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Stability of Tridiagonal Systems of Linear Equations

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1. **Introduction.** The purpose of this note is to give some sufficient conditions for the stability of tridiagonal systems of linear equations. This study was suggested by a recent paper by T. Torii [2]. We consider a one-parameter family of systems of linear equations

(1) $$A(h)x(h)=f(h) \qquad (0<h<h_0)$$

where $x(h)$ and $f(h)$ denote real (or complex) n-dimensional column vectors and $A(h)$ denotes an $n\times n$ tridiagonal matrix

(2) $$A(h)=\begin{pmatrix} b_1(h) & c_1(h) & & & & 0 \\ a_2(h) & b_2(h) & c_2(h) & & & \\ & a_3(h) & b_3(h) & c_3(h) & & \\ & & \cdot & \cdot & \cdot & \\ & & & \cdot & \cdot & \cdot \\ & & & a_{n-1}(h) & b_{n-1}(h) & c_{n-1}(h) \\ 0 & & & & a_n(h) & b_n(h) \end{pmatrix}.$$

The dimension n depends on the parameter h. We suppose that

(3) $$n(h) \to \infty \qquad \text{as } h \to 0.$$

Such a family will appear when we want to solve certain differential equations by finite difference methods. In that case the parameter h will stand for the mesh size. We may consider a family with a finite number of parameters but a similar treatment is possible. We assume that $A(h)$ is inversible for each $0<h<h_0$ and furthermore that

(4) $$\sup_{h>0} \| A(h) \| < +\infty.$$

Here the norm $\| T \|$ of an $n\times n$ matrix $T=(t_{ij})$ is given by

(5) $$\| T \| = \max_{1\leq i\leq n} \sum_{j=1}^{n} | t_{ij} | .$$

In accordance with Torii [2], we say that the tridiagonal systems (1) (or, more precisely, the family, $A(h)$) are stable if

(6) $$\sup_{h>0} \| A(h)^{-1} \| < +\infty.$$

Torii [2] has given a formula for expressing $\| A(h)^{-1} \|$ in terms of fundamental solutions of certain homogeneous system associated with the matrix $A(h)$ and discussed conditions under which (6) holds. Our conditions for (6) given in the present paper are expressed in terms of the coefficients $a_k(h)$, $b_k(h)$ and $c_k(h)$

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themselves. Therefore our conditions may be easier to handle than Torii's. Here we shall not discuss unstable cases as well as other problems relating to the capacity of computers.

2. **Basic observations.** In order to estimate the norm of the matrix $A(h)^{-1}$, we factorize $A(h)$ into two triangular matrices $L(h)$ and $R(h)$ as follows:

(7) $$A(h)=L(h)R(h)$$

where

$$L(h)=\begin{pmatrix} p_1(h) & & & & 0 \\ q_2(h) & p_2(h) & & & \\ & q_3(h) & p_3(h) & & \\ & & \cdot & \cdot & \\ 0 & & & q_n(h) & p_n(h) \end{pmatrix} \text{ and } R(h)=\begin{pmatrix} r_1(h) & s_1(h) & & & 0 \\ & r_2(h) & s_2(h) & & \\ & & \cdot & \cdot & \\ & & & r_{n-1}(h) & s_{n-1}(h) \\ 0 & & & & r_n(h) \end{pmatrix}.$$

Then we have

(8) $$\begin{aligned} a_k(h)&=q_k(h)r_{k-1}(h) && 2\leqq k\leqq n(h),\\ b_1(h)&=p_1(h)r_1(h) && \\ b_k(h)&=p_k(h)r_k(h)+q_k(h)s_{k-1}(h) && 2\leqq k\leqq n(h),\\ c_k(h)&=p_k(h)s_k(h) && 1\leqq k\leqq n(h)-1. \end{aligned}$$

