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Note on the stability of a presemistar operation

Akira Okabe*

Abstract

In [8] Matsuda has investigated stability of a semistar operation. In this paper we extend the notion of stability of a semistar operation to the presemistar operation case and we shall study stability properties of presemistar operations.

1. Introduction

Throughout D will be an integral domain with quotient field K. Let $\mathcal{K}(D)$ be the set of all nonzero D-submodules of K. Each member of $\mathcal{K}(D)$ is called a *Kaplansky fractional ideal* of D or a *K-fractional ideal* of D. Let $\mathcal{F}(D)$ be the set of all *nonzero fractional ideals* of D and let $\mathcal{F}_f(D)$ be the set of all *nonzero finitely generated fractional ideals* of D. We denote the set of all *nonzero integral ideals* of D by $\mathcal{I}(D)$.

First we shall recall the definition of a presemistar operation which has been defined in [14].

Let $\star$ be a self-map of $\mathcal{K}(D)$. Then $\star$ is called a *presemistar operation* on D, if the following three conditions are satisfied:

(E) $E \subseteq E^\star$ for all $E \in \mathcal{K}(D)$;

(OP) If $E \subseteq F$, then $E^\star \subseteq F^\star$ for all $E, F \in \mathcal{K}(D)$;

(T) $(aE)^\star = aE^\star$ for all $a \in K \setminus \{0\}$ and all $E \in \mathcal{K}(D)$.

As in [14], we say that a self-map $\star$ of $\mathcal{K}(D)$ has *Extension Property* (resp. *Order Preservation Property*, *Transportability Property*) if $\star$ satisfies condition (E) (resp. (OP), (T)).

A self-map $\star$ of $\mathcal{K}(D)$ is called a *semistar operation* on D, if it is a presemistar operation on D and satisfies the following condition:

(I) $(E^\star)^\star = E^\star$ for all $E \in \mathcal{K}(D)$.

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*Professor Emeritus, Oyama National College of Technology, Oyama, Tochigi 323-0806, Japan (a.okabe58@aw.wakwak.com)

We say that a self-map $\star$ of $\mathcal{K}(D)$ has *Idempotence Property* if $\star$ satisfies condition (I).

Here we list some representative examples of semistar operations.

If we set $E^{\bar{d}} = E$ (resp. $E^{\bar{e}} = K$) for all $E \in \mathcal{K}(D)$, then the map $E \mapsto E^{\bar{d}}$ (resp. $E \mapsto E^{\bar{e}}$) is a semistar operation on D and is called *the $\bar{d}$-operation* (resp. *the $\bar{e}$-operation*).

For each $E \in \mathcal{K}(D)$, we set $E^{-1} = \{x \in K \mid xE \subseteq D\}$ and $E^{\bar{v}} = (E^{-1})^{-1}$. Then the map $E \mapsto E^{\bar{v}}$ is a semistar operation on D and is called *the $\bar{v}$-operation*. Here it is easily seen that $E^{-1} = \{0\}$ for all $E \in \mathcal{K}(D) \setminus \mathcal{F}(D)$ and therefore $E^{\bar{v}} = K$ for all $E \in \mathcal{K}(D) \setminus \mathcal{F}(D)$.

Let $\star$ be a self-map of $\mathcal{F}(D)$ such that $D^\star = D$. Then $\star$ is called a *star operation* on D if the conditions (E), (OP), (T), and (I) hold for all $a \in K \setminus \{0\}$ and all E, F in $\mathcal{F}(D)$. Each star operation $\star$ on D can be extended to a semistar operation $\star^e$ on D as shown in [12, Proposition17].

If $E \in \mathcal{F}(D)$, then $E^{-1} \in \mathcal{F}(D)$ and so $E^{\bar{v}} \in \mathcal{F}(D)$. Hence, if we set $E^v = E^{\bar{v}}$ (resp. $E^d = E^{\bar{d}}$) for all $E \in \mathcal{F}(D)$, then the self-map v (resp. d) of $\mathcal{F}(D)$ is a star operation on D, because $D^{\bar{v}} = D^{\bar{d}} = D$ holds. The map v (resp. d) is called *the v-operation* (resp. *the d-operation*) on D.

For any $E, F \in \mathcal{K}(D)$, the set $\{x \in K \mid xF \subseteq E\}$ is denoted by $E : F$ and for each $E \in \mathcal{K}(D)$, the set $D : E$ is also denoted by E^{-1}.

As in [14], the set of all presemistar operations (resp. all semistar operations) on D is denoted by $\mathbf{PS}(D)$ (resp. $\mathbf{S}(D)$).

2. Definition of stability

In this paper, we denote the set of positive integers by $\mathbf{N}$ and the set of non-negative integers by