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Some results on a conjecture regarding Mori domain

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Abstract

Based on the famous Mori-Nagata Theorem: The integral closure of a noetherian domain is a Krull domain, similar assertion was conjectured for Mori domain as follows: The complete integral closure of a Mori domain is a Krull domain.

The conjecture is positive for a noetherian domain, Krull domain, a semi normal Mori domain [6] and Mori domains for which $(D : D^\star) \neq 0$.

In general, as M. Roitman has noted [26], the conjecture is not true.

In this paper, an attempt is being made, among other things, to prove that the conjecture is true for a one dimensional Mori domain and for a finite dimensional AV- Mori domain. On the other hand, using the idea of conductor ideals, a simplified proof is given that the conjecture is true for semi normal Mori domains with nonzero pseudo radical.

Introduction

Let D Be an integral domain with quotient field K. Let$F(D)$ be the set of non-zero fractional ideals of D. A mapping $I \to I^\star$ of $F(D)$ into $F(D)$ is called a $\star$-operation on D, if the following conditions are satisfied:

i). $(a)^\star = (a)$ and $(aI)^\star = a(I)^\star$, for $0 \neq a \in K$

ii). $I \subseteq I^\star$ and $I \subseteq J \Rightarrow I^\star \subseteq J^\star$

iii). $I^\star = I^{\star\star}$

A non-zero fractional ideal I is called a $\star$- ideal if $I = I^\star$ and I is said to be a finite type if for each $I^\star$ of $F(D)$, $I^\star = \cup_\lambda I_\lambda^\star$, where $\{I_\lambda\}$ is a family of nonzero finitely generated fractional ideals of D contained in I.

The simplest example of a $\star$- operation is the identity mapping.

Another well known $\star$- operation is the v- operation given by $I_v = (I^{-1})^{-1} = \cap\{xD : I \subseteq xD, 0 \neq x \in K\}$ where $I^{-1} = (D : I) = \{x \in K : xI \subseteq D\}$.

A v-ideal is called divisorial ideal. A v-ideal with $(I : I) = I^{-1}$ is called strongly divisorial ideal.

An integral domain satisfying the ascending chain condition (ACC) on divisorial integral domains is called a Mori domain. If R is a subring of S and $s \in S$, then s is called almost integral over if all powers of s belong to a finite R-submodules of S.

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The set $C(R) := R^{\star}$ of all elements of S which are almost integral over R is called the complete integral closure of R in S. The ring $C(R) := R^{\star}$ is integrally closed [17, Theorem 13.1]. If $R = R^{\star}$, then R is called completely integrally closed.

If D is a noetherian domain, the concepts of integral closure and that of complete integral closure are the same.

Clearly, noetherian domain and Krull domain are Mori domains. But the converse is not true. For instance, if (V, M) is a discrete valuation ring (DVR) with canonical projection $\pi : V \longrightarrow V/M$ with $[V/M : K] = \infty$, then $A := \pi^{-1}(K)$ is a Mori, non-noetherian domain [2, Theorem 3.2].

On the other hand, D is a Krull domain iff D is completely integrally closed Mori domain.

Some of the pitfalls of a Mori domain in contrast to noetherian domain are the following:

i). Mori domain do not necessarily satisfy the principal ideal theorem (PIT). For instance, the domain $A = K[xy^n : n > 0] = K + xK[x, y]$, where K is a field and x, y are indeterminates, does not satisfy the principal ideal theorem. Clearly,$M = (xy^n : n \geq 0)$ is a minimal ideal over xA, while $htM = 2 \neq 1$

ii). The primary decomposition theorem is not satisfied by Mori domain [20].

iii). Unlike noetherian and Krull domains, the ring of polynomials (power series) over a Mori domain is not necessarily a Mori domain [25].

iv). The complete integral closure is not necessarily completely integrally closed [19, Cf. Example 91].

v). Unlike the integral closure, the complete integral closure can change Krull dimension [19, Cf. Example 88].

vi). The Cohen type criterion is valid with additional assumption given as follows: Let $D = D^{\star}$. Then D is a Krull domain iff every prime ideal is v-finite [23]. But even if every prime ideal is v-finite, then D is not necessarily Mori [24].