This factorization device is known to be the Gauss method (cf. Henrici [1 ; p. 352]). Putting $L(h)^{-1}=(c_{ij}(h))$ and $R(h)^{-1}=(d_{ij}(h))$, we get, after a simple computation,

$$c_{ij}(h)=\begin{cases} 0 & i<j \\ 1/p_i(h) & i=j \\ (-1)^{i-j}\dfrac{q_{j+1}(h)q_{j+2}(h)\cdots q_i(h)}{p_j(h)p_{j+1}(h)\cdots p_i(h)} & i>j \end{cases}$$

and

$$d_{ij}(h)=\begin{cases} 0 & i>j \\ 1/r_i(h) & i=j \\ (-1)^{j-i}\dfrac{s_i(h)s_{i+1}(h)\cdots s_{j-1}(h)}{r_i(h)r_{i+1}(h)\cdots r_j(h)} & i<j. \end{cases}$$

Since we have $A(h)^{-1}=R(h)^{-1}L(h)^{-1}$ and therefore $\|A(h)^{-1}\|\leqq \|R(h)^{-1}\|\ \|L(h)^{-1}\|$, the condition (6) holds if

(9) $$\sup_{h>0}\|L(h)^{-1}\|<+\infty$$

and

(10) $$\sup_{h>0}\|R(h)^{-1}\|<+\infty.$$

Using the formula (5) for $L(h)^{-1}$, we get

$$\|L(h)^{-1}\|=\max_{1\leqq k\leqq n(h)}(1+u_k(h)+u_k(h)u_{k-1}(h)+\cdots+u_k(h)u_{k-1}(h)\cdots u_2(h))/|p_k(h)|$$

where $u_k(h)=|q_k(h)/p_{k-1}(h)|$ for $2\leqq k\leqq n(h)$. So a sufficient condition for (9) is the following:

(11) $$\inf_{0<h,\ 1\leqq k\leqq n(h)}|p_k(h)|>0 \quad\text{and}\quad \limsup u_k(h)<1$$

where the "lim sup" is taken along the directed set $\{(k, h):\ 0<h<h_0$ and $1\leqq k\leqq n(h)\}$ with the partial ordering $(k, h)\leqq(k', h')$, meaning that $k\leqq k'$ and $h\geqq h'$.

On the other hand, we can write $\| L(h)^{-1} \|$ in the form

$$\| L(h)^{-1} \| = \max_{1 \leqq k \leqq n(h)} (| p_k(h)^{-1} | + u'_k(h) | p_{k-1}(h) |^{-1} + u'_k(h) u'_{k-1}(h) | p_{k-2}(h) |^{-1} + \cdots + u'_k(h) u'_{k-1}(h) \cdots u'_2(h) | p_1(h) |^{-1})$$

where $u'_k(h) = | q_k(h)/p_k(h) |$, so that the second condition in (11) can be replaced by lim sup $u'_k(h) < 1$. A similar observation can be made for $R(h)^{-1}$ and we get the following

THEOREM 1. *The tridiagonal systems* (1) *are stable if any one of the following cases occurs*: (i) (P_1), (P_2), (P_3), (P_4); (ii) (P_1), (P_2), (P_3), (P_4'); (iii) (P_1), (P_2), (P_3'), (P_4); (iv) (P_1), (P_2), (P_3'), (P_4'); *where*

(P_1) $\inf [| p_k(h) | : 0 < h < h_0, 1 \leqq k \leqq n(h)] > 0$,

(P_2) $\inf [| r_k(h) | : 0 < h < h_0, 1 \leqq k \leqq n(h)] > 0$,

(P_3) $\limsup | q_{k+1}(h)/p_k(h) | < 1$,

(P_3') $\limsup | q_k(h)/p_k(h) | < 1$,

(P_4) $\limsup | s_k(h)/r_{k+1}(h) | < 1$,

(P_4') $\limsup | s_k(h)/r_k(h) | < 1$.

In what follows, we shall try to get some conditions on the coefficients $a_k(h)$, $b_k(h)$, $c_k(h)$ themselves which imply one of the four cases mentioned in Theorem 1.

