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A Remark on Interpolation on Finite Open Riemann Surfaces

Morisuke HASUMI*

Let Ω be a region on a Riemann surface R whose boundary $\partial\Omega$ consists of finitely many analytic simple closed curves $\gamma_1, \dots, \gamma_m$, with $\Omega \cap \partial\Omega$ compact. Such an Ω is usually known as a finite open Riemann surface. Let $\{z_k\}$ be a sequence of distinct points on Ω. Then it is called an interpolating sequence if, for any bounded sequence $\{c_k\}$ of complex numbers, there exists a bounded holomorphic function h on Ω such that $h(z_k)=c_k$ for all k. The following theorem has been proved by Stout [2; 3].

THEOREM. *For $j=1, \dots, m$, let γ'_j be a simple closed curve in Ω which, together with γ_j, bounds an annular region A_j in Ω, such that the A_j have disjoint closures. Then a discrete sequence $E=\{z_k\}$ in Ω is an interpolating sequence if and only if $E_j=A_j \cap E$ is an interpolation set for A_j for each j.*

The purpose of this note is to derive this theorem from some recent results of Hoffman [1]. The proof is short but it is not satisfactory because of the quite heavy machinary on which it is based.

Let $H^\infty(\Omega)$ be the usual Banach algebra of bounded holomorphic functions on Ω and $M(H^\infty(\Omega))$ the maximal ideal space of $H^\infty(\Omega)$. It is shown in [1] the following: (1) A discrete sequence S in Ω is interpolating if and only if its closure in $M(H^\infty(\Omega))$ contains only those points p in $M(H^\infty(\Omega))$ which belong to non-trivial Gleason parts for $H^\infty(\Omega)$. (2) A point p in $M(H^\infty(\Omega))$ belongs to a trivial Gleason part if and only if, for any $f \in H^\infty(\Omega)$ with $\hat{f}(p)=0$, there exist $g, h \in H^\infty(\Omega)$ such that $\hat{g}(p)=\hat{h}(p)=0$ and $f=gh$.

In order to prove the theorem, suppose first that E is an interpolating sequence in Ω. Then $E_j=E \cap A_j$ for each j is also interpolating in A_j.

Suppose conversely that each E_j is interpolating in A_j. By taking smaller A_j if necessary, we may assume all γ_j' to be analytic, and this means that the closure of each E_j in $M(H^\infty(A_j))$ contains only those points which belong to non-trivial Gleason parts for $H^\infty(A_j)$. Now we suppose, on the contrary, that E is not interpolating in Ω. Then its closure in $M(H^\infty(\Omega))$ contains at least one p_0 belonging to some trivial Gleason part of $M(H^\infty(\Omega))$. It is easy to see that p_0 is already contained in the closure in $M(H^\infty(\Omega))$ of some E_j, because p_0 lies off Ω and thus deletion of any finite number of points from E does not affect the situation. So we may assume that E_1 contains a

*Department of Mathematics, Ibaraki University, Mito, Japan

Cauchy net N in $M(H^\infty(\Omega))$ converging to p_0 and that N converges in R to a point $\lambda_0\in\partial\Omega$ and is contained in a coordinate disk U in R around λ_0 with $U\cap\Omega\subset A_1$. On the other hand, the net N has an accumulation point q_0 in the space $M(H^\infty(U\cap\Omega))$. Let f be any function in $H^\infty(U\cap\Omega)$ that satisfies $\hat{f}(q_0)=0$. As Hoffman [1] remarks, there is a factorization $f=f_1f_2$ where $f_1\in H^\infty(\Omega)$ and $f_2\in H^\infty(U\cap\Omega)$ is analytic on $U\cap\Omega$ with $f_2(\lambda_0)=1$. Then we have $\hat{f}(q_0)=0=\hat{f}_1(q_0)\hat{f}_2(q_0)=\hat{f}_1(q_0)f_2(\lambda_0)=\hat{f}_1(q_0)$. So $\hat{f}_1(p_0)=\lim_{\omega\in N}f_1(\omega)=\hat{f}_1(q_0)=0$. As p_0 belongs to a trivial Gleason part for $H^\infty(\Omega)$, there exist g_1, $h_1\in H^\infty(\Omega)$ such that $\hat{g}_1(p_0)=\hat{h}_1(p_0)=0$ and $f_1=g_1h_1$ on Ω. As we have also $\hat{g}_1(q_0)=\lim_{\omega\in N}g_1(\omega)=\hat{g}_1(p_0)=0$ and similarly $\hat{h}_1(q_0)=0$, we see that the point q_0 belongs to a trivial Gleason part for $H^\infty(U\cap\Omega)$. So $E_1\cap U$ cannot be an interpolating sequence in $U\cap\Omega$. Hence E_1 cannot be interpolating in the larger domain A_1, contradicting our original assumption.

References

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