Theorem (Mori-Nagata). The integral closure of a noetherian domain is a Krull domain.

Based on the Mori-Nagata theorem, the following conjecture was posed:

Conjecture. The complete integral closure of a Mori domain is a Krull domain.

The conjecture is true for noetherian domain, seminormal domain [6], Krull domain and Mori domains for which $(D : D^{\star}) \neq (0)$ [5].

But in general, the conjecture is not true [26, Proposition 4.17].

In this paper, a number of facts, which are relevant to find positive answers to the conjecture will be considered. Throughout the discussion, D denotes an integral domain.

Theorem 1 ([3, Prop. 2.9])**.** *Let D be a Mori domain. If $D_M(D)$, the set of maximal divisorial ideals of D, is a finite set, then $D_M(D)$ is exactly the set of maximal ideals of D.*

For a Mori domain D, $D_M(D)$ can be decomposed in to two complementary sets $I(D)$ and $S(D)$ where $I(D) = \{P \in D_M(D) \mid D_P \text{ is a DVR}\}$ and $S(D) = \{P \in D_M(D) \mid P^{-1} = (P : P)\}$ [4].

Theorem 2 ([4, Prop. 3.3]). *Let D be a Mori domain.Then*
i). $B = \cap\{D_P \mid P \in I(D)\}$ is a Krull domain,
ii). $C = \cap\{D_Q \mid Q \in S(D)\}$ is a strongly Mori domain,
iii). $D = B \cap C$.

Theorem 3 ([21]). *If P is a prime ideal of height one of a Mori domain D , then P is a v-ideal of D.*

Theorem 4 ([4, Prop.3.2]). *Let D be a Mori domain. Then*
i). If I is a non-zero ideal of D , then $ID_P = D_P$ for all but finitely many $P \in D_M(D)$.
ii). $D = \cap\{D_P : P \in D_M(D)\}$ and this decomposition has a finite character, i.e., if $0 \neq x \in D$ then x is a unit of D_P for all but finitely many $P \in D_M(D)$.

Theorem 5 ([18]). *Let $\{D_\alpha\}$ be a family of domains such that $D \subseteq D_\alpha \subseteq K$ where K is the quotient field of D. If $D = \cap_\alpha D_\alpha$, then $D^\star \subseteq \cap D^\star_\alpha$; equality holds if $\{D_\alpha\}$ has a finite character.*

In the remaining section, we will prove that the conjecture is true by imposing some additional conditions on a Mori domain.

Definition 1. A domain D with quotient field K is called pseudo-valuation domain if P is a prime ideal of D, for $x, y \in K, xy \in P \Rightarrow x \in P$ or $y \in P$.

Theorem 6. *Let (D, M) be a quasi-local one dimensional Mori domain with $(M : M)$ a pseudo valuation domain. Then $D^\star$is a Krull domain.*

Proof. By ACC conditions on divisorial ideals, a maximum divisorial ideal, say, M' exists. By theorem 1, $D_M(D) = \text{Max}(D) \Rightarrow M' = M$. Then by theorem 2, there are two cases to be considered:

Case 1. $M \in I(D)$. In this case, $D = D_M$ is a discrete valuation ring [4, Theorem 2.5].

Case 2. $M \in S(D)$. In this case, $M^{-1} = (M : M)$ is a Mori domain [5, Cor.11]. Since M is an ideal of D and M^{-1}, $(0) \subset M \subseteq (D : M^{-1}) = (D : (M : M)) \Rightarrow D^\star = (M^{-1})^\star = (M : M)^\star$. But then $(M : M)$ is a pseudo-valuation domain implies that $D^\star = (M^{-1})^\star = (M : M)^\star$ is a rank one valuation domain and hence it is a Krull domain [16].