3. Conditions on the moduli of $a_k(h)$, $b_k(h)$ and $c_k(h)$. In this section, we obtain some relations between the moduli of the coefficients $a_k(h)$, $b_k(h)$ and $c_k(h)$ which imply the stability of the systems (1). First we show

THEOREM 2. *If there exist constants* γ, δ *and* ε *such that*

$$0 \leqq \gamma < 1, \quad 0 \leqq \delta < 1, \quad 0 < \varepsilon$$

and, for all $0 < h < h_0$,

(12) $| b_1(h) | \geqq \varepsilon$, $| b_k(h) | - \gamma | c_{k-1}(h) | \geqq \varepsilon$ *for* $2 \leqq k \leqq n(h)$,

(13) $| a_2(h) | \leqq \gamma | b_1(h) |$, $| a_{k+1}(h) | \leqq \gamma (| b_k(h) | - \gamma | c_{k-1}(h) |)$ *for* $2 \leqq k \leqq n(h)$,

(14) $(1 + \gamma\delta) | c_k(h) | \leqq \delta | b_{k+1}(h) |$ *for* $1 \leqq k \leqq n(h)$,

then the systems (1) *are stable and we have*

$$\| A(h)^{-1} \| \leqq \frac{1}{\varepsilon(1-\gamma)(1-\delta)}.$$

Proof. To see this, we put $p_1(h) = p_2(h) \cdots = 1$. This is clearly possible. We first fix an h so that we do not write the parameter h explicitly. We want to show that $| q_k(h) | \leqq \gamma$ for all $2 \leqq k \leqq n(h)$. Clearly, $| q_2 | = | a_2/r_1 | = | a_2/b_1 | \leqq \gamma$ by (13). Supposing $| q_k | \leqq \gamma$ for $k < n(h)$, we have

$$| q_{k+1} | = | a_{k+1} | / | b_k - q_k c_{k-1} | \leqq | a_{k+1} | / (| b_k | - \gamma | c_{k-1} |) \leqq \gamma$$

in view of (12) and (13). By induction we get the desired result. Since $p_k = 1$ for all k, we thus see that (P_3) and (P_3') are true. Moreover, $| r_1 | = | b_1 | \geqq \varepsilon$ and

$$| r_k | = | b_k - q_k c_{k-1} | \geqq | b_k | - | q_k | \, | c_{k-1} | \geqq | b_k | - \gamma | c_{k-1} | \geqq \varepsilon$$

for $2 \leqq k \leqq n(h)$ by (12), so that (P_2) is true. Finally we have

$$|s_k/r_{k+1}| = |c_k/(b_{k+1}-q_{k+1}c_k)| \leqq |c_k|/(|b_{k+1}|-\gamma|c_k|) \leqq \delta$$

for $1 \leqq k \leqq n(h)$ by (14). So we have (P_4). Hence our conditions imply the case (i) of Theorem 1 and therefore the systems (1) are stable. It is easy to see from the estimates for $|q_k|$, $|r_k|$, $|s_k/r_{k+1}|$ that we have the desired bound for $\|A(h)^{-1}\|$. Q. E. D.

THEOREM 3. *If there exist constants* γ, δ *and* ε *such that*

$$0 \leqq \gamma < 1,\ 0 \leqq \delta < 1,\ 0 < \varepsilon$$

and, together with (12) *and* (13),

(15) $\quad |c_k(h)| \leqq \delta(|b_k(h)| - \gamma|c_{k-1}(h)|) \qquad$ *for* $2 \leqq k \leqq n(h)$

is true, then the systems (1) *are stable and the estimate for* $\|A(h)^{-1}\|$ *is the same as in Theorem* 2.

Proof. We see as before that (12) and (13) imply (P_1), (P_2), (P_3) (and (P_3')). Now it follows from (15) that (we again omit h)

$$|s_k/r_k| = |c_k/(b_k - q_k c_{k-1})| \leqq |c_k|/(|b_k| - \gamma|c_{k-1}|) \leqq \delta$$

because $|q_k| \leqq \gamma$. So we have (P_4') and consequently the case (ii) of Theorem 1. Q. E. D.

Similarly we get the following two theorems.

