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ABSTRACT. The object of our research is a *piecewise Riemannian 2-polyhedron* which is a combinatorial 2-polyhedron such that each 2-simplex is isometric to a triangle bounded by three smooth curves on some Riemannian 2-manifold. In the previous paper [4], which is a joint work with J. Itoh, we have introduced the concept of *total curvature* for piecewise Riemannian 2-polyhedra and proved a generalized Gauss-Bonnet theorem and a generalized Cohn-Vossen theorem. In this paper, we shall give a definition of flatness of piecewise Riemannian 2-polyhedra and characterize them.

§1. Introduction.

"Curvature" is one of the most important tools to investigate "Geometry" of manifolds. For our research object "polyhedra," the concept of "Curvature" has been introduced and remarkable results are obtained by Banchoff [3] for any dimensional compact piecewise linear polyhedra, and by Ballman-Brin [1] and Ballman-Buyalo [2] for 2-dimensional cocompact piecewise Riemannian polyhedra. My interest is based particularly on the study of noncompact case from the view point of total curvature. In our previous paper [4] with J. Itoh, we have defined two kinds of total curvature for noncompact piecewise Riemannian 2-polyhedra, total curvature and weak total curvature, which both coincide with the usual definitions for Riemannian manifolds or compact 2-polyhedra. It is naturally and easily seen that a generalized Gauss-Bonnet theorem holds under these total curvatures. Furthermore, in [4], we have shown the difference between the geometric meanings of these two kinds of total curvature, and under the assumption of admitting total curvature (not weak total curvature) we have proved a generalized Cohn-Vossen theorem.

The aim of my research is to clarify the meaning of "Curvature" of polyhedra and characterize them in terms of curvature. In this paper, as a first step of this research direction, we shall define the flatness of polyhedra and classify

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flat polyhedra. Namely, under this assumption of flatness, we can obtain the following theorems.

Theorem 1. *If X is a flat 2-polyhedron, then X is a Cat(0)-domain.*

Theorem 2. *Let X be a complete, simply connected, flat 2-polyhedron without boundary. Then X is a product of a tree and a real number* $\mathbf{R}$.

The proofs of Theorem 1 is based only on combinatorial method independent of the results in our previous paper [4], and then we have Theorem 2 by applying Nagano's characterization theorem in [5].

§2. Preliminaries.

In this section, we will review the definitions of a piecewise Riemannian 2-polyhedron and the total curvature of it after [4], because it seems that these concepts are not familiar to you. For the definitions of Cat(0)-domain and so on, refer to the Nagano's paper [5].

First we will introduce the definition of a piecewise Riemannian 2-polyhedron. Let X be a 2-dimensional locally finite simplicial complex. In what follows, we also denote the point-set of union of all the simplices of X, the *polyhedron* of X, by the same symbol X. The metric d on a 2-dimensional polyhedron (simply 2-polyhedron) X is defined as satisfying the following conditions, which is called a *piecewise Riemannian* metric on X.

For each 2-simplex Δ, we take a metric d_Δ on it such that (Δ, d_Δ) is isometric to some triangle bounded by a piecewise smooth simple closed curve on a Riemannian 2-manifold whose broken points are corresponding to vertices of Δ. Here we choose a metric such that the induced metric on a 1-simplex adjacent to some 2-simplices is independent of the choice of adjacent 2-simplices. For each 1-simplex which is not a proper face of any 2-simplex, we may choose any metric. Now we define the metric d by

$$d(x,y) := \inf\{\, l(\gamma) \mid \gamma \text{ is a piecewise smooth curve from } x \text{ to } y\},$$

where $l(\gamma)$ is the length of γ, for any $x, y \in X$.

Definition 1. We call such a space (X, d) a *piecewise Riemannian 2-polyhedron* (simply PR 2-polyhedron).

A piecewise Riemannian 2-polyhedron X is said to be *piecewise linear* (simply PL) if each 2-simplex is isometric to a geodesic triangle on the Euclidean plane $\mathbf{R}^2$.

An i-simplex $\mathcal{E}$ of a polyhedron X is called a free face if there is just one $(i+1)$-simplex of X which contains $\mathcal{E}$ as a face. For a piecewise Riemannian 2-polyhedron X, the closure of the point-set of union of free faces is called the boundary of X and denoted by $\mathcal{B}X$. The complement of it, $X \setminus \mathcal{B}X$, is called the interior of X and denoted by $\mathcal{I}X$. It is clear that these definitions are independent of the choice of divisions of X.