Corollary 7. *Let D be a one dimensional Mori domain with $(P : P)$ a pseudo-valuation domain for every $P \in Spec(D)$. Then $D^\star$ is a Krull domain.*

Proof. Let $\{P_i : i \in I\} = Spec(D)$ with $htP = 1$. Since D is a Mori domain, by theorem 3, P_i is divisorial for all i . Since D is one dimensional, it follows that $\{P_i : i \in I\} = D_M(D)$, the set of maximal ideals of D . But then $D = \cap_i D_{P_i}$ by theorem 4. Consequently, since $D = \cap_i D_{P_i}$ has a finite character, using theorem 5, $D^\star = \cap D^\star_{P_i}$. Hence $D^\star$ is a Krull domain since $D^\star_{P_i}$ is a Krull domain by theorem 6 and an intersection of Krull domains is a Krull domain.

Theorem 8 ([12]). *Let D be a domain. Then $D \subseteq T$ satisfies Lying over (LO) for each proper non-trivial over ring T of D iff D is quasi local and* $\dim D \leq 1$.

Theorem 9 ([9]). *For a domain D with quotient field K, the following are equivalent:*

1). D is quasi local i- domain and $\dim D \leq 1$.
2). (D, T) is a survival pair for each ring T such that $D \subset T \subseteq K$.
3). (D, T) is a Lying over pair for each ring T such that $D \subset T \subseteq K$.

Remark 10. For a one dimensional quasi- local domain D, theorem 8 implies part (3) of theorem 9. But then, since D is an i-domain, i.e., $\bar{D}$ is a rank one valuation domain and hence completely integrally closed [17]. Consequently, $D^{\star\star} = D^{\star}$.

To see this,

$$D \subseteq \bar{D} \subseteq D^{\star} \Rightarrow D^{\star} \subseteq (\bar{D})^{\star} = \bar{D} \subseteq D^{\star} \Rightarrow D^{\star\star} \subseteq (\bar{D})^{\star} = \bar{D} \subseteq D^{\star} \Rightarrow D^{\star\star} = D^{\star}$$

Lemma 11. *If (D, M) is one dimensional quasi-local Mori domain, then $D^{\star}$ is a Krull domain.*

Proof. Since D is a Mori domain and M is a maximal divisorial ideal of D, the following two cases could be considered:

Case 1. $M \in I(D)$, the set of invertible ideals of D. In this case, $D = D_M$ is a discrete valuation ring, DVR [4], and hence a Krull domain.

Case 2). $M \in S(D)$. But then M^{-1} is a Mori domain [5].

If $Spec(M^{-1}) \cap S(M^{-1}) = \emptyset$, then M^{-1} is completely integrally closed Mori domain [8]. Consequently, M^{-1} is a Krull domain [5].

But then, by remark 10, $(0) \subset M \subseteq (D : M^{-1}) = (D : (M : M)) \Rightarrow D^{\star} = (M^{-1})^{\star} = (M : M)^{\star}$ is a Krull domain being a completely integrally closed Mori domain.

On the other hand, if $Spec(M^{-1}) \cap S(M^{-1}) \neq \emptyset$, then we proceed as follows: Since the pseudo radical of D is M, the pseudo radical of $D^{\star}$ is non-zero. Furthermore, $D^{\star} = \bar{D^{\star}}$ [17] implies $D^{\star}$ is integrally closed and hence semi normal domain. Consequently, using [7], $D^{\star\star} = (P_1 : P_1)$ where $P_1 := \cap_{P \in Spec(D^{\star})} P$. Using remark 10, $D^{\star\star} = D^{\star}$. Hence, it is sufficient to show that $D^{\star}$ is a Mori domain.