THEOREM 4. *If there exist constants* γ, δ *and* ε *such that*

$$0 \leqq \gamma < 1,\ 0 \leqq \delta < 1,\ 0 < \varepsilon$$

and

(16) $\quad |b_1(h)| \geqq \varepsilon,\ |b_k(h)| - \gamma|a_k(h)| \geqq \varepsilon \qquad$ *for* $2 \leqq k \leqq n(h)$,

(17) $\quad |c_1(h)| \leqq \gamma|b_1(h)|,\ |c_k(h)| \leqq \gamma(|b_k(h)| - \gamma|a_k(h)|) \qquad$ *for* $2 \leqq k \leqq n(h)$,

(18) $\quad (1+\gamma\delta)|a_k(h)| \leqq \delta|b_k(h)| \qquad$ *for* $2 \leqq k \leqq n(h)$,

then the systems (1) *are stable and the estimate for* $\|A(h)^{-1}\|$ *is the same as in Theorem* 2.

THEOREM 5. *If there exist constants* γ, δ *and* ε *such that*

$$0 \leqq \gamma < 1,\ 0 \leqq \delta < 1,\ 0 < \varepsilon$$

and, together with (16) *and* (17),

(19) $\quad |a_{k+1}(h)| \leqq \delta(|b_k(h)| - \gamma|a_k(h)|) \qquad$ *for* $2 \leqq k \leqq n(h)$

is true, then the systems (1) *are stable and the estimate for* $\|A(h)^{-1}\|$ *is the same as in Theorem* 2.

These theorems can be proved by setting $r_1(h) = r_2(h) = \cdots = 1$ and we omit the proof. Obviously, if we assign other non-zero values to $p_1(h)$, $p_2(h)$,... (or to $r_1(h)$, $r_2(h)$, ...) which satisfy (P_1) (or (P_2)), then we get theorems similar to Theorems 2 through 5.

COROLLARY 1. *If* $|a_2(h)| = |a_3(h)| = \cdots = a$, $|b_1(h)| = |b_2(h)| = \cdots = b$, $|c_1(h)| = |c_2(h)| = \cdots = c$, *and if* $a + c < b$, *then the systems* (1) *are stable and we have*

$$\| A(h)^{-1} \| \leqq \frac{1}{b-a-c}\ .$$

Proof. First we suppose that $c \leqq a$. If $c=0$, then we can take $\gamma=a/b$, $\delta=0$ and $\varepsilon=b$ in Theorem 2 so that we get the corollary. Now let $c>0$. If we put $f(t)=ct^2-bt+a$, then $f(0)=a\geqq 0$ and $f(1)=c-b+a<0$ by assumption. So the smaller root of the equation $f(t)=0$, which we call γ_0, satisfies $0 \leqq \gamma_0 < 1$ and the other root, γ_1 say, is larger than 1. We have

$$\gamma_0=(b-(b^2-4ac)^{1/2})/2c$$

and $$\gamma_1=(b+(b^2-4ac)^{1/2}/)2c.$$

If we put

$$\varepsilon_0=b-\gamma_0 c=(b+(b^2-4ac)^{1/2})/2, \text{ and}$$

$$\delta_0=c/(b-\gamma_0 c)=2c/(b+(b^2-4ac)^{1/2}),$$

then $\gamma=\gamma_0$, $\delta=\delta_0$, and $\varepsilon=\varepsilon_0$ satisfy the conditions in Theorem 2 and a simple computation proves the corollary. Q. E. D.

In particular, this corollary can be applied to the case in which $a_2(h)=a_3(h)=\cdots=a$, $b_1(h)=b_2(h)\cdots=b$ and $c_1(h)=c_2(h)=\cdots=c$. Torii [2] has shown, among others, that, if a, b and c are real, and if $|a+c|<|b|$, then the systems (1) are stable. When a and c have the same sign, our corollary covers this result. In the next section, we shall find conditions which will imply Torii's result in case $ac<0$.

