac{(\gamma x)_i}{\langle x\rangle_\gamma^4}(A(t)x)_\alpha = \frac{1}{\langle x\rangle_\gamma^2}\Big(A_{\alpha i}(t) - 2\frac{(\gamma x)_i}{\langle x\rangle_\gamma^2}(A(t)x)_\alpha\Big),$$

the left hand side of the equation (2.10) is equal to

$$\begin{aligned}
&\sum_{i,j=1}^{n} \frac{\gamma^{ij}}{\langle x\rangle_\gamma^4 \rho(\langle x\rangle_\gamma)} \Big(A_{\alpha i}(t) - 2\frac{(\gamma x)_i}{\langle x\rangle_\gamma^2}(A(t)x)_\alpha\Big)\Big(A_{\beta j}(t) - 2\frac{(\gamma x)_j}{\langle x\rangle_\gamma^2}(A(t)x)_\beta\Big) \\
&= \frac{1}{\langle x\rangle_\gamma^4 \rho(\langle x\rangle_\gamma)} \sum_{i,j=1}^{n} \Big(\gamma^{ij} A_{\alpha i}(t) A_{\beta j}(t) - 2\frac{A_{\alpha i}(t)\gamma^{ij}(\gamma x)_j}{\langle x\rangle_\gamma^2}(A(t)x)_\beta \\
&\quad - 2\frac{A_{\beta j}(t)\gamma^{ij}(\gamma x)_i}{\langle x\rangle_\gamma^2}(A(t)x)_\alpha + 4\frac{\gamma^{ij}(\gamma x)_i(\gamma x)_j}{\langle x\rangle_\gamma^4}(A(t)x)_\alpha (A(t)x)_\beta\Big) \\
&= \frac{1}{\langle x\rangle_\gamma^4 \rho(\langle x\rangle_\gamma)} \Big(({}^tA(t)\gamma^{-1}A(t))_{\alpha\beta} - 2\frac{(A(t)x)_\alpha}{\langle x\rangle_\gamma^2}(A(t)x)_\beta \\
&\quad - 2\frac{(A(t)x)_\beta}{\langle x\rangle_\gamma^2}(A(t)x)_\alpha + 4\frac{\gamma(x,x)}{\langle x\rangle_\gamma^4}(A(t)x)_\alpha (A(t)x)_\beta\Big) \\
&= \frac{({}^tA(t)\gamma^{-1}A(t))_{\alpha\beta}}{\langle x\rangle_\gamma^4 \rho(\langle x\rangle_\gamma)}, \quad (1 \leqq \alpha, \beta \leqq n).
\end{aligned}$$

Therefore we have the following matrix equation

$$A(t)\gamma^{-1\,t}A(t) = f_0'(t)\frac{\langle x\rangle_\gamma^4 \rho(\langle x\rangle_\gamma)}{\sigma(\langle x\rangle_\gamma^{-2}\langle A(t)x\rangle_\eta)}\eta^{-1} \quad (t,x) \in D, \tag{2.11}$$

which is equivalent to

$${}^t\big(\langle x\rangle_\gamma^{-2}A(t)\big)\eta\big(\langle x\rangle_\gamma^{-2}A(t)\big) = f_0'(t)\frac{\rho(\langle x\rangle_\gamma)}{\sigma(\langle x\rangle_\gamma^{-2}\langle A(t)x\rangle_\eta)}\gamma, \quad (t,x) \in D.$$

Then we have

$$f_0'(t) = \frac{\sigma(\langle x\rangle_\gamma^{-2}\langle A(t)x\rangle_\eta)\eta(\langle x\rangle_\gamma^{-2}A(t)x, \langle x\rangle_\gamma^{-2}A(t)x)}{\rho(\langle x\rangle_\gamma)\gamma(x,x)} > 0 \quad (t,x) \in D,$$

because $\gamma(x,x) > 0$ and $\eta(\langle x\rangle_\gamma^{-2}A(t)x, \langle x\rangle_\gamma^{-2}A(t)x) > 0$ follow from the conditions $(t,x) \in D \subset \mathbb{R} \times M_0$ and $f(t,x) = (f_0(t), \langle x\rangle_\gamma^{-2}A(t)x) \in \mathbb{R} \times N_0$.

Since the left hand side is independent of x, we can define the function $\nu(t)$ by

$$\nu(t) = \Big(f_0'(t)\frac{\langle x\rangle_\gamma^4 \rho(\langle x\rangle_\gamma)}{\sigma(\langle x\rangle_\gamma^{-2}\langle A(t)x\rangle_\eta)}\Big)^{1/2}, \quad t \in I_0. \tag{2.12}$$

Then ν is a strictly positive C^∞-function on I_0 and satisfies

$$A(t)\gamma^{-1\,t}A(t) = \nu(t)^2\eta^{-1}. \tag{2.13}$$

Put $R(t) = \nu(t)^{-1}A(t)$. Then $R(t) \in O_{\gamma,\eta}(n)$ for all $t \in I_0$ and the equations

$$\langle R(t)x\rangle_\eta = \langle x\rangle_\gamma, \quad \langle A(t)x\rangle_\eta = \nu(t)\langle x\rangle_\gamma, \quad (t,x) \in I_0 \times \mathbb{R}^n \tag{2.14}$$

hold as before. Substituting (2.13) and (2.14) into (2.11), we have

$$\frac{1}{\langle x\rangle_\gamma^4\rho(\langle x\rangle_\gamma)}\nu(t)^2\eta^{-1}=f_0'(t)\frac{1}{\sigma(\langle x\rangle_\gamma^{-2}\nu(t)\langle x\rangle_\gamma)}\eta^{-1}, \tag{2.15}$$

and hence

$$\sigma(\frac{\nu(t)}{\langle x\rangle_\gamma})=\frac{f_0'(t)\langle x\rangle_\gamma^4}{\nu(t)^2}\rho(\langle x\rangle_\gamma)\quad (t,x)\in D. \tag{2.16}$$

Putting $r=\langle x\rangle_\gamma$, we have (2.5). □

If (f,φ) be a caloric morphism such that f is of form (a):

$$f(t,x)=(f_0(t),A(t)x).$$

Then f is expressed as

$$f(t,x)=(f_0(t),f_1(t,x),\dots,f_n(t,x)),$$
$$f_\alpha(t,x)=\sum_{j=1}^n\nu(t)R_{\alpha j}(t)x_j,\quad \alpha=1,2,\dots,n.$$

Their derivatives are given by

$$\begin{aligned}\frac{\partial f_\alpha}{\partial t}&=\sum_{j=1}^n(\nu'(t)R_{\alpha j}(t)+\nu(t)R_{\alpha j}'(t))x_j,\\ \frac{\partial f_\alpha}{\partial x_j}&=\nu(t)R_{\alpha j}(t)\end{aligned} \tag{2.17}$$

for $\alpha,j=1,2,\dots,n$.

Lemma 2.1. *Let ρ and σ be two strictly positive C^1-functions defined on the intervals J_ρ and J_σ in $\mathbb{R}_+$, respectively. Let μ and ν be two strictly positive C^1-functions defined on an interval I. Let E be a domain in $J_\rho\times\mathbb{R}_+$.*

(1) Assume that ρ, σ, μ, ν satisfy the equation

$$\sigma(\nu(t)r)=\mu(t)\rho(r),\quad (t,r)\in E. \tag{2.18}$$

If $\nu'(t)\neq 0$ on an interval I', then there exist constants $p_1,p_2\in\mathbb{R}_+$ and $q\in\mathbb{R}$ such that

$$\rho(r)=p_1r^q\quad (r\in J_\rho'),\qquad \sigma(s)=p_2s^q\quad (s\in J_\sigma'),$$
$$\mu(t)=\frac{p_2}{p_1}\nu(t)^q\quad (t\in I'),$$

where $J_\rho':=\{r;(t,r)\in E,\,t\in I'\}$ and $J_\sigma':=\{\nu(t)r;(t,r)\in E,\,t\in I'\}$.

(2) Assume that ρ, σ, μ and ν satisfy the equation

$$\sigma(\frac{\nu(t)}{r}) = \mu(t)r^4\rho(r), \quad (t,r) \in E. \tag{2.19}$$

If $\nu'(t) \neq 0$ on an interval I', then there exist constants $p_1, p_2 > 0$ and $q \in \mathbb{R}$ such that

$$\rho(r) = p_1 r^q \quad (r \in J'_\rho), \qquad \sigma(s) = p_2 s^{-q-4} \quad (s \in J'_\sigma),$$
$$\mu(t) = \frac{p_2}{p_1}\nu(t)^{-q-4} \quad (t \in I'),$$

where $J'_\rho := \{r; (t,r) \in E,\, t \in I'\}$ and $J'_\sigma := \{\frac{\nu(t)}{r}; (t,r) \in E,\, t \in I'\}$.

Proof. First we show (1). Differentiating (2.18) by r and by t, we have the equations

$$\sigma'(\nu(t)r)\nu(t) = \mu(t)\rho'(r), \quad \sigma'(\nu(t)r)\nu'(t)r = \mu'(t)\rho(r), \quad (t,r) \in E.$$

Since $\nu'(t) \neq 0$ on I', these equations yield

$$\frac{\mu'(t)\rho(r)}{\nu'(t)r}\nu(t) = \mu(t)\rho'(r), \quad (t,r) \in E_1,$$

where $E_1 = \{(t,x) \in E; t \in I'\}$, and hence

$$\frac{\mu'(t)\nu(t)}{\mu(t)\nu'(t)} = \frac{r\rho'(r)}{\rho(r)}, \quad (t,r) \in E_1. \tag{2.20}$$

Therefore, the both sides of the equation (2.20) are equal to a constant q, so that

$$\frac{r\rho'(r)}{\rho(r)} = q, \qquad r \in J'_\rho,$$
$$\frac{\mu'(t)}{\mu(t)} = q\frac{\nu'(t)}{\nu(t)}, \quad t \in I',$$

where $J'_\rho = \{r; (t,r) \in E_1\}$. The solutions of these equations are

$$\begin{aligned} \rho(r) &= p_1 r^q, && r \in J'_\rho, \\ \mu(t) &= c\nu(t)^q, && t \in I' \end{aligned} \tag{2.21}$$

with some positive constants p_1 and c. Substituting (2.21) into (2.18), we have

$$\sigma(\nu(t)r) = cp_1\nu(t)^q r^q,$$

and hence

$$\sigma(s) = cp_1 s^q, \quad s \in J'_\sigma,$$

where $J'_\sigma = \{\nu(t)r; (t,r) \in E_1\}$. We have the statement (1) by putting $p_2 = cp_1$.