Recall that $M \in S(D)$ and $D^{\star} = \cup\{I^{-1} : I \in S(D)\}$. Clearly, $I^{-1} = (D : I) = (M : I), \forall I \in S(D), MD^{\star} \subseteq M$, i.e., M is an ideal of $D^{\star}$. It remains to show that M is a prime ideal of $D^{\star}$. To see this, let $r_1^{\star}, r_2^{\star} \in D^{\star}$ such that $r_1^{\star} \cdot r_2^{\star} \in M$. If $r_i^{\star} \in D$, for $i = 1, 2$, then we are done. If not, $\exists 0 \neq d_i \in D$ such that $d_i(r_i^{\star})^n \in D, \forall n > 0$. In particular, $d_1 r_1^{\star} \in D$, $d_2 r_2^{\star} \in D$.

Notice that $d_1 r_1^{\star} d_2 r_2^{\star} \in M \Rightarrow d_1 r_1^{\star} \in M$ or $d_2 r_2^{\star} \in M$.

In a Mori domain, every divisorial prime ideal has the form $(a) : b$, for suitably chosen $a, b \in D$ [20].

Consequently, $\exists a$, $b \in D$ such that $M = (a) : b = (a) : (b)$. But then, with out loss of generality, we may assume that $d_1 r_1^{\star} \in M = (a) : (b) \Rightarrow d_1 r_1^{\star} b \in (a) \Rightarrow r_1^{\star} \in M = (a) : (b) = (a) : (bd_1)$.

To see this, we proceed as follows: $b | bd_1 \Rightarrow (bd_1) \subseteq (b) \Rightarrow (a) : (b) \subseteq (a) : (bd_1)$. Notice that $(a) : (bd_1)$ is a divisorial ideal and M is a maximal ideal implies that $M = (a) : (b) \subseteq (a) : (bd_1) \subseteq M$. Hence M is a prime ideal of D.

On the other hand,

$$M \subseteq P_1 := \cap_{(0)\neq P\in Spec(D^\star)} P \Rightarrow D^{\star\star} = (P_1 : P_1) \subseteq (M : M) = M^{-1} \subseteq D^\star \subseteq D^{\star\star}$$

by [7] and [14], respectively.

But then $D^\star$ is a Krull domain being a completely integrally closed Mori domain.

Theorem 12. *The complete integral closure of a one dimensional Mori domain is a Krull domain.*

Proof. Let D be a one dimensional Mori domain with $Spec(D) = \{P_i\}$. Using [4], we have $D = \cap_i D_{P_i} \Rightarrow D^\star = (\cap_i D_{P_i})^\star = \cap_i {D_{P_i}}^\star$ by [18] and hence $D^\star$ is a Krull domain being an intersection of Krull domains by lemma 11.

Definition 2. An integral domain D is an AV- domain if for every $P \in Spec(D)$, $PD_P = P$.

Theorem 13. *Let D be an AV- Mori domain of dimension $n > 1$ with a maximal ideal M. Then $D^\star$ is a Krull domain.*

Proof. Since (D, M) is a quasi-local Mori domain [1] of dimension $n > 1$, $M \in S(D)$ by [15].

Clearly, $(0) \subset M \subseteq (D : M^{-1}) = (D : (M : M)) \Rightarrow D^\star = (M^{-1})^\star = (M : M)^\star$.

Case i. If $Spec(M^{-1}) \cap S(M^{-1}) = \emptyset$, then M^{-1} is completely integrally closed [8]. Furthermore, $M^{-1}$is a Mori domain [5] implies that $D^\star = M^{-1}$ is a Krull domain.

Case ii. If $Spec(M^{-1}) \cap S(M^{-1}) \neq \emptyset$, then $\exists \bar{P} \in Spec(M^{-1}) \cap S(M^{-1})$. Since $\dim D > 1$, we can assume that there exist a chain of prime ideals $(0) \subset P_1 \subset P_2 \subset P_3 \subset \ldots \subset P_n = M \subset D$ with $P_i \neq M$, for $i \neq n$.