Corollary 2. *Suppose that we have two families of tridiagonal matrices $\{A(h)\}$ and $\{A'(h)\}$ with $n(h)=n'(h)$ for all $0<h<h_0$. If $\{A(h)\}$ satisfies the conditions of any one of Theorems 2–5, and if*

$$|a_k'(h)| \leqq |a_k(h)|,\quad |b_k'(h)| \geqq |b_k(h)|,\quad |c'_k(h)| \leqq |c_k(h)|$$

for all possible h and k, then the family $\{A'(h)\}$ is also stable and $\| A'(h)^{-1}\| \leqq \| A(h)^{-1}\|$ for $0<h<h_0$. In particular, if there exist positive numbers a, b, c such that $a+c<b$ and $|a_k(h)|\leqq a$, $|b_k(h)|\geqq b$, $|c_k(h)|\leqq c$, then the family $\{A(h)\}$ is stable and $\| A(h)^{-1}\| \leqq 1/(b-a-c)$ for $0<h<h_0$.

This is trivial and so we omit the proof.

4. **Further conditions for stability**. We begin with the proof of Torii's theorem mentioned in the last section.

Theorem 6 (Torii [2]). *Let $a_2(h)=a_3(h)=\cdots=a$, $b_1(h)=b_2(h)=\cdots=b$, and $c_1(h)=c_2(h)=\cdots=c$. If a, b and c are real, aud if $|a+c|<|b|$, then the systems* (1) *are stable.*

Proof. If $ac\geqq 0$, then Corollary 1 in the last section implies this theorem so that we assume $ac<0$.

We suppose that $a>0$, $b>0$ and $c<0$. Putting $p_1(h)=p_2(h)=\ldots=1$ in (8), we get, by omitting h,

$a=q_k r_{k-1}$ $(2\leqq k)$, $b=r_1$, $b=r_k+cq_k$ $(2\leqq k)$, $c=s_k (1\leq k)$. So if we put $g(t)=a/(b-ct)$, then $q_{k+1}=g(q_k)$ for $k\geqq 2$. It follows easily from our assumption on a, b, c

and $q_2=a>0$ that $q_2, q_3, \ldots$ are all positive and tend to the limit q_∞ that is given by $g(q_\infty)=q_\infty$, i. e., $q_\infty=(b-(b^2-4ac)^{1/2})/2c$. As $0<q_\infty<1$, we see that (P_3) is true. On the other hand, $r_1=b$, $r_k=b-cq_k\geqq b$ for $k\geqq 2$. So (P_2) also holds. The sequence $\{r_k\}$ tends to $r_\infty=b-cq_\infty=(b+(b^2-4ac)^{1/2})/2$. As $-a-c<b$, we see by a simple computation that $|s_k/r_k|=|c|/r_k\to|c|/r_\infty<1$. Thus (P_4) holds and therefore the systems (1) are stable.

A similar argument works for other combinations of signs for a, b and c. This establishes the theorem. Q. E. D.

The following theorem gives a partial generalization of Theorem 6.

THEOREM 7. *Suppose that* $a_k(h)\geqq 0$, $b_k(h)\geqq 0$, $c_k(h)\leqq 0$, *and the following conditions are satisfied*:

(20) $\alpha\equiv\inf[(b_k(h)-(b_k(h)^2-4a_{k+1}(h)c_{k-1}(h))^{1/2})/2c_{k-1}(h): c_{k-1}(h)\neq 0]>0,$

(21) $\beta\equiv\limsup(b_k(h)-(b_k(h)^2-4a_{k+1}(h)c_{k-1}(h))^{1/2})/2c_{k-1}(h)<1$

the limit being taken for such k *as* $c_{k-1}(h)\neq 0$,

(22) $\gamma\equiv\sup[a_{k+1}(h)/b_k(h): c_{k-1}(h)\neq 0]<+\infty,$

(23) $\sigma\equiv\limsup a_{k+1}(h)/b_k(h)<1$

the limit being taken for such k *as* $a_k(h)c_{k-1}(h)=0$,

(24) $\tau\equiv\limsup a_{k+2}(h)b_k(h)/(b_k(h)b_{k+1}(h)-a_{k+1}(h)c_k(h))<1$

the limit being taken for such k *as* $a_k(h)c_{k-1}(h)=0$ *and* $a_{k+1}(h)c_k(h)\neq 0$,