Next we prove the statement (2). Differentiating (2.19) by r and by t, we have the equations

$$-\sigma'(\frac{\nu(t)}{r})\frac{\nu(t)}{r^2} = \mu(t)(r^4\rho'(r) + 4r^3\rho(r)), \quad \sigma'(\frac{\nu(t)}{r})\frac{\nu'(t)}{r} = \mu'(t)r^4\rho(r), \quad (t,r) \in E.$$

Since $\nu'(t) \neq 0$ on I', these equations yield

$$\mu(t)(r^4\rho'(r) + 4r^3\rho(r)) = -\mu'(t)r^4\rho(r)\frac{\nu(t)}{\nu'(t)r}, \quad (t,r) \in E_1,$$

where $E_1 = \{(t,x) \in E; t \in I'\}$, and hence

$$\frac{r\rho'(r)}{\rho(r)} = -4 - \frac{\mu'(t)\nu(t)}{\mu(t)\nu'(t)}, \quad (t,r) \in E_1. \tag{2.22}$$

Therefore, both sides of the equation (2.22) are equal to a constant q, so that

$$\begin{aligned} \frac{r\rho'(r)}{\rho(r)} &= q, & r \in J'_\rho, \\ \frac{\mu'(t)}{\mu(t)} &= -(q+4)\frac{\nu'(t)}{\nu(t)}, & t \in I', \end{aligned}$$

where $J'_\rho = \{r; (t,r) \in E_1\}$. The solutions of these equations are

$$\begin{aligned} \rho(r) &= p_1 r^q, & r \in J'_\rho, \\ \mu(t) &= c\nu(t)^{-q-4}, & t \in I' \end{aligned} \tag{2.23}$$

with some positive constants p_1 and c. Substituting (2.23) into (2.19), we have

$$\sigma(\frac{\nu(t)}{r}) = cp_1\Big(\frac{\nu(t)}{r}\Big)^{-q-4},$$

and hence

$$\sigma(s) = cp_1 s^{-q-4}, \quad s \in J'_\sigma,$$

where $J'_\sigma = \{\frac{\nu(t)}{r}; (t,r) \in E_1\}$. We have the statement (2) by putting $p_2 = cp_1$. □

3. Lemmas

The following lemma enables us to reduce the case (b) to the case (a).

Lemma 3.1. *(1) Assume that* $\sigma(\frac{\nu}{r}) = \frac{\lambda r^4}{\nu^2}\rho(r)$ *holds for* $r \in J_\rho$ *with some positive constants* ν *and* λ*. Then for each* $R \in O_{\gamma,\eta}(n)$*, the inversion* $(j,1)$ *with*

$$j(t,x) = (\lambda t, \frac{\nu R x}{\langle x\rangle_\gamma^2})$$

is a caloric morphism from $\mathbb{R}\times M$ *to* $\mathbb{R}\times N$.

(2) If $\rho(r)=p_1r^q$ *and* $\sigma(s)=p_2s^{-q-4}$, *then for each* $R\in O_{\gamma,\eta}(n)$, *the inversion* $(j,1)$ *with*

$$j(t,x)=(\frac{p_2}{p_1}t,\frac{Rx}{\langle x\rangle_\gamma^2})$$

is a caloric morphism from $\mathbb{R}\times M$ *to* $\mathbb{R}\times N$.

Proof. (1) Clearly, $(j,1)$ satisfies the equations (E-1) and (E-3). We shall show the equation (E-2). For simplicity, we put $y=Rx$. Since $j_\alpha(t,x)=\dfrac{\nu(Rx)_\alpha}{\langle x\rangle_\gamma^2}=\dfrac{\nu y_\alpha}{\langle x\rangle_\gamma^2}$, we have

$$\begin{aligned}
\sum_{i=1}^n\frac{x_i}{\langle x\rangle_\gamma}\frac{\partial j_\alpha}{\partial x_i}&=\nu\sum_{i=1}^n\frac{x_i}{\langle x\rangle_\gamma}\Big(\frac{R_{\alpha i}}{\langle x\rangle_\gamma^2}-2\frac{y_\alpha(\gamma x)_i}{\langle x\rangle_\gamma^4}\Big)=\nu\Big(\frac{y_\alpha}{\langle x\rangle_\gamma^3}-2\frac{y_\alpha\gamma(x,x)}{\langle x\rangle_\gamma^5}\Big)\\
&=\nu\Big(\frac{y_\alpha}{\langle x\rangle_\gamma^3}-2\frac{y_\alpha\langle x\rangle_\gamma^2}{\langle x\rangle_\gamma^5}\Big)=-\nu\frac{y_\alpha}{\langle x\rangle_\gamma^3},\\
\sum_{i,l=1}^n\gamma^{il}\frac{\partial^2 j_\alpha}{\partial x_i\partial x_l}&=\sum_{i,l=1}^n\gamma^{il}\nu\Big(-2\frac{R_{\alpha i}(\gamma x)_l}{\langle x\rangle_\gamma^4}-2\frac{R_{\alpha l}(\gamma x)_i}{\langle x\rangle_\gamma^4}-2\frac{y_\alpha\gamma_{il}}{\langle x\rangle_\gamma^4}+8\frac{y_\alpha(\gamma x)_i}{\langle x\rangle_\gamma^5}\frac{(\gamma x)_l}{\langle x\rangle_\gamma}\Big)\\
&=\frac{2\nu}{\langle x\rangle_\gamma^4}\sum_{i,l=1}^n\gamma^{il}\Big[-R_{\alpha i}(\gamma x)_l-R_{\alpha l}(\gamma x)_i-y_\alpha\Big(\gamma_{il}-4\frac{(\gamma x)_i(\gamma x)_l}{\langle x\rangle_\gamma^2}\Big)\Big]\\
&=\frac{2\nu y_\alpha}{\langle x\rangle_\gamma^4}\Big(-2-n+4\frac{\gamma(x,x)}{\langle x\rangle_\gamma^2}\Big)=2(2-n)\nu\frac{y_\alpha}{\langle x\rangle_\gamma^4},\\
\Delta_g j_\alpha&=\frac{1}{\rho(\langle x\rangle_\gamma)}\sum_{i,l=1}^n\gamma^{il}\frac{\partial^2 j_\alpha}{\partial x_i\partial x_l}+\frac{n-2}{2}\frac{\rho'(\langle x\rangle_\gamma)}{\rho(\langle x\rangle_\gamma)^2}\sum_{i=1}^n\frac{x_i}{\langle x\rangle_\gamma}\frac{\partial j_\alpha}{\partial x_i}\\
&=\frac{2(2-n)\nu}{\rho(\langle x\rangle_\gamma)}\frac{y_\alpha}{\langle x\rangle_\gamma^4}-\frac{n-2}{2}\frac{\rho'(\langle x\rangle_\gamma)}{\rho(\langle x\rangle_\gamma)^2}\nu\frac{y_\alpha}{\langle x\rangle_\gamma^3}
\end{aligned}$$

and

$$\begin{aligned}
\sum_{l,k=1}^n g(\nabla_g j_l,\nabla_g j_k)\cdot{}^h\Gamma_{lk}^\alpha\circ j&=\lambda\frac{2-n}{2}\frac{\sigma'(\langle\nu y\rangle_\eta/\langle x\rangle_\gamma^2)}{\sigma(\langle\nu y\rangle_\eta/\langle x\rangle_\gamma^2)^2}\frac{(\nu y)_\alpha/\langle x\rangle_\gamma^2}{\langle\nu y\rangle_\eta/\langle x\rangle_\gamma^2}\\
&=\lambda\frac{2-n}{2}\frac{\sigma'(\nu/\langle x\rangle_\gamma)}{\sigma(\nu/\langle x\rangle_\gamma)^2}\frac{y_\alpha}{\langle x\rangle_\gamma}.
\end{aligned}$$

Differentiating the equation $\sigma(\nu/r)^{-1}=\dfrac{\nu^2}{\lambda r^4}\rho(r)^{-1}$ by r, we have

$$\frac{\sigma'(\nu/r)}{\sigma(\nu/r)^2}(-\frac{\nu}{r^2})=\frac{4\nu^2}{\lambda r^5\rho(r)}+\frac{\nu^2\rho'(r)}{\lambda r^4\rho(r)^2},\quad r\in J_\rho,$$

and hence

$$\lambda\frac{2-n}{2}\frac{\sigma'(\nu/\langle x\rangle_\gamma)}{\sigma(\nu/\langle x\rangle_\gamma)^2}\frac{y_\alpha}{\langle x\rangle_\gamma}=\frac{2(n-2)\nu y_\alpha}{\langle x\rangle_\gamma^4\rho(\langle x\rangle_\gamma)}+\frac{n-2}{2}\frac{\nu\rho'(\langle x\rangle_\gamma)y_\alpha}{\langle x\rangle_\gamma^3\rho(\langle x\rangle_\gamma)^2}.$$

Thus we have

$$\begin{aligned}
&\Delta_g j_\alpha + 2g(\nabla_g \log\varphi, \nabla_g j_\alpha) + \sum_{l,k=1}^{n} g(\nabla_g j_l, \nabla_g j_k)\cdot {}^h\Gamma^\alpha_{lk}\circ j \\
&\quad = \frac{2(2-n)\nu y_\alpha}{\langle x\rangle_\gamma^4\rho(\langle x\rangle_\gamma)} - \frac{n-2}{2}\frac{\nu\rho'(\langle x\rangle_\gamma)y_\alpha}{\langle x\rangle_\gamma^3\rho(\langle x\rangle_\gamma)^2} + \frac{2(n-2)\nu y_\alpha}{\langle x\rangle_\gamma^4\rho(\langle x\rangle_\gamma)} + \frac{n-2}{2}\frac{\nu\rho'(\langle x\rangle_\gamma)y_\alpha}{\langle x\rangle_\gamma^3\rho(\langle x\rangle_\gamma)^2} \\
&\quad = 0 = \frac{\partial j_\alpha}{\partial t}, \quad \langle x\rangle_\gamma \in J_\rho.
\end{aligned}$$

We have (E-2).