Since there exists a one to one correspondence between $\{\bar{P} | \bar{P} \in Spec(M^{-1}),\ \bar{P} \neq M\}$ and $\{P | P \in Spec(D),\ P \neq M\}$, we can assume that $\bar{P} = (P_i : M)$, for some $i \neq n$. In such a case, the above correspondence is depicted as follows: $\bar{P} = (P_i : M) \leftrightarrow P_i = D \cap (P_i : M)$, for $i \neq n$ and $M^{-1}_{\bar{P}} = D_{P_i}$ with $\dim D_{P_i} < n$ [13, 22].

We will prove the theorem by induction. The case $n = 1$ follows from theorem 12; Suppose the assertion is true for Mori domain of dimension $< n$.

Since D is an AV-domain, for each $P_i \in Spec(D)$, $P_i D_{P_i} = P_i$ and $(0) \subset P_i \subseteq (D : D_{P_i}) \Rightarrow D^\star = D^\star_{P_i}$, for $i \neq n$.

Consequently, by induction assumption, $D^\star$ is a Krull domain.

V. Barucci has proved that the complete integral closure of a seminormal domain is a Krull domain [6].

Using the idea of conductors, we give a simplified proof to show that the complete integral closure of a semi normal Mori domain having a non zero pseudo radical is a Krull domain.

Lemma 14. *If (D, M) is one dimensional quasi-local semi normal Mori domain, then $D^\star$ is a Krull domain.*

Proof. Since D is quasi-local, the pseudo-radical of D is M. Notice that if M is invertible, i.e., $M \in I(D)$, then $D = D_M$ is a discrete valuation ring, DVR, and hence a Krull domain [4].

On the other hand, if $M \in S(D)$, using [7], $D^\star = (M : M) = M^{-1}$ is a Mori domain [5]. Furthermore,$(0) \subset M \subseteq (D : D^\star) \subseteq (D : M^{-1}) \Rightarrow D^\star = D^{\star\star} = M^{-1}$.

Hence $D^\star$ is a Krull domain being completely integrally closed Mori domain.

Theorem 15. *Let D be a one dimensional semi normal Mori domain, then $D^\star$ is a Krull domain.*

Proof. The proof follows as in theorem 12 and lemma 14.

Theorem 16. *Let D be a semi normal Mori domain with a non zero pseudo radical. then $D^\star$ is a Krull domain.*

Proof. Since $D^\star$ and D are semi normal [11], and $I = I^\star$ [10], where $(0) \neq I := \cap_{P_i \in Spec(D)} P_i$ and $(0) \neq I^\star := \cap_{P_i^\star \in Spec(D^\star)} P_i^\star$ by [7], we have $D^{\star\star} = (I^\star : I^\star) = (I : I) = D^\star$.

Considering I, if $P_i \in S(D)$, $\forall P_i \in Spec(D)$, then $I \in S(D)$ [8]. Consequently, $D^{\star\star} = (I^\star : I^\star) = (I : I) = I^{-1} = D^\star$ implies that $D^\star$ is a Krull domain being completely integrally closed Mori domain.

On the other hand, if $\exists P \in Spec(D)$ such that $P \notin S(D)$, then D_P is a discrete valuation ring [15]. But then,D_P being a valuation domain, $PD_P = P$ implies that $D_P = (P : P)$ and hence $(P : P)$ is a rank one valuation ring.

Consequently, since P is a common ideal of D, D_P and $(P : P)$, we have $D^{\star\star} = D^\star = D_P^\star = (P : P)^\star = (P : P)$ implies that $D^\star$ is a Krull domain.

In the above discussion, it can be seen that $(D : D^\star) \neq (0)$. For further investigation, One can consider the case, if the conjecture is valid for Mori domains with non-zero pseudo-radical ideal.

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