(25) $\kappa\equiv\limsup a_{k+1}(h)/(b_k(h)-c_{k-1}(h)\ \delta)<1$

where $\delta=\gamma\alpha^2/(\alpha^2+\gamma-\alpha)$ *and the limit is taken for such* k *as* $c_{k-1}(h)\neq 0$,

(26) $\xi\equiv\limsup|c_k(h)|/b_k(h)<1$

the limit being taken for such k *as* $a_k(h)c_{k-1}(h)=0$,

(27) $\eta\equiv\limsup b_{k-1}(h)|c_k(h)|/(b_{k-1}(h)b_k(h)-a_k(h)c_{k-1}(h))<1$

the limit being taken for such k *as* $a_k(h)c_{k-1}(h)\neq 0$ *and* $a_{k-1}(h)c_{k-2}(h)=0$,

(28) $\zeta\equiv\limsup|c_k(h)|/(b_k(h)-c_{k-1}(h)\delta)<1$

the limit being taken for such k *as* $c_{k-1}(h)\neq 0$,

(29) $\inf b_k(h)>0.$

Then, the systems (1) *are stable.*

Proof. We put $p_1(h)=p_2(h)=\ldots=1$ in (8). Then (P_1) is trivially true. Next we shall show that $q_k(h)\geqq 0$ for all $2\leqq k\leqq n(h)$. Since $r_1(h)=b_1(h)>0$, $q_2(h)=a_2(h)/b_1(h)\geqq 0$. Since $A(h)$ and therefore $R(h)$ are inversible, $r_k(h)=b_k(h)-c_{k-1}(h)q_k(h)\neq 0$ so that $q_{k+1}(h)=a_{k+1}(h)/(b_k(h)-c_{k-1}(h)q_k(h))$. So, if $q_k(h)\geqq 0$, then $q_{k+1}(h)\geqq 0$ as far as $k+1\leqq n(h)$. Hence, by induction, $q_k(h)\geqq 0$ for all $2\leqq k\leqq n(h)$. Thus,

$$\inf_{k\geqq 1}|r_k|=\inf_{k\geqq 1}(b_k-c_{k-1}q_k)\geqq\inf_{k\geqq 1}b_k>0$$

by (29), so that (P_2) also holds.

We shall show (P_3): $\limsup q_k(h)<1$. In order to avoid notational complication, we omit the parameter h below. By our hypothesis, there exist an $\varepsilon>0$ and an N such that, for $k>N$,

(30) $0<\alpha\leqq(b_k-(b_k^2-4a_{k+1}c_{k-1})^{1/2})/2c_{k-1}\leqq\beta+\varepsilon<1$, if $c_{k-1}\neq 0$

(31) $a_{k+1}/b_k \leqq \sigma+\varepsilon<1$, if $a_k c_{k-1}=0$,

(32) $a_{k+1}b_{k-1}/(b_{k-1}b_k-a_k c_{k-1}) \leqq \tau+\varepsilon<1$, if $a_k c_{k-1} \neq 0$, $a_{k-1}c_{k-2}=0$,

and

(33) $a_{k+1}/(b_k-c_{k-1}\delta) \leqq \kappa+\varepsilon<1$, if $c_{k-1} \neq 0$.

We put $\rho=\max(\beta+\varepsilon, \sigma+\varepsilon, \tau+\varepsilon, \kappa+\varepsilon)$. Let $k \geqq N+m$, where m is sufficiently large and will be determined later.

(a) If $a_k c_{k-1}=0$, then $q_{k+1}=a_{k+1}/b_k \leqq \sigma+\varepsilon \leqq \rho$ by (31).

(b) If $a_{k+1}=0$, then $q_{k+1}=0<\rho$.

(c) If $a_{k+1} \neq 0$, $a_k c_{k-1} \neq 0$ but $a_{k-1}c_{k-2}=0$, then $q_k=a_k/b_{k-1}$, so that

$$\begin{aligned} q_{k+1}&=a_{k+1}/(b_k-c_{k-1}q_k)=a_{k+1}/(b_k-c_{k-1}(a_k/b_{k-1})) \\ &=a_{k+1}b_{k-1}/(b_{k-1}b_k-a_k c_{k-1}) \leqq \tau+\varepsilon \leqq \rho \end{aligned}$$

by (32).