To show (E-4), first we remark

$$j_0'(t)(h^{\alpha\beta}\circ j) = \lambda\frac{1}{\sigma(\langle x\rangle_\gamma^{-2}\nu(t)\langle y\rangle_\eta)}\eta^{\alpha\beta} = \lambda\frac{1}{\sigma(\nu(t)\langle x\rangle_\gamma^{-1})}\eta^{\alpha\beta}, \quad 1\leqq \alpha,\beta\leqq n.$$

On the other hand, equations

$$\begin{aligned}
g(\nabla_g j_\alpha, \nabla_g j_\beta) &= \frac{1}{\rho(\langle x\rangle_\gamma)}\sum_{i,l=1}^{n}\gamma^{il}\frac{\partial j_\alpha}{\partial x_i}\frac{\partial j_\beta}{\partial x_l} \\
&= \sum_{i,l=1}^{n}\frac{\gamma^{il}}{\langle x\rangle_\gamma^4\rho(\langle x\rangle_\gamma)}\Big(\nu R_{\alpha i} - 2\frac{(\nu y)_\alpha(\gamma x)_i}{\langle x\rangle_\gamma^2}\Big)\Big(\nu R_{\beta l} - 2\frac{(\nu y)_\beta(\gamma x)_l}{\langle x\rangle_\gamma^2}\Big) \\
&= \frac{\nu^2(R\gamma^{-1}{}^tR)_{\alpha\beta}}{\langle x\rangle_\gamma^4\rho(\langle x\rangle_\gamma)} - 2\frac{\nu^2(R\gamma^{-1}\gamma x)_\alpha y_\beta}{\langle x\rangle_\gamma^6\rho(\langle x\rangle_\gamma)} \\
&\qquad - 2\frac{\nu^2 y_\alpha(R\gamma^{-1}\gamma x)_\beta}{\langle x\rangle_\gamma^6\rho(\langle x\rangle_\gamma)} + 4\frac{\nu^2\gamma^{-1}(\gamma x,\gamma x)y_\alpha y_\beta}{\langle x\rangle_\gamma^8\rho(\langle x\rangle_\gamma)} \\
&= \nu^2\Big[\frac{(\eta^{-1})_{\alpha\beta}}{\langle x\rangle_\gamma^4\rho(\langle x\rangle_\gamma)} - 2\frac{y_\alpha y_\beta}{\langle x\rangle_\gamma^6\rho(\langle x\rangle_\gamma)} - 2\frac{y_\alpha y_\beta}{\langle x\rangle_\gamma^6\rho(\langle x\rangle_\gamma)} + 4\frac{y_\alpha y_\beta}{\langle x\rangle_\gamma^6\rho(\langle x\rangle_\gamma)}\Big] \\
&= \frac{\nu^2\eta^{\alpha\beta}}{\langle x\rangle_\gamma^4\rho(\langle x\rangle_\gamma)}, \qquad 1\leqq\alpha,\beta\leqq n
\end{aligned}$$

hold. By assumption,

$$\frac{\nu^2}{\langle x\rangle_\gamma^4\rho(\langle x\rangle_\gamma)} = \lambda\frac{1}{\sigma(\nu/\langle x\rangle_\gamma)}, \quad \langle x\rangle_\gamma\in J_\rho.$$

Thus we have the equation (E-4):

$$g(\nabla_g j_\alpha, \nabla_g j_\beta) = j_0'(t)(h^{\alpha\beta}\circ j).$$

Therefore $(j,1)$ is a caloric morphism.

(2) is a special case of (1). □

Lemma 3.2. *Let (f,φ) be a caloric morphism on a domain $D\subset\mathbb{R}\times M$ such that f is of form $f(t,x) = (f_0(t), \nu(t)R(t)x)$, where $\nu(t)$ is a strictly positive C^∞-function and $R(t)$ is an $O_{\gamma,\eta}(n)$-valued C^∞-function. We put*

$$S(t) = \gamma R(t)^{-1}R'(t).$$

Then $S(t)$ is skew-symmetric and the following statements hold.

(1) φ satisfies the following equations on D:

$$\nabla_g \log\varphi = \frac{\nu'(t)}{2\nu(t)}x + \frac{1}{2}\gamma^{-1}S(t)x, \quad \nabla_x \log\varphi = \frac{\rho(\langle x\rangle_\gamma)}{2}\Big(\frac{\nu'(t)}{\nu(t)}\gamma + S(t)\Big)x, \tag{3.1}$$

$$\Delta_g \log\varphi = \frac{n}{4}\frac{\nu'(t)}{\nu(t)}\Big(\frac{\langle x\rangle_\gamma \rho'(\langle x\rangle_\gamma)}{\rho(\langle x\rangle_\gamma)} + 2\Big), \tag{3.2}$$

$$g(\nabla_g \log\varphi, \nabla_g \log\varphi) = \frac{\rho(\langle x\rangle_\gamma)}{4}\Big\{\Big(\frac{\nu'(t)}{\nu(t)}\Big)^2\langle x\rangle_\gamma^2 + (x, {}^tS(t)\gamma^{-1}S(t)x)\Big\}, \tag{3.3}$$

where $\nabla_x = \Big(\frac{\partial}{\partial x_1}, \ldots, \frac{\partial}{\partial x_n}\Big)$.

(2) If $n \geqq 3$, then $R'(t) = O$ for all $t \in I_0$ and hence the equations in (3.1) *are*

$$\nabla_g \log\varphi = \frac{\nu'(t)}{2\nu(t)}x, \quad \nabla_x \log\varphi = \frac{\rho(\langle x\rangle_\gamma)}{2}\frac{\nu'(t)}{\nu(t)}\gamma x, \tag{3.4}$$

and (3.3) *is*

$$g(\nabla_g \log\varphi, \nabla_g \log\varphi) = \frac{\rho(\langle x\rangle_\gamma)}{4}\Big(\frac{\nu'(t)}{\nu(t)}\Big)^2\langle x\rangle_\gamma^2. \tag{3.5}$$

(3) If $R'(t) \neq 0$ on an interval I', then $n = 2$ and $\rho(r) = pr^{-2}$ holds for all $r \in J'_\rho = \{\langle x\rangle_\gamma; (t,x) \in D, t \in I'\}$ with some constant $p > 0$.

Proof. First of all, we remark that the matrix $S(t)$ is skew-symmetric. In fact, $S(t) + {}^tS(t) = \gamma R^{-1}(t)R'(t) + {}^tR'(t){}^tR^{-1}(t)\gamma = {}^tR(t)\eta R'(t) + {}^tR'(t)\eta R(t) = ({}^tR(t)\eta R(t))' = \gamma' = O$, because $\gamma = {}^tR(t)\eta R(t)$ follows from $R(t) \in O_{\gamma,\eta}(n)$.

First we prove (1). By (2.1), (2.2) and (2.17), we have

$$\begin{aligned}
\Delta_g f_\alpha &= \frac{1}{\rho(\langle x\rangle_\gamma)}\sum_{i,j=1}^{n}\gamma^{ij}\frac{\partial^2 f_\alpha}{\partial x_i \partial x_j} + \frac{n-2}{2}\frac{\rho'(\langle x\rangle_\gamma)}{\rho(\langle x\rangle_\gamma)^2}\sum_{j=1}^{n}\frac{x_j}{\langle x\rangle_\gamma}\frac{\partial f_\alpha}{\partial x_j} \\
&= \frac{n-2}{2}\frac{\rho'(\langle x\rangle_\gamma)}{\rho(\langle x\rangle_\gamma)^2}\sum_{j=1}^{n}\frac{x_j}{\langle x\rangle_\gamma}\frac{\partial f_\alpha}{\partial x_j} \\
&= \frac{n-2}{2}\frac{\rho'(\langle x\rangle_\gamma)}{\langle x\rangle_\gamma \rho(\langle x\rangle_\gamma)^2}\nu(t)\sum_{i=1}^{n}R_{\alpha i}(t)x_i
\end{aligned} \tag{3.6}$$

and

$$\begin{aligned}
2g(\nabla_g \log\varphi, \nabla_g f_\alpha) &= \frac{2}{\rho(\langle x\rangle_\gamma)}\sum_{j,k=1}^{n}\gamma^{jk}\frac{\partial \log\varphi}{\partial x_j}\frac{\partial f_\alpha}{\partial x_k} \\
&= \frac{2}{\rho(\langle x\rangle_\gamma)}\sum_{j,k=1}^{n}\frac{\partial \log\varphi}{\partial x_j}\nu(t)\gamma^{jk}R_{\alpha k}(t)
\end{aligned} \tag{3.7}$$

for $\alpha = 1, 2, \ldots, n$. The formula (2.3) implies

$$\begin{aligned}\sum_{j,k=1}^{n} \big(g(\nabla_g f_j, \nabla_g f_k) \cdot {}^h\Gamma_{jk}^{\alpha} \circ f\big)(t,x) &= f_0'(t) \frac{2-n}{2} \frac{\sigma'(\langle f(t,x)\rangle_\eta)}{\sigma(\langle f(t,x)\rangle_\eta)^2} \frac{f_\alpha(t,x)}{\langle f(t,x)\rangle_\eta} \\ &= f_0'(t) \frac{2-n}{2} \frac{\sigma'(\nu(t)\langle x\rangle_\gamma)}{\sigma(\nu(t)\langle x\rangle_\gamma)^2} \frac{\sum_{i=1}^n \nu(t) R_{\alpha i}(t) x_i}{\nu(t)\langle x\rangle_\gamma} \\ &= f_0'(t) \frac{2-n}{2} \frac{\sigma'(\nu(t)\langle x\rangle_\gamma)}{\langle x\rangle_\gamma \sigma(\nu(t)\langle x\rangle_\gamma)^2} \sum_{i=1}^n R_{\alpha i}(t) x_i\end{aligned} \tag{3.8}$$

for $\alpha = 1, 2, \ldots, n$. On the other hand, differentiating (2.4) by r, we have

$$\frac{f_0'(t)\sigma'(\nu(t)\langle x\rangle_\gamma)}{\sigma(\nu(t)\langle x\rangle_\gamma)^2} = \frac{\nu(t)\rho'(\langle x\rangle_\gamma)}{\rho(\langle x\rangle_\gamma)^2}. \tag{3.9}$$