For a point p on a piecewise Riemannian 2-polyhedron X, we denote by $\mathcal{R}_p$ the set of all minimizing geodesics emanating from p. For $\alpha, \beta \in \mathcal{R}_p$ we define the *angle* at p as follows: For an arbitrarily constant k, we denote by $M(k)$ the 2-dimensional space form of constant sectional curvature k. For a geodesic triangle $\Delta(\alpha(s)p\beta(t))$, let $\widetilde{\Delta}(\alpha(s)p\beta(t))$ be a geodesic triangle sketched in $M(k)$ whose corresponding edges have same length as $\Delta(\alpha(s)p\beta(t))$, and let $\widetilde{\angle}_k(\alpha(s)p\beta(t))$ be the angle at p of $\widetilde{\Delta}(\alpha(s)p\beta(t))$. Then it is clear that the limit

$$\angle_p(\alpha,\beta) := \lim_{s,t\to 0} \widetilde{\angle}_k(\alpha(s)p\beta(t))$$

exists, which is independent of the choice of k. We call it the *angle* at p subtended by α and β. It is easily seen that the angle $\angle_p$ is a pseudo-metric on $\mathcal{R}_p$ and induces an equivalence relation $\sim$ defined as follows: $\alpha \sim \beta$ if and only if $\angle_p(\alpha,\beta) = 0$. The completion of the metric space $(\mathcal{R}_p/_\sim, \angle_p)$ is called the *space of directions* at p, and denoted by $(\Sigma_p, \angle_p)$.

For a subset Y of X, let

$$\mathcal{R}_p^Y := \{\alpha \in \mathcal{R}_p \mid \alpha([0,\epsilon]) \subset Y \text{ for some } \epsilon > 0\}.$$

The *space of directions with respect to* Y, denoted by Σ_p^Y, is the completion of the metric space $\mathcal{R}_p^Y/_\sim$.

Next, for a point p on a piecewise Riemannian 2-polyhedron X, we will introduce two curvatures (the *regular curvature* and the *singular curvature*) at p. The *regular curvature* $K(p)$ is defined by

$$K(p) := \begin{cases} \text{the Gaussian curvature at } p & \\ & \text{if } p \text{ is on some open 2-simplex of } X, \\ 0 & \text{otherwise.} \end{cases}$$

For a compact piecewise Riemannian 2-polyhedron X, the *regular total curvature* of X is defined as the integral of the regular curvature K on X and denoted by $e_{reg}(X)$. In other words, it is expressed as follows: Let $C(\Delta)$ be the total curvature of a 2-simplex Δ. Note here that Δ has a Riemannian metric. Then

$$e_{reg}(X) := \sum_{\Delta:\text{2-simplex}} C(\Delta).$$

Now, fix a subdivision of X in which p is a vertex. Then the *singular curvature* $k(p)$ at p is defined by

$$k(p) = \pi(2 - \chi(LK(p))) - L(\Sigma_p),$$

where $\chi(LK(p))$ is the Euler characteristic of the point-set of the linked complex $LK(p)$ of p and L is the 1-dimensional Hausdorff measure on Σ_p. By definition, $LK(p)$ is the sum of simplices σ on X such that the cone with vertex p and base σ is also a simplex of X. It is clear that $k(p) = 0$ if p is not a vertex of the original division of X.

Furthermore we will define the *singular total curvature* of X. For a pair (c, Δ) of a 2-simplex Δ and its face c, $\int_c \kappa d_\Delta$ is defined as the integral of a geodesic curvature κ on c. Then we define the *singular total curvature* $e_{sing}(X)$ of X by

$$e_{sing}(X) := \sum_{p \in \mathcal{I}X} k(p) + \sum_{(c,\Delta)} \int_c \kappa d_\Delta,$$

where the summation of the second term is taken over all pairs (c, Δ) of an open 1-simplex $c \subset \mathcal{I}X$ and a 2-simplex Δ adjacent to c.

Now we define the total curvature as follows.

Definition 2. The total curvature $C(X)$ is defined by

$$C(X) := e_{reg}(X) + e_{sing}(X).$$

Then we have the following generalized Gauss-Bonnet theorem.

Theorem A(Theorem 3.1 in [4]). *Let X be a compact piecewise Riemannian 2-polyhedron. Then we have*

$$C(X) + \sum_{p \in \mathcal{B}X} k(p) + \int_{\mathcal{B}X} \kappa d_X = 2\pi\chi(X),$$

where $\chi(X)$ is the Euler characteristic of X.

Finally we will introduce the total curvature of a noncompact complete piecewise Riemannian 2-polyhedron X.

Definition 3. Let $\{D_i\}$ be an increasing sequence of compact subpolyhedra of X such that $\cup D_i = X$. If a limit $\lim_{i\to\infty} C(D_i)$ exists on $[-\infty, \infty]$ and is independent of the choice of $\{D_i\}$, then it is called the *total curvature* of X and is denoted by $C(X)$. If $C(X)$ exists, then X is said to admit total curvature.

Definition 4. A noncompact piecewise Riemannian 2-polyhedron X is said to be *finitely connected*, if it is homeomorphic to a compact piecewise Riemannian 2-polyhedron $\widetilde{X}$ with finitely many points $\{p_1, \dots, p_n\}$ removed.

For a finitely connected 2-polyhedron X as above, let L_i be the point-set of the linked complex $LK(p_i)$ of a removed point p_i on $\widetilde{X}$. We may assume that $L_i \cap L_j = \emptyset$ for $i \neq j$ by taking a subdivision if necessary. Let X_∞ be the disjoint union of $\{L_i\}$. Then there is a large compact set D on X such that $X \setminus D$ is homeomorphic to $X_\infty \times \mathbf{R}$. Since X is homotopic to D, we define the Euler characteristic $\chi(X)$ as $\chi(D) = \chi(\widetilde{X}) - n + \chi(X_\infty)$. Note that $\widetilde{X}$ is a finite polyhedron but X is not so, that is, the structure of X as a polyhedron is quite different from that of $\widetilde{X}$. Now, we have the following theorem of a Cohn-Vossen type.

Theorem B(Theorem 4.1 in [4]). *Let X be a finitely connected noncompact complete piecewise Riemannian 2-polyhedron without free faces admitting total curvature. Then we have*

$$C(X) \leqq 2\pi\chi(X) - \pi\chi(X_\infty).$$

§3. Characterization of flat 2-polyhedra.

It is clear that the total curvature $C(X)$ of a non-compact piecewise Riemannian 2-polyhedron X is finite if and only if for any $\epsilon > 0$, there exists a compact subpolyhedron K of X such that $|C(X) - C(Y)| < \epsilon$ for any compact subpolyhedron $Y \supset K$. That is the regular curvature $K(p)$ and the singular curvature $k^E(p)$ at any point $p \in X \setminus K$ and subpolyhedron E of X with $p \in \mathcal{I}E$ are nearly zero, and the geodesic curvatures on essential edges are also nearly zero, where an essential edge means a 1-simplex adjacent to more than three 2-simplices. It is so complicated to characterize the structure of finite total curvature. Now, in a Riemannian case, this condition means asymptotically flatness. Motivated this observation, we shall define a flatness as follows and as a first approach we will characterize in the following simple case.

Definition 5. Let X be a PR 2-polyhedron. If X is a piecewise linear 2-polyhedron, and if, for any point $x \in \mathcal{I}X$ and any subpolyhedron E of X with $x \in \mathcal{I}E$, the singular curvature $k^E(x)$ of E at x is equal to 0, then X is said to be *flat.*

We shall note that a flat 2-polyhedron is not necessarily finitely connected. For example, let X be a PR 2-polyhedron consisting of a flat planes $\{(x_1, x_2, x_3) \in \mathbf{R}^3 \,|x_3 = 0\}$ and $\{(x_1, x_2, x_3) \in \mathbf{R}^3 \,|x_2 \in \mathbf{N}\}$. Then X is an infinitely connected flat 2-polyhedron. In this section, we will give a classification of flat 2-polyhedra. To begin with, as a local structure, we have the following

Lemma 1. *Let X be a PR 2-polyhedron and $x \in \mathcal{I}X$. If, for any subpolyhedron E of X with $x \in \mathcal{I}E$, the singular curvature $k^E(x)$ of E at x equals to 0, then each connected component of the space of directions Σ_x contains an embedded circle S, and the length of any embedded circle is equal to π.*

Proof. Fix a subdivision of X on which x is a vertex. Since $x \in \mathcal{I}X$, each connected component of the linked complex $LK(x)$ contains an embedded circle S_0. Let $E = S_0 * x$ be a cone with vertex x and base S_0, and S be the space of directions of E at x. It is clear that S is either homeomorphic to a circle or one point set. Then the singular curvature $k^E(x)$ of E at x is satisfies

$$k^E(x) = (2 - \chi(S_0))\pi - L(S) = 0,$$

and hence, since $\chi(S_0) = 0$, we have $L(S) = 2\pi$. That is, S is isometric to a standard circle. □

Furthermore, from this property, we can make the shape of the space of directions clear. For any natural number n, let θ_n be the bipartite graph consisting of two vertices and n edges of length π which connect the two vertices. Naturally θ_2 is a standard circle.

Proposition 1. *If X is a flat PR 2-polyhedron, for any point $x \in \mathcal{I}X$ there is a natural number $n \geqq 2$ such that the space of directions Σ_x of X at x is θ_n.*

Proof. Let $Sing(X)$ be the set of all points $x \in X$ such that any neighborhood of x is not homeomorphic to an open disk. Note that $Sing(X) \supset \mathcal{B}X$. If $x \in X \setminus Sing(X)$, then the space of direction Σ_x is isometric to a standard circle S^1. So we assume that $x \in Sing(X) \cap \mathcal{I}X$. In this case we will divide our observation in three steps. First step is to show the following

claim 1. Σ_x is connected.

In fact, we assume that there are two distinct connected components on Σ_x. Then we can take disjoint circles S_1 and S_2 such that S_1 is on one component and S_2 is on the other. Note that $L(S_1) = L(S_2) = 2\pi$ by Lemma 1. Let E_1, E_2 be subpolyhedra of X such that $\Sigma_x^{E_i} = S_i$. Naturally we may assume that E_i is homeomorphic to a closed disk, and then a subpolyhedron $E := E_1 \cup E_2$ is homeomorphic to a suspension of $S_1 \cup S_2$. Hence we have

$$k^E(x) = (2 - \chi(LK^E(x)))\pi - L(\Sigma_x^E) = -2\pi \neq 0,$$

which is a contradiction.

Next we shall consider two distinct circles S_1, S_2 on Σ_x.

claim 2. $S_1 \cap S_2$ is either two antipodal points or a closed segment with length π. That is, $S_1 \cup S_2$ is either θ_3 or θ_4.

In fact, if $S_1 \cap S_2 = \emptyset$, then a contradiction is led in the same way as claim 1. If $S_1 \cap S_2$ is connected, then we can show that it is not one point set but a segment of length π as follows.

Let E be a subpolyhedron such that $\Sigma_x^E = S_1 \cup S_2$, and then, we may assume E is homeomorphic to a cone of $S_1 \cup S_2$. If $S_1 \cap S_2$ is one point set, then $k^E(x) = -\pi \neq 0$, a contradiction. Hence it is a segment. Let $e := S_1 \cap S_2$, and e_i be the closure of $S_i \setminus e$ $(i = 1, 2)$. Then $L(S_i) = L(e) + L(e_i) = 2\pi$. Furthermore, since $e_1 \cup e_2$ is also an embedded circle on Σ_x, $L(e_1) + L(e_2) = 2\pi$. Therefore we have $L(e) = L(e_1) = L(e_2) = \pi$.

If $S_1 \cap S_2$ has more than one components, then it is two points set. In fact, for any two components e_1 and e_2, $S_i \setminus (e_1 \cup e_2)$ also consists of just two connected components, whose closures are denoted by f_i, g_i $(i = 1, 2)$. First we will assume that either $\partial f_1 = \partial f_2$ or $\partial f_1 = \partial g_2$. It is sufficient to consider the former case. Then $f_1 \cup f_2$ and $g_1 \cup g_2$ are also isometric to a standard circle. Hence we have $L(f_1) = L(g_1) = L(f_2) = L(g_2) = \pi$ and $L(e_1) = L(e_2) = 0$. This means e_1 and e_2 are the antipodal points of S_1 and of S_2. Otherwise, there are four embedded standard circles except S_1 and S_2. Namely we have

$$L(S_1) = L(f_1) + L(g_1) + L(e_1) + L(e_2) = 2\pi, \tag{1}$$

$$L(S_2) = L(f_2) + L(g_2) + L(e_1) + L(e_2) = 2\pi, \tag{2}$$

$$L(f_1) + L(f_2) + L(e_i) = 2\pi, \tag{3}$$

$$L(f_1) + L(g_2) + L(e_j) = 2\pi, \tag{4}$$

$$L(g_1) + L(f_2) + L(e_j) = 2\pi, \tag{5}$$

$$L(g_1) + L(g_2) + L(e_i) = 2\pi, \tag{6}$$

for some $\{i,j\} = \{1,2\}$. We may assume that $i = 1$ and $j = 2$. Then adding up (3) + (6) − {(1) + (2)}, we have $L(e_2) = 0$. In the same way, $L(e_1) = 0$ and $L(f_1) = L(g_1) = L(f_2) = L(g_2) = \pi$. Therefore in any case, the number of connected components is just two, and they are the antipodal points of S_1 and of S_2.

Now we will assume that there are three different embedded circles S_1, S_2 and S_3. By Claim 2, there is the unique pair of points $\{v_i, w_i\} \subset S_i \cap S_3$ with $\angle_x(v_i, w_i) = \pi$ $(i = 1,2)$. Let $\overline{v_i w_i}$ be the half segment on S_i satisfying $\overline{v_i w_i} \cap S_3 = \{v_i, w_i\}$. If $\{v_1, w_1\} \neq \{v_2, w_2\}$, then we can take a embedded circle containing half segments $\overline{v_1 w_1}$ and $\overline{v_2 w_2}$, whose length is greater than 2π. This is a contradiction, and hence $\{v_1, w_1\} = \{v_2, w_2\}$. By induction, this completes the proof. □

This proposition immediately leads us the following theorems.

Theorem 1. *If X is flat, then X is a Cat(0)-domain.*

Proof. For any point $x \in \mathcal{I}X$, by Proposition 1, there is a number n such that $\Sigma_x = \theta_n$. Since X is a PL-polyhedron, this means that a sufficiently small neighborhood U of x is isometric to an n flat open half disks D_n identified the boundaries. Naturally D_2 is a standard open disk. It is clear that D_n is of nonpositive curvature (in the meaning of a comparison theorem), and hence X is a Cat(0)-domain. □

Theorem 2. *Let X be a complete, simply connected, flat 2-polyhedron without boundary. Then X is a product of a tree and a real number* $\mathbf{R}$.

Proof. Since X is a complete and simply connected Cat(0)-domain, then X is a Hadamard space. Furthermore the diameter of the space of directions at any point $x \in X$ is π by Proposition 1. This implies that the diameter of the ideal boundary $X(\infty)$ of X is also π. In fact, if not, there are two rays γ_1 and γ_2 such that $Td(\gamma_1(\infty), \gamma_2(\infty)) > \pi$, where Td is a Tits metric on $X(\infty)$. Then we can take a point $p \in X$ such that $\angle_p(\tilde{\gamma}_1'(0), \tilde{\gamma}_2'(0)) > \pi$, where $\tilde{\gamma}_i$ is an asymptotic ray to γ_i emanating from p. This is a contradiction. Applying Nagano's characterization theorem (Theorem A in [5]) of Hadamard space X with $diam(X(\infty)) = \pi$ and noting Proposition 1, we have X is a product of a tree and $\mathbf{R}$. □

Remark 1. Nagano's characterization theorem (Theorem A in [5]) is stated as follows: Let X be a locally compact, geodesically complete Hadamard 2-space such that the diameter of $(X(\infty)), Td)$ is equal to π. Then X is isometric to either the product of two trees, the Euclidean cone over $(X(\infty)), Td)$, or a thick Euclidean building of dimension 2 of type A_2, B_2, or G_2.

Remark 2. It is notable that if X is a Hadamard manifold with $diam(X(\infty)$, $Td) = \pi$, it is not necessarily flat in our sense and we might have a point with positive singular curvature. We shall illustrate such an example. Let T_n be a tree consisting of n edges, n free vertices and one essential vertex. Since each edge is adjacent to the essential vertex, we call it a central vertex of T_n. Let X be a product $T_n \times T_m$ and $v := (v_n, v_m) \in X$, where v_n is a central vertex of T_n.

Then X is a PL 2-polyhedron and the singular curvature $k(v)$ at v is equal to $(n-2)(m-2)\pi/2$. Therefore if $n, m > 2$, then $k(v)$ is positive.

So I wonder that the concept of singular curvature at a point is not compatible with the concept of curvature from the point of view of comparison theorem, but it seems natural as a concept of curvature to take the information of the topological structure in consideration.

§4. Appendix.

I am wondering that the definition of flatness in Section 2 was too strong. In our definition, as seen in Lemma 1, any embedded circle of a space of directions at any point has a length of 2π and is not allowed taking a side way. Even if the diameter of a space of directions of Hadamard space is equal to π, the length of an embedded circle is not necessarily 2π and there may exist some longer embedded circles. From this observation, it may be more suitable that polyhedra is defined to be flat if the diameter and the injective radius of the space of directions at any point is equal to π, instead of the condition concerning a singular curvature. Under such a definition of flatness, it is easily seen that a flat PR 2-polyhedron is a Hadamard space such that the diameter of its ideal boundary is equal to π, and hence the classification of flat polyhedra is the same as Nagano's classification of Hadamard 2-spaces.

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