(d) Suppose in general that $a_{k+1}a_k a_{k-1} \ldots a_{k-i} \neq 0$, $c_{k-1}c_{k-2} \ldots c_{k-i-1} \neq 0$ and $a_{k-i-1}c_{k-i-2}=0$ for some i such as $1 \leqq i < m$. Then, by (31), $q_{k-i}=a_{k-i}/b_{k-i-1} \leqq \sigma+\varepsilon <1$. As we shall see below, this implies that $q_{k-i+1} \geqq \delta$. Moreover (32) implies that

$$\begin{aligned} q_{k-i+1}&=a_{k-i+1}/(b_{k-i}-c_{k-i-1}q_{k-i}) \\ &=a_{k-i+1}b_{k-i-1}/(b_{k-i-1}b_{k-1}-a_{k-i}c_{k-i-1}) \leqq \tau+\varepsilon<1. \end{aligned}$$

So we have $\delta \leqq q_{k-i+1} \leqq \rho$. It follows easily that $\delta \leqq q_l \leqq \rho$ for all $k-i+1 \leqq l \leqq k+1$ and in particular $\delta \leqq q_{k+1} \leqq \rho$.

We note that the hyperbola

$$y=f_1(x)=\gamma\alpha^2/(\alpha^2+(\gamma-\alpha)x)$$

passes the points $(0, \gamma)$ and (α, α) and the hyperbola,

$$y=f_2(x)=a_j/(b_{j-1}-c_{j-2}x), \qquad \text{with } j=k-i+1,$$

cuts the y-axis at $y=a_j/b_{j-1}$ ($\leqq \gamma$ by (22)) and the line $y=x$ at

$$x=(b_{j-1}-(b_{j-1}^2-4a_j c_{j-2})^{1/2})/2c_{j-2} \geqq \alpha.$$

So these two curves intersect for some x between 0 and α and therefore we have $f_1(x) \leqq f_2(x)$ for $x \geqq \alpha$. Thus,

$$q_{k-i+1}=f_2(q_{k-i})>f_2(1) \geqq f_1(1)=\delta,$$

which we had to show.

(e) Finally we suppose that $a_{k+1}a_k a_{k-1} \ldots a_{k-m} \neq 0$ and $c_{k-1}c_{k-2} \ldots c_{k-m-1} \neq 0$. If $\gamma \leqq 1$, then $q_N=q_{k-m} \leqq \gamma \leqq 1$. This implies $\delta \leqq q_{N+1} \leqq \gamma \leqq 1$ and therefore $\delta \leqq q_{N+2} \leqq \rho$. Consequently $\delta \leqq q_{k+1} \leqq \rho$.

If $\gamma>1$, then we argue as follows: we put

$$f_3(x)=(\gamma\delta-(\gamma-\kappa-\varepsilon)x)/\delta$$

and define two sequences $\{\mu_1, \mu_2, \ldots\}$, $\{\upsilon_1, \upsilon_2, \ldots\}$ by setting inductively

$$\upsilon_1=\gamma, \ \mu_j=f_1(\upsilon_j), \ \upsilon_{j+1}=f_3(\mu_j) \text{ for } j \geqq 1.$$

It is easy to see that $\{\upsilon_j\}$ is decreasing, $\{\mu_j\}$ is increasing, and, after a finite number of steps——m_0 steps, say——we shall come to the situation in which $\mu_{m_0}<\delta \leqq \mu_{m_0+1}$ and $\kappa+\varepsilon<\upsilon_{m_0+1} \leqq 1 \leqq \upsilon_{m_0}$. If $\delta \leqq q_N \leqq 1$, then $\delta \leqq q_{N+1} \leqq \rho$, so that, as we remarked in (d), $\delta \leqq q_{k+1} \leqq \rho$. If $q_N>1$, then $q_{N+1}<\delta$. So we may assume without loss of generality that $0<q_N<\delta$. Then we have successively the following:

$$\delta<\alpha<q_{N+1}<f_3(q_N)<f_3(0)=\gamma=\upsilon_1,$$
$$\kappa+\varepsilon\geqq q_{N+2}>f_1(\upsilon_1)=\mu_1,$$
$$\delta<q_{N+3}<f_3(\mu_1)=\upsilon_2,$$
$$\kappa+\varepsilon\geqq q_{N+4}>f_1(\upsilon_2)=\mu_2,$$
$$\cdots\cdots\cdots\cdots\cdots\cdots$$
$$k+\varepsilon\geqq q_{N+2m_0}>f_1(\upsilon_{m_0})=\mu_{m_0}, \text{ and}$$
$$\delta<q_{N+2m_0+1}\leqq f_3(\mu_{m_0})=\upsilon_{m_0+1}\leqq 1.$$

So we have $\delta<q_{N+2m_0+2}\leqq\rho$ and thus $\delta\leqq q_{k+1}\leqq\rho$ provided $N+2m_0+1\leqq k\leqq n(h)-1$.

$(a)-(e)$ cover all cases that can occur; so we conclude that $q_k(h)\leqq\rho$ for all $N+2m_0+2\leqq k\leqq n(h)$. Hence $\lim\sup|q_k(h)|<1$ and the property (P_3) is established.

In order to show (P_4'), we take N so large and ε so small that, together with (30), (31), (32), and (33), the following are also true: for $k>N$,

(34) $|c_k|/b_k\leqq\xi+\varepsilon<1$, if $a_kc_{k-1}=0$,

(35) $b_{k-1}|c_k|/(b_{k-1}b_k-a_kc_{k-1})\leqq\eta+\varepsilon<1$, if $a_kc_{k-1}\neq 0$, $a_{k-1}c_{k-2}=0$,

(36) $|c_k|/(b_k-c_{k-1}\delta)\leqq\zeta+\varepsilon<1$, if $c_{k-1}\neq 0$.

Then we argue just as before: we take m large and let $k\geqq N+m$.

(a') If $a_kc_{k-1}=0$, then $q_k=0$ so that $|s_k/r_k|=|c_k|/b_k\leqq\xi+\varepsilon$, by (34).

(b') If $c_k=0$, then $|s_k/r_k|=0$.

(c') If $c_k\neq 0$, $a_kc_{k-1}\neq 0$ but $a_{k-1}c_{k-2}=0$, then $q_k=a_k/b_{k-1}$, so that, by (35),

$$|s_k/r_k|=|c_k|/(b_k-c_{k-1}q_k)=b_{k-1}|c_k|/(b_{k-1}b_k-a_kc_{k-1})\leqq\eta+\varepsilon.$$

(d') Suppose that $a_ka_{k-1}\dots a_{k-i}\neq 0$ and $c_kc_{k-1}\dots c_{k-i-1}\neq 0$ but $a_{k-i-1}c_{k-i-2}=0$ where $1\leqq i<m$. Then the argument in (d) proves that $\delta\leqq q_k\leqq\rho$. So, by (36),

$$|s_k/r_k|=|c_k|/(b_k-c_{k-1}q_k)\leqq|c_k|(b_k-c_{k-1}\delta)\leqq\zeta+\varepsilon.$$

(e') Suppose finally that $a_ka_{k-1}\dots a_{k-m}\neq 0$ and $c_kc_{k-1}\dots c_{k-m-1}\neq 0$. As was shown in (e), we have $\delta\leqq q_k\leqq\rho$ if we take $m\geqq 2m_0+1$. So we also have $|s_k/r_k|\leqq\zeta+\varepsilon$.

Summing up, we see that $|s_k/r_k|\leqq\max(\xi+\varepsilon, \eta+\varepsilon, \zeta+\varepsilon)<1$ for all large $k\leqq n(h)$. Hence we have (P_4') as desired. Thus the case (ii) (or (iv)) in Theorem 1 happens and so the systems (1) is stable. Q. E. D.

We can obtain an analogous theorem for the case $a_k(h)\leqq 0$, $b_k(h)\geqq 0$, $c_k(h)\geqq 0$, but we do not state it here. It is clear that Theorem 6 for $a>0$, $b>0$, $c<0$ is included in Theorem 7.

References

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