Substituting (2.17), (3.6), (3.7), (3.8) and (3.9) into (E-2), we have

$$\sum_{j=1}^{n} (\nu'(t) R_{\alpha j}(t) + \nu(t) R_{\alpha j}'(t)) x_j = \frac{2\nu(t)}{\rho(\langle x\rangle_\gamma)} \sum_{j,k=1}^{n} \gamma^{jk} \frac{\partial \log\varphi}{\partial x_j} R_{\alpha k}(t),$$

and hence

$$\frac{\nu'(t)}{2\nu(t)} R(t)x + \frac{1}{2} R'(t)x = R(t)\nabla_g \log\varphi.$$

Therefore we have

$$\nabla_g \log\varphi = \frac{\nu'(t)}{2\nu(t)} x + \frac{1}{2}\gamma^{-1} S(t) x$$

and

$$\nabla_x \log\varphi = \frac{\rho(\langle x\rangle_\gamma)}{2} \Big(\frac{\nu'(t)}{\nu(t)}\gamma + S(t)\Big) x,$$

which are equations (3.1). We also have

$$\begin{aligned}\Delta_g \log\varphi &= \sum_{i=1}^{n} \frac{1}{\rho(\langle x\rangle_\gamma)^{\frac{n}{2}}} \frac{\partial}{\partial x_i} \Big(\rho(\langle x\rangle_\gamma)^{\frac{n}{2}} \frac{1}{2} \Big[\frac{\nu'(t)}{\nu(t)} x_i + (\gamma^{-1} S(t) x)_i\Big]\Big) \\ &= \sum_{i=1}^{n} \frac{n\rho'(\langle x\rangle_\gamma)(\gamma x)_i}{4\rho(\langle x\rangle_\gamma)\langle x\rangle_\gamma} \Big[\frac{\nu'(t)}{\nu(t)} x_i + (\gamma^{-1} S(t) x)_i\Big] + \frac{1}{2} \sum_{i=1}^{n} \Big[\frac{\nu'(t)}{\nu(t)} \delta_{ii} + \sum_{j=1}^{n} (\gamma^{-1} S(t))_{ij} \delta_{ij}\Big] \\ &= \frac{n}{4} \frac{\rho'(\langle x\rangle_\gamma)}{\rho(\langle x\rangle_\gamma)} \Big(\frac{\nu'(t)}{\nu(t)} \frac{\langle x\rangle_\gamma^2}{\langle x\rangle_\gamma} + \frac{S(t)(x,x)}{\langle x\rangle_\gamma}\Big) + \frac{n}{2} \frac{\nu'(t)}{\nu(t)} + \frac{1}{2} \sum_{i,j=1}^{n} \gamma^{ij} S_{ji}(t),\end{aligned}$$

where $S(t)(x,x) = \sum_{i,j=1}^n S_{ij}(t) x_i x_j$. Since $S(t)$ is skew-symmetric and γ^{-1} is symmetric, $S(t)(x,x) = 0$ and $\sum_{i,j=1}^n \gamma^{ij} S_{ji}(t) = 0$. Therefore we have the equation (3.2).

Substituting (3.1) into (2.2), we have (3.3):

$$g(\nabla_g \log\varphi, \nabla_g \log\varphi) = \rho(\langle x\rangle_\gamma)\frac{1}{4}\gamma(\frac{\nu'(t)}{\nu(t)}x + \gamma^{-1}S(t)x, \frac{\nu'(t)}{\nu(t)}x + \gamma^{-1}S(t)x)$$
$$= \frac{\rho(\langle x\rangle_\gamma)}{4}\Big\{\Big(\frac{\nu'(t)}{\nu(t)}\Big)^2\langle x\rangle_\gamma^2 + (x, {}^tS(t)\gamma^{-1}S(t)x)\Big\}.$$

Thus we have the statement (1).

Next we proceed to prove the statement (2). Differentiating the latter equation of (3.1),

$$\frac{\partial \log\varphi}{\partial x_j} = \frac{\rho(\langle x\rangle_\gamma)}{2}(\frac{\nu'(t)}{\nu(t)}y_j + \sum_{k=1}^n S_{jk}(t)x_k), \quad j = 1, 2, \dots, n,$$

by x_i $(i \neq j)$, where $y = \gamma x$ and $S_{jk}(t)$ is the (j,k) element of the matrix $S(t)$, we have

$$\frac{\partial}{\partial x_i}\frac{\partial \log\varphi}{\partial x_j} = \frac{\rho'(\langle x\rangle_\gamma)}{2\langle x\rangle_\gamma}(\frac{\nu'(t)}{\nu(t)}y_i y_j + \sum_{k=1}^n S_{jk}(t)y_i x_k) + \frac{\rho(\langle x\rangle_\gamma)}{2}(\frac{\nu'(t)}{\nu(t)}\gamma_{ji} + S_{ji}(t)).$$

We also have

$$\frac{\partial}{\partial x_j}\frac{\partial \log\varphi}{\partial x_i} = \frac{\rho'(\langle x\rangle_\gamma)}{2\langle x\rangle_\gamma}(\frac{\nu'(t)}{\nu(t)}y_j y_i + \sum_{k=1}^n S_{ik}(t)y_j x_k) + \frac{\rho(\langle x\rangle_\gamma)}{2}(\frac{\nu'(t)}{\nu(t)}\gamma_{ij} + S_{ij}(t)).$$

Since $\dfrac{\partial}{\partial x_i}\dfrac{\partial \log\varphi}{\partial x_j} = \dfrac{\partial}{\partial x_j}\dfrac{\partial \log\varphi}{\partial x_i}$ for each $i, j = 1, 2, \dots, n$ with $i \neq j$,

$$\frac{\rho'(\langle x\rangle_\gamma)}{\langle x\rangle_\gamma}\sum_{k=1}^n S_{jk}(t)y_i x_k + \rho(\langle x\rangle_\gamma)S_{ji}(t) = \frac{\rho'(\langle x\rangle_\gamma)}{\langle x\rangle_\gamma}\sum_{k=1}^n S_{ik}(t)y_j x_k + \rho(\langle x\rangle_\gamma)S_{ij}(t)$$

holds. Then we have

$$2S_{ij}(t) = \frac{\rho'(\langle x\rangle_\gamma)}{2\langle x\rangle_\gamma \rho(\langle x\rangle_\gamma)}(y_i\sum_{k=1}^n S_{jk}(t)x_k - y_j\sum_{k=1}^n S_{ik}(t)x_k)$$

for each $i, j = 1, 2, \dots, n$ with $i \neq j$, and hence

$$S_{ij}(t) = \frac{\rho'(\langle x\rangle_\gamma)}{2\langle x\rangle_\gamma \rho(\langle x\rangle_\gamma)}(y_i z_j - z_i y_j), \tag{3.10}$$

where we put $z = Sx$. Let $n \geqq 3$. Then for each fixed $t \in I_0$ and each triple indices i, j, k with $1 \leqq i < j < k \leqq n$, the equation (3.10) implies

$$S_{ij}(t)y_k + S_{jk}(t)y_i + S_{ki}(t)y_j = 0$$

for all (y_i, y_j, y_k) in an open subset of $\mathbb{R}^3$. This implies $(S_{ij}(t), S_{jk}(t), S_{ki}(t)) = 0$ for each $1 \leqq i < j < k \leqq n$, because γ is non-degenerate. Therefore $S(t) = O$, and hence $R'(t) = O$ for all $t \in I_0$. Thus we have the statement (2).

Finally, assume that $R'(t) \neq 0$ on an interval I'. Then (2) yields $n = 2$. Hence $S(t) = \begin{pmatrix} 0 & S_{12}(t) \\ -S_{12}(t) & 0 \end{pmatrix}$ and $z = (S_{12}(t)x_2, -S_{12}(t)x_1)$. Then the equation (3.10) implies

$$\begin{aligned} S_{12}(t) &= \frac{\rho'(r)}{2r\rho(r)}\{y_1(-S_{12}(t)x_1) - S_{12}(t)x_2y_2\} = -\frac{\rho'(r)}{2r\rho(r)}S_{12}(t)(x, \gamma x) \\ &= -\frac{r\rho'(r)}{2\rho(r)}S_{12}(t), \end{aligned}$$

where we put $r = \langle x \rangle_\gamma$. Since $S_{12}(t) \neq 0$ for $t \in I'$, $-\dfrac{r\rho'(r)}{2\rho(r)} = 1$ and hence $\rho(r) = pr^{-2}$ holds for all $r \in J'_\rho = \{\langle x \rangle_\gamma; (t, x) \in D, t \in I'\}$, which shows (3). □

4. Some special cases

Before the proof of Theorem 1.1, we deal with the case that ρ has the form $\rho(r) = p_1 r^q$ in this section. The following Proposition 4.1 corresponds to the cases 1-a and 1-b of Theorem 1.1. To state the results, we introduce the two dimensional polar coordinate with respect to γ. Since γ is a real symmetric matrix, there exists an orthogonal matrix U such that $\gamma = {}^tU \begin{pmatrix} \alpha & 0 \\ 0 & \beta \end{pmatrix} U$ ($\alpha > 0$, $\beta \neq 0$). If we put $B = \begin{pmatrix} \sqrt{\alpha} & 0 \\ 0 & \sqrt{|\beta|} \end{pmatrix} U$ and $\tilde{x} = Bx$, then $\det B = \sqrt{|\det \gamma|}$,

$$\langle x \rangle_\gamma^2 = \alpha(Ux)_1^2 + \beta(Ux)_2^2 = \begin{cases} \tilde{x}_1^2 + \tilde{x}_2^2, & \det \gamma > 0, \\ \tilde{x}_1^2 - \tilde{x}_2^2, & \det \gamma < 0, \end{cases}$$

and

$$\begin{aligned} \frac{\partial}{\partial x_1}\frac{\tilde{x}_2}{\tilde{x}_1} &= \frac{\partial}{\partial x_1}\Big(\frac{B_{21}x_1 + B_{22}x_2}{B_{11}x_1 + B_{12}x_2}\Big) = \frac{-x_2 \det B}{\tilde{x}_1^2} = \frac{-\sqrt{|\det \gamma|}}{\tilde{x}_1^2}x_2, \\ \frac{\partial}{\partial x_2}\frac{\tilde{x}_2}{\tilde{x}_1} &= \frac{\partial}{\partial x_2}\Big(\frac{B_{22}x_2 + B_{21}x_1}{B_{12}x_2 + B_{11}x_1}\Big) = \frac{x_1 \det B}{\tilde{x}_1^2} = \frac{\sqrt{|\det \gamma|}}{\tilde{x}_1^2}x_1 \end{aligned}$$

hold. The polar coordinate (r, θ) with respect to γ is defined by

$$r = \langle x \rangle_\gamma, \text{ and } \theta = \begin{cases} \arctan \dfrac{\tilde{x}_2}{\tilde{x}_1}, & \det \gamma > 0, \\ \\ \operatorname{arctanh} \dfrac{\tilde{x}_2}{\tilde{x}_1}, & \det \gamma < 0. \end{cases}$$

Note that for each point $x = (r, \theta) \in M$, the polar coordinate of the inversion $\dfrac{x}{\langle x\rangle_\gamma^2}$ is equal to (r^{-1}, θ), because $\langle \dfrac{x}{\langle x\rangle_\gamma^2}\rangle_\gamma = \dfrac{1}{\langle x\rangle_\gamma}$ and $\dfrac{x}{\langle x\rangle_\gamma^2}$ is a scholar multiple of x. Then

$$\nabla_x \theta = \frac{\sqrt{|\det\gamma|}}{\langle x\rangle_\gamma^2}\begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} x \tag{4.1}$$

holds in any case. In fact, if $\det\gamma > 0$,

$$\frac{\partial\theta}{\partial x_1} = \frac{\tilde{x}_1^2}{\tilde{x}_1^2 + \tilde{x}_2^2}\frac{\partial}{\partial x_1}\frac{\tilde{x}_2}{\tilde{x}_1} = -\frac{\sqrt{|\det\gamma|}}{\langle x\rangle_\gamma^2}x_2, \quad \frac{\partial\theta}{\partial x_2} = \frac{\tilde{x}_1^2}{\tilde{x}_1^2 + \tilde{x}_2^2}\frac{\partial}{\partial x_2}\frac{\tilde{x}_2}{\tilde{x}_1} = \frac{\sqrt{|\det\gamma|}}{\langle x\rangle_\gamma^2}x_1,$$

and if $\det\gamma < 0$,

$$\frac{\partial\theta}{\partial x_1} = \frac{\tilde{x}_1^2}{\tilde{x}_1^2 - \tilde{x}_2^2}\frac{\partial}{\partial x_1}\frac{\tilde{x}_2}{\tilde{x}_1} = -\frac{\sqrt{|\det\gamma|}}{\langle x\rangle_\gamma^2}x_2, \quad \frac{\partial\theta}{\partial x_2} = \frac{\tilde{x}_1^2}{\tilde{x}_1^2 - \tilde{x}_2^2}\frac{\partial}{\partial x_2}\frac{\tilde{x}_2}{\tilde{x}_1} = \frac{\sqrt{|\det\gamma|}}{\langle x\rangle_\gamma^2}x_1.$$

Now we state the proposition.

Proposition 4.1. *Let $n = 2$ and $\rho(r) = p_1 r^{-2}$ ($p_1 \in \mathbb{R}_+$).*

(1) If there exists a caloric morphism (f, φ) such that f is of form (a)*, then $\sigma(s) = p_2 s^{-2}$ with some $p_2 \in \mathbb{R}_+$ and*

$$f(t,x) = \Big(\frac{p_2}{p_1}t + d, ce^{at}R_0 e^{t\gamma^{-1}\left(\begin{smallmatrix} 0 & -b \\ b & 0\end{smallmatrix}\right)}x\Big),$$
$$\varphi(t,r,\theta) = Cr^{\frac{1}{2}ap_1}\exp\frac{p_1}{2}\Big(\frac{b}{\sqrt{|\det\gamma|}}\theta + \frac{1}{2}(a^2 + \frac{b^2}{\det\gamma})t\Big).$$

Especially, $\nu(t) = ce^{at}$ where ν is the function defined in (2.7)*.*

(2) If there exists a caloric morphism (f, φ) such that f is of form (b)*, then $\sigma(s) = p_2 s^{-2}$ with some $p_2 \in \mathbb{R}_+$ and*

$$f(t,x) = \Big(\frac{p_2}{p_1}t + d, ce^{at}\langle x\rangle_\gamma^{-2}R_0 e^{t\gamma^{-1}\left(\begin{smallmatrix} 0 & -b \\ b & 0\end{smallmatrix}\right)}x\Big),$$
$$\varphi(t,r,\theta) = Cr^{-\frac{1}{2}ap_1}\exp\frac{p_1}{2}\Big(\frac{b}{\sqrt{|\det\gamma|}}\theta + \frac{1}{2}(a^2 + \frac{b^2}{\det\gamma})t\Big).$$

Especially, $\nu(t) = ce^{at}$ where ν is the function defined in (2.12)*.*

In both cases, $a, b, d \in \mathbb{R}$, $c, C \in \mathbb{R}_+$, $R_0 \in O_{\gamma,\eta}(2)$ and (r, θ) is the polar coordinate of $\mathbb{R}^2$ with respect to γ.

Proof. Let D be the domain of f. (2.4) implies that for all $(t, r) \in E = \{(t, \langle x\rangle_\gamma) \in \mathbb{R}\times\mathbb{R}_+; (t,x) \in D\}$,

$$\sigma(\nu(t)r) = \frac{f_0'(t)}{\nu(t)^2}p_1 r^{-2}$$

holds. Put $s = \nu(t)r$. Then

$$s^2\sigma(s) = f_0'(t)p_1, \quad (t,s) \in E' = \{(t, \nu(t)r) \in \mathbb{R} \times \mathbb{R}_+; (t,r) \in E\}.$$

Hence $s^2\sigma(s)$ and $f_0'(t)p_1$ equal to a constant $p_2 \in \mathbb{R}_+$. Therefore $\sigma(s) = p_2 s^{-2}$ and $f_0(t) = \dfrac{p_2}{p_1}t + d$ with $d \in \mathbb{R}$.

By Lemma 3.2 (1), $\log\varphi$ satisfies the equation

$$\nabla_x \log\varphi = \frac{p_1\langle x\rangle_\gamma^{-2}}{2}\Big(\frac{\nu'(t)}{\nu(t)}\gamma x + S(t)x\Big) = \frac{p_1}{2}\frac{\nu'(t)}{\nu(t)}\nabla_x \log\langle x\rangle_\gamma + \frac{p_1}{2\langle x\rangle_\gamma^2}S(t)x.$$

Since $S(t)$ is skew-symmetric and $n = 2$, $S(t) = \begin{pmatrix} 0 & -s(t) \\ s(t) & 0 \end{pmatrix}$, where we put $s(t) = S_{21}(t)$ for simplicity. By (4.1), we have

$$\frac{p_1 s(t)}{2\sqrt{|\det\gamma|}}\nabla_x\theta = \frac{p_1 s(t)}{2\langle x\rangle_\gamma^2}\begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} x = \frac{p_1}{2\langle x\rangle_\gamma^2}S(t)x,$$

and hence

$$\nabla_x \log\varphi = \nabla_x\Big(\frac{p_1}{2}\frac{\nu'(t)}{\nu(t)}\log\langle x\rangle_\gamma + \frac{p_1 s(t)}{2\sqrt{|\det\gamma|}}\theta\Big).$$

Therefore, there exists a C^∞-function $\psi(t)$ such that

$$\log\varphi(t,r,\theta) = \frac{p_1}{2}\frac{\nu'(t)}{\nu(t)}\log r + \frac{p_1 s(t)}{2\sqrt{|\det\gamma|}}\theta + \psi(t). \tag{4.2}$$

On the other hand, φ satisfies the equation (E-1). Since $\varphi > 0$, (E-1) is equivalent to

$$\frac{\partial\log\varphi}{\partial t} - \Delta_g\log\varphi - g(\nabla_g\log\varphi, \nabla_g\log\varphi) = 0. \tag{4.3}$$

By (4.2), we have

$$\frac{\partial\log\varphi}{\partial t} = \frac{p_1}{2}\Big(\frac{\nu'(t)}{\nu(t)}\Big)'\log r + \frac{p_1 s'(t)}{2\sqrt{|\det\gamma|}}\theta + \psi'(t).$$

By Lemma 3.2, we have

$$\Delta_g\log\varphi = \frac{n}{4}\frac{\nu'(t)}{\nu(t)}\Big(\frac{\langle x\rangle_\gamma\rho'(\langle x\rangle_\gamma)}{\rho(\langle x\rangle_\gamma)} + 2\Big) = \frac{n}{4}\frac{\nu'(t)}{\nu(t)}(-2+2) = 0, \tag{4.4}$$

$$g(\nabla_g\log\varphi, \nabla_g\log\varphi) = \frac{p_1}{4\langle x\rangle_\gamma^2}\Big[\Big(\frac{\nu'(t)}{\nu(t)}\Big)^2\langle x\rangle_\gamma^2 + (x, s(t)^2\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}\gamma^{-1}\begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}x)\Big]. \tag{4.5}$$

Since ${}^t\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}\gamma^{-1}\begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} = \dfrac{1}{\det\gamma}\gamma$, we have

$$\begin{aligned} g(\nabla_g\log\varphi, \nabla_g\log\varphi) &= \frac{p_1}{4\langle x\rangle_\gamma^2}\Big\{\Big(\frac{\nu'(t)}{\nu(t)}\Big)^2\langle x\rangle_\gamma^2 + \frac{s(t)^2}{\det\gamma}(x, \gamma x)\Big\} \\ &= \frac{p_1}{4}\Big\{\Big(\frac{\nu'(t)}{\nu(t)}\Big)^2 + \frac{s(t)^2}{\det\gamma}\Big\}. \end{aligned}$$

Substitute these equations into (4.3). Then we have

$$\frac{p_1}{2}\Big(\frac{\nu'(t)}{\nu(t)}\Big)'\log r+\frac{p_1 s'(t)}{2\sqrt{|\det\gamma|}}\theta+\psi'(t)-\frac{p_1}{4}\Big\{\Big(\frac{\nu'(t)}{\nu(t)}\Big)^2+\frac{s(t)^2}{\det\gamma}\Big\}=0. \tag{4.6}$$

Therefore we obtain a system of differential equations

$$\begin{cases}\Big(\dfrac{\nu'(t)}{\nu(t)}\Big)'=0,\\ s'(t)=0,\\ \psi'(t)=\dfrac{p_1}{4}\Big[\Big(\dfrac{\nu'(t)}{\nu(t)}\Big)^2-\dfrac{s(t)^2}{\det\gamma}\Big],\end{cases}$$

because the coefficients of $\log r$ and θ in (4.6) must be equal to 0. The solution of this system is

$$\begin{cases}\nu(t)=ce^{at},\\ s(t)=b,\\ \psi(t)=\dfrac{p_1}{4}(a^2-\dfrac{b^2}{\det\gamma})t+C_0,\end{cases} \tag{4.7}$$

where $a,b,C_0\in\mathbb{R}$ and $c\in\mathbb{R}_+$. Note that $a=0$ if and only if $\nu'(t)=0$ for all t. Substituting (4.7) into (4.2), we have

$$\log\varphi(t,r,\theta)=\frac{1}{2}ap_1\log r+\frac{p_1}{2\sqrt{|\det\gamma|}}b\theta+\frac{p_1}{4}(a^2+\frac{b^2}{\det\gamma})t+C,$$

and

$$S(t)=\begin{pmatrix}0 & -b\\ b & 0\end{pmatrix}. \tag{4.8}$$

Therefore

$$\varphi(t,r,\theta)=Cr^{\frac{1}{2}ap_1}\exp\Big(\frac{p_1}{2\sqrt{|\det\gamma|}}b\theta+\frac{p_1}{4}(a^2+\frac{b^2}{\det\gamma})t\Big).$$

Now choose a number $t_0\in\mathbb{R}$ such that $\{t=t_0\}\cap D\neq\emptyset$. Since $S(t)=\gamma R(t)^{-1}R'(t)$, $R(t)$ satisfies the differential equation

$$\gamma R(t)^{-1}R'(t)=\begin{pmatrix}0 & -b\\ b & 0\end{pmatrix}$$

by (4.8). The solution of this equation is

$$\begin{aligned}R(t)&=R(t_0)\exp(t-t_0)\gamma^{-1}\begin{pmatrix}0 & -b\\ b & 0\end{pmatrix}\\ &=R_0\exp t\gamma^{-1}\begin{pmatrix}0 & -b\\ b & 0\end{pmatrix},\end{aligned}$$

where $R_0 = R(t_0)\exp(-t_0)\gamma^{-1}\begin{pmatrix} 0 & -b \\ b & 0 \end{pmatrix}$. Thus we have

$$f(t,x) = (\frac{p_2}{p_1}t + d, ce^{at}R_0 e^{t\gamma^{-1}\left(\begin{smallmatrix} 0 & -b \\ b & 0 \end{smallmatrix}\right)}x),$$
$$\varphi(t,r,\theta) = Cr^{\frac{1}{2}ap_1}\exp\frac{p_1}{2}\Big(\frac{b}{\sqrt{|\det\gamma|}}\theta + \frac{1}{2}(a^2 + \frac{b^2}{\det\gamma})t\Big)$$

for all $(t,x) \in D$. This shows (1).

The assertion (2) is reduced to (1) by the composition with an inversion. In fact, Lemma 3.1 implies that the inversion $(j,1)$, where

$$j(t,x) = (t, \frac{x}{\langle x\rangle_\gamma^2}),$$

is a caloric morphism from $(\mathbb{R}\times M, p_1 r^{-2}\gamma)$ to itself. Then the composition $(f\circ j, 1\cdot(\varphi\circ j)) = (f\circ j, \varphi\circ j)$ of $(j,1)$ and (f,φ), is a caloric morphism. The mapping $f\circ j$ is of form (a), because

$$(f\circ j)(t,x) = (f_0(t), \nu(t)\langle x\rangle_\gamma^2 R(t)\frac{x}{\langle x\rangle_\gamma^2}) = (f_0(t), \nu(t)R(t)x).$$

By (1), we have

$$(f\circ j)(t,x) = (\frac{p_2}{p_1}t + d, ce^{at}R_0 e^{t\gamma^{-1}\left(\begin{smallmatrix} 0 & -b \\ b & 0 \end{smallmatrix}\right)}x),$$
$$(\varphi\circ j)(t,r,\theta) = Cr^{\frac{1}{2}ap_1}\exp\frac{p_1}{2}\Big(\frac{b}{\sqrt{|\det\gamma|}}\theta + \frac{1}{2}(a^2 + \frac{b^2}{\det\gamma})t\Big)$$

for all $(t,x) \in j^{-1}(D)$. Since $j^{-1} = j$ and $j(t,r,\theta) = (t, r^{-1}, \theta)$,

$$f(t,x) = (f\circ j)(j(t,x)) = (\frac{p_2}{p_1}t + d, ce^{at}\langle x\rangle_\gamma^{-2}R_0 e^{t\gamma^{-1}\left(\begin{smallmatrix} 0 & -b \\ b & 0 \end{smallmatrix}\right)}x),$$
$$\varphi(t,r,\theta) = (\varphi\circ j)(j(t,r,\theta)) = C(\frac{1}{r})^{\frac{1}{2}ap_1}\exp\frac{p_1}{2}\Big(\frac{b}{\sqrt{|\det\gamma|}}\theta + \frac{1}{2}(a^2 + \frac{b^2}{\det\gamma})t\Big).$$

This completes the proof. □

The next proposition corresponds to the cases 2-a and 2-b of Theorem 1.1.

Proposition 4.2. *Let $n \geqq 3$ and $\rho(r) = p_1 r^{-2}$ ($p_1 \in \mathbb{R}_+$).*

(1) If there exists a caloric morphism (f,φ) such that f is of form (a)*, then $\sigma(s) = p_2 s^{-2}$ with some $p_2 \in \mathbb{R}_+$ and*

$$f(t,x) = (\frac{p_2}{p_1}t + d, ce^{at}R_0 x),$$
$$\varphi(t,x) = C\langle x\rangle_\gamma^{\frac{1}{2}ap_1}\exp\Big(\frac{p_1}{4}a^2 t\Big).$$

Especially, $\nu(t) = ce^{at}$, *where* ν *is the function defined in* (2.7).

(2) If there exists a caloric morphism (f, φ) *such that* f *is of form* (b), *then* $\sigma(s) = p_2 s^{-2}$ *with some* $p_2 \in \mathbb{R}_+$ *and*

$$f(t,x) = (\frac{p_2}{p_1}t + d, ce^{at}\langle x\rangle_\gamma^{-2}R_0x),$$

$$\varphi(t,x) = C\langle x\rangle_\gamma^{-\frac{1}{2}ap_1}\exp\Big(\frac{p_1}{4}a^2t\Big).$$

Especially, $\nu(t) = ce^{at}$, *where* ν *is the function defined in* (2.12).

In both cases, $a, d \in \mathbb{R}$, $c, C \in \mathbb{R}_+$ *and* $R_0 \in O_{\gamma,\eta}(n)$.

Proof. By the same argument as in the proof of the above proposition, $f_0(t) = \dfrac{p_2}{p_1}t + d$ and $\sigma(s) = p_2 s^{-2}$ hold with some $p_2 \in \mathbb{R}_+$ and $d \in \mathbb{R}$.

By Lemma 3.2 (2), $R(t)$ is a constant R_0 and $\log\varphi$ satisfies the equation

$$\frac{\partial \log\varphi}{\partial x_j} = \frac{p_1}{2\langle x\rangle_\gamma^2}\frac{\nu'(t)}{\nu(t)}(\gamma x)_j, \quad j = 1, \ldots, n,$$

because $n \geqq 3$. Therefore φ is a function of $\langle x\rangle_\gamma$, i.e.

$$\varphi(t,x) = \varphi(t, \langle x\rangle_\gamma),$$

and

$$\frac{\partial\log\varphi}{\partial r} = \frac{p_1\nu'(t)}{2\nu(t)}\frac{1}{r},$$

and hence

$$\log\varphi(t,r) = \frac{p_1\nu'(t)}{2\nu(t)}\log r + \psi(t). \tag{4.9}$$

By (E-1) and (4.3),

$$\frac{\partial\log\varphi}{\partial t} - \Delta_g\log\varphi - g(\nabla_g\log\varphi, \nabla_g\log\varphi) = 0.$$

From Lemma 3.2 and (4.9), it follows that

$$\frac{\partial\log\varphi}{\partial t} = \frac{p_1}{2}\Big(\frac{\nu'(t)}{\nu(t)}\Big)'\log r + \psi'(t),$$

$$\Delta_g\log\varphi = \frac{n(q+2)}{2}\frac{\nu'(t)}{\nu(t)} = 0,$$

$$g(\nabla_g\log\varphi, \nabla_g\log\varphi) = \frac{p_1}{4}\langle x\rangle_\gamma^{q+2}\Big(\frac{\nu'(t)}{\nu(t)}\Big)^2 = \frac{p}{4}\Big(\frac{\nu'(t)}{\nu(t)}\Big)^2.$$

Hence, we have the equation

$$\frac{p_1}{2}\Big(\frac{\nu'(t)}{\nu(t)}\Big)'\log r + \psi'(t) - \frac{p_1}{4}\Big(\frac{\nu'(t)}{\nu(t)}\Big)^2 = 0.$$

Therefore we obtain a system of differential equations

$$\begin{cases} \Big(\dfrac{\nu'}{\nu}\Big)' = 0, \\ \psi' = \dfrac{p_1}{4}\Big(\dfrac{\nu'}{\nu}\Big)^2. \end{cases}$$

The solution is

$$\begin{cases} \nu(t) = ce^{at}, \\ \psi(t) = \dfrac{p_1 a^2}{4} t + C_0, \end{cases} \tag{4.10}$$

where $a, C_0 \in \mathbb{R}$ and $c \in \mathbb{R}_+$. Note that $a = 0$ if and only if $\nu'(t) = 0$ for some t. Substituting (4.10) into (4.9), we have

$$\log \varphi(t,r) = \frac{ap_1}{2} \log r + \frac{p_1 a^2}{4} t + C_0.$$

Thus we have

$$f(t,x) = (t + d, ce^{at} R_0 x), \quad \varphi(t,x) = C\langle x\rangle_\gamma^{\frac{1}{2}ap_1} \exp\Big(\frac{p_1}{4} a^2 t\Big)$$

for all $(t,x) \in D$. We have shown the first statement (1). By composing the inversion $(j,1)$ as in the proof of Proposition 4.1, we have (2). This completes the proof. □

The next proposition corresponds to the cases 3-a and 3-b of Theorem 1.1.

Proposition 4.3. *Let $\rho(r) = p_1 r^q$ ($p_1 \in \mathbb{R}_+$, $q \in \mathbb{R}$, $q \neq -2$).*

(1) If there exists a caloric morphism (f,φ) such that f is of form (a)*, then $\sigma(s) = p_2 s^q$ ($p_2 \in \mathbb{R}_+$) and*

$$f(t,x) = \Big(\frac{p_2}{p_1}\frac{ct+d}{at+b}, |at+b|^{-2/(q+2)} R_0 x\Big),$$
$$\varphi(t,x) = \frac{C}{|at+b|^{n/2}} \exp\Big[-\frac{pa\langle x\rangle_\gamma^{q+2}}{(q+2)^2(at+b)}\Big],$$

where $a,b,c,d, \in \mathbb{R}$ ($bc - ad = 1$), $C \in \mathbb{R}_+$ and $R_0 \in O_\gamma(n)$. Especially, $\nu(t) = |at+b|^{-2/(q+2)}$ where ν is the function defined in (2.7).

(2) If there exists a caloric morphism (f,φ) such that f is of form (b)*, then $\sigma(s) = p_2 s^{-q-4}$ ($p_2 \in \mathbb{R}_+$) and*

$$f(t,x) = \Big(\frac{p_2}{p_1}\frac{ct+d}{at+b}, |at+b|^{2/(q+2)} \langle x\rangle_\gamma^{-2} R_0 x\Big),$$
$$\varphi(t,x) = \frac{C}{|at+b|^{n/2}} \exp\Big[-\frac{p_1 a\langle x\rangle_\gamma^{q+2}}{(q+2)^2(at+b)}\Big],$$

where $a,b,c,d \in \mathbb{R}$ ($bc - ad = 1$), $C \in \mathbb{R}_+$ and $R_0 \in O_\gamma(n)$. Especially, $\nu(t) = |at+b|^{2/(q+2)}$ where ν is the function defined in (2.12).

Proof. Since $q \neq -2$, $R(t)$ is a constant R_0 and equations

$$\frac{\partial \log \varphi}{\partial x_j} = \frac{p_1 \langle x \rangle_\gamma^q}{2} \frac{\nu'(t)}{\nu(t)} (\gamma x)_j, \quad j = 1, \ldots, n$$

hold by Lemma 3.2 (3). As in the proof of Proposition 4.2, φ is a function of $\langle x \rangle_\gamma$, i.e., $\varphi(t, x) = \varphi(t, \langle x \rangle_\gamma)$, and hence there exists a C^∞-function $\psi(t)$ such that

$$\log \varphi(t, r) = \frac{p_1}{2(q+2)} \frac{\nu'(t)}{\nu(t)} r^{q+2} + \psi(t), \tag{4.11}$$

and then

$$\frac{\partial \log \varphi}{\partial t} = \frac{p_1}{2(q+2)} \Big(\frac{\nu'(t)}{\nu(t)} \Big)' r^{q+2} + \psi'(t).$$

By (3.2) and (3.5) we have

$$\Delta_g \log \varphi = \frac{n}{4} \frac{\nu'(t)}{\nu(t)} (q+2),$$
$$g(\nabla_g \log \varphi, \nabla_g \log \varphi) = \frac{p_1}{4} \Big(\frac{\nu'(t)}{\nu(t)} \Big)^2 r^{q+2},$$

respectively. Substituting these into (E-1), we have

$$\frac{p_1}{2(q+2)} \Big[\Big(\frac{\nu'(t)}{\nu(t)} \Big)' - \frac{q+2}{2} \Big(\frac{\nu'(t)}{\nu(t)} \Big)^2 \Big] r^{q+2} + \psi' - \frac{n(q+2)}{4} \Big(\frac{\nu'(t)}{\nu(t)} \Big)' = 0.$$

Therefore we obtain a system of differential equations

$$\begin{cases} \Big(\dfrac{\nu'(t)}{\nu(t)} \Big)' - \dfrac{q+2}{2} \Big(\dfrac{\nu'(t)}{\nu(t)} \Big)^2 = 0, \\ \psi' - \dfrac{n(q+2)}{4} \Big(\dfrac{\nu'(t)}{\nu(t)} \Big)' = 0. \end{cases}$$

The solution is

$$\begin{cases} \nu(t) = |at + b|^{-2/(q+2)}, \\ \psi(t) = \log |at + b|^{-n/2} + C_0, \end{cases} \tag{4.12}$$

where $a, b, C_0 \in \mathbb{R}$. Note that, $a = 0$ if and only if $\nu'(t) = 0$ for some t. Substituting (4.12) into (4.11), we have

$$\log \varphi(t, r) = -\frac{p_1 a}{(q+2)^2 (at+b)} r^{q+2} + \log |at + b|^{-n/2} + C_0.$$

On the other hand, (2.4):

$$\sigma(\nu(t) r) = \frac{f_0'(t)}{\nu(t)^2} p_1 r^q, \quad (t, r) \in E = \{(t, \langle x \rangle_\gamma); (t, x) \in D\},$$

where D is the domain of f, implies

$$s^{-q}\sigma(s) = p_1 f_0'(t)\nu(t)^{-q-2} = p_1(at+b)^2 f_0'(t).$$

Hence $s^{-q}\sigma(s)$ and $p_1(at+b)^2 f_0'(t)$ equal to a constant $p_2 \in \mathbb{R}_+$. Therefore $f_0(t) = \dfrac{p_2}{p_1}\dfrac{ct+d}{at+b}$, where $c, d \in \mathbb{R}$ with $bc - ad = 1$. Consequently,

$$f(t,x) = (\frac{p_2}{p_1}\frac{ct+d}{at+b}, |at+b|^{-2/(q+2)}R_0 x)$$

and

$$\varphi(t,x) = \frac{C}{|at+b|^{n/2}} \exp\Big[-\frac{p_1 a\langle x\rangle_\gamma^{q+2}}{(q+2)^2(at+b)}\Big]$$

for all $(t,x) \in D$, where $C = e^{C_0} \in \mathbb{R}_+$. This shows (1).

The assertion (2) is reduced to (1) by the composition with an inversion. By (2.5):

$$\sigma(\frac{\nu(t)}{r}) = \frac{f_0'(t)r^4}{\nu(t)^2} p_1 r^q$$

for $(t,r) \in E = \{(t,\langle x\rangle_\gamma); (t,x) \in D\}$, where D is the domain of f, we have

$$s^{q+4}\sigma(s) = p_1 f_0'(t)\nu(t)^{q+2}.$$

Hence $s^{q+4}\sigma(s)$ and $p_1 f_0'(t)\nu(t)^{q+2}$ equal to a constant $p_2 \in \mathbb{R}_+$. Therefore $\sigma(s) = p_2 s^{-q-4}$ and $f_0'(t) = \dfrac{p_2}{p_1}\nu(t)^{-q-2}$. We put $q' = -q-4$. Then $q = -q'-4$ and $\rho(r) = p_1 r^{-q'-4}$. Fix $t_0 \in I_0$. Apply Lemma 3.1 (2) for $\sigma(r) = p_2 r^{q'}$, $\rho(s) = p_1 s^{-q'-4}$ and $R(t_0)^{-1} \in O_{\eta,\gamma}(n)$. Then the inversion $(j,1)$ with

$$j(\tau,\xi) = (\tau, \frac{R(t_0)^{-1}\xi}{\langle\xi\rangle_\eta^2})$$

is a caloric morphism from $\mathbb{R}\times N$ to $\mathbb{R}\times M$. Then the composition $(j\circ f, \varphi\cdot(1\circ f)) = (j\circ f, \varphi)$ of $(j,1)$ and (f,φ), is a caloric morphism from D to $\mathbb{R}\times M$. The mapping $j\circ f$ is of form (a), because

$$(j\circ f)(t,x) = (f_0(t), \frac{1}{\nu(t)}\langle x\rangle_\gamma^2 R(t_0)^{-1}R(t)\frac{x}{\langle x\rangle_\gamma^2}) = (f_0(t), \frac{1}{\nu(t)}R(t_0)^{-1}R(t)x).$$

Note that $R(t_0)^{-1}R(t) \in O_{\gamma,\gamma}$. Hence (1) implies

$$(j\circ f)(t,x) = (\frac{p_2}{p_1}\frac{ct+d}{at+b}, |at+b|^{-2/(q+2)}R_1 x)$$

and

$$\varphi(t,x) = \frac{C}{|at+b|^{n/2}} \exp\Big[-\frac{p_1 a\langle x\rangle_\gamma^{-(q+2)}}{(q+2)^2(at+b)}\Big]$$

for all $(t,x) \in D$, where $a,b,c,d \in \mathbb{R}$ $(bc-ad=1)$, $C \in \mathbb{R}_+$ and $R_1 \in O_{\gamma,\gamma}$. Since $j^{-1}(t,x) = (t, \dfrac{R(t_0)x}{\langle x\rangle_\gamma^2})$, we obtain

$$f(t,x) = (j^{-1} \circ (j \circ f))(t,x) = (\frac{p_2}{p_1}\frac{ct+d}{at+b}, |at+b|^{2/(q+2)} \langle x\rangle_\gamma^{-2} R_0 x),$$

where $R_0 := R(t_0)R_1 \in O_{\gamma,\eta}$. Thus we have (2). This completes the proof. □

5. Proof of the main result

Proof of Theorem 1.1. Let (f,φ) be a caloric morphism from a domain $D \subset \mathbb{R} \times M$ to $\mathbb{R} \times N$ such that the mapping f has the form (a) or (b). By Proposition 2.2, we have

$$f(t,x) = (f_0(t), \nu(t)R(t)x), \quad (t,x) \in D,$$
$$\sigma(\nu(t)r) = \frac{f_0'(t)}{\nu(t)^2}\rho(r), \qquad (t,r) \in E = \{(t, \langle x\rangle_\gamma) \in \mathbb{R}^2; (t,x) \in D\}$$

in the case (a) or

$$f(t,x) = (f_0(t), \langle x\rangle_\gamma^{-2}\nu(t)R(t)x), \quad (t,x) \in D,$$
$$\sigma(\nu(t)r) = \frac{f_0'(t)}{\nu(t)^2}\rho(r), \qquad (t,r) \in E = \{(t, \langle x\rangle_\gamma) \in \mathbb{R}^2; (t,x) \in D\}$$

in the case (b), where $\nu(t)$ is a strictly positive C^∞-function and $R(t)$ is an $O_{\gamma,\eta}(n)$-valued C^∞-function.

Assume that the function $\nu(t)$ is not constant. We shall prove that (f,φ) is one of the cases 1-a, 1-b, 2-a, 2-b, 3-a or 3-b. Let I' be a connected component of the open set $\{t \in I_0; \nu'(t) \neq 0\}$ and let $J_\rho' = \{\langle x\rangle_\gamma; (t,x) \in D, t \in I'\}$. Then by Proposition 2.2 and Lemma 2.1, $\rho(r) = p_1 r^q$ on J_ρ'. By Propositions 4.1, 4.2 and 4.3, $\nu'(t)$ has one of the following forms

$$\nu'(t) = cae^{at},$$
$$\nu'(t) = \frac{-2a}{(q+2)}|at+b|^{-2/(q+2)-1},$$
$$\nu'(t) = \frac{2a}{(q+2)}|at+b|^{2/(q+2)-1},$$

with $a \neq 0$ on I', since we assumed that ν is not constant. Then the above expression of $\nu'(t)$ shows that $\nu'(t) \neq 0$ on the closure of I' in I_0 in all of the above cases. Hence, $I' = I_0$, because I_0 is connected. Therefore $(t, \langle x\rangle_\gamma) \in I' \times J_\rho'$ for all $(t,x) \in D$ and $\rho(r) = p_1 r^q$ for all r. Again by Propositions 4.1, 4.2 and 4.3, (f,φ) is one of the cases 1-a, 1-b, 2-a, 2-b, 3-a or 3-b.

Next, we deal with the case that ν is constant. Because of the preceding argument, we may exclude the case that $\rho(r)$ has the form $\rho(r) = pr^q$. We first consider the case (a). By Lemma 3.2 (3), $R'(t) = 0$. Moreover, by (3.1), we have $\nabla_x \log\varphi = 0$ because

$\nu'(t) = 0$. Therefore $R(t)$ is a constant matrix R_0 and φ depends only on t. Since φ satisfies (E-1), φ is a positive constant C. On the other hand, (2.4) in Proposition 2.2 implies $\sigma(\nu r) = \frac{f_0'(t)}{\nu^2}\rho(r)$. Therefore $f_0'(t) = \frac{\nu^2\sigma(\nu r)}{\rho(r)}$ is a positive constant λ. Thus we have $\sigma(\nu r) = \frac{\lambda}{\nu^2}\rho(r)$ and $f_0(t) = \lambda t + d$ with some $d \in \mathbb{R}$. Therefore

$$f(t,x) = (\lambda t + d, \nu R_0 x), \quad \varphi(t,x) = C. \tag{5.1}$$

This is the case 4-a.

Finally, we consider the case (b). Since ν is constant, f_0' is equal to a constant λ and $\sigma(\frac{\nu}{r}) = \lambda\frac{r^4}{\nu^2}\rho(r)$ holds by the same argument as above. Then we have $f_0(t) = \lambda t + d$ with some $d \in \mathbb{R}$ and

$$\rho(\frac{\nu}{r}) = \frac{1}{\lambda}\frac{r^4}{\nu^2}\sigma(r).$$

Fix $t_0 \in I_0$. Apply Lemma 3.1 (1) for $\sigma(r)$, $\rho(s)$ and $R(t_0)^{-1} \in O_{\eta,\gamma}(n)$. Then the inversion $(j,1)$ with

$$j(\tau,\xi) = (\frac{1}{\lambda}\tau, \frac{\nu R(t_0)^{-1}\xi}{\langle\xi\rangle_\eta^2}),$$

is a caloric morphism from $\mathbb{R}\times N$ to $\mathbb{R}\times M$. Then $(j\circ f,\varphi)$, the composition of $(j,1)$ and (f,φ), is a caloric morphism from D to $\mathbb{R}\times M$. The mapping $j\circ f$ is of form (a):

$$(j\circ f)(t,x) = (t + \frac{d}{\lambda}, R(t_0)^{-1}R(t)x).$$

Note that $R(t_0)^{-1}R(t) \in O_{\gamma,\gamma}$. Hence by (5.1), we have

$$(j\circ f)(t,x) = (t + \frac{d}{\lambda}, R_1 x), \quad \varphi(t,x) = C, \qquad (t,x) \in D,$$

where $C \in \mathbb{R}_+$ and $R_1 \in O_{\gamma,\gamma}$. Since $j^{-1}(t,x) = (\lambda t, \frac{\nu R(t_0)x}{\langle x\rangle_\gamma^2})$, we obtain

$$f(t,x) = (j^{-1}\circ(j\circ f))(t,x) = (\lambda t + d, \frac{\nu R_0 x}{\langle x\rangle_\gamma^2}),$$

where $R_0 := R(t_0)R_1 \in O_{\gamma,\eta}(n)$. This is the case 4-b.

Thus we have completed the proof of Theorem 1.1. □

Acknowledgements: The author would like to express his gratitude to the referee for his valuable comments.

References

[1] B. Fuglede, *Harmonic morphisms between semi-riemannian manifolds*, Ann. Acad. Sci. Fenn. Math., **21** (1996), 31–50.

[2] T. Ishihara, *A mapping of Riemannian manifolds which preserves harmonic functions*, J. Math. Kyoto Univ., **19** (1979), no. 2, 215–229.

[3] H. Leutwiler, *On Appell transformation*, Potential theory, J. Král, J. Lukeš, I. Netuka, J. Veselý eds., Plenum, New York, 1988, 215–222.

[4] K. Shimomura, *The determination of caloric morphisms on Euclidean domains*, Nagoya Math J., **158** (2000), 133–166.

[5] M. Nishio and K. Shimomura, *Caloric morphisms on semi-euclidean space*, Rev. Roumaine. Math. Pures Appl. **47** (2002), 727–736.

[6] M. Nishio and K. Shimomura, *A characterization of caloric morphisms between manifolds*, Ann. Acad. Sci. Fenn. Math., **28** (2003), 111–122.

[7] K. Shimomura, *Caloric morphisms with respect to radial metrics on* $\mathbb{R}^n \setminus \{0\}$, Math. J. Ibaraki Univ., **35** (2003), 35–53.

[8] K. Shimomura, *Caloric morphisms with respect to radial metrics on semi-euclidean spaces*, Math. J. Ibaraki Univ., **37** (2005), 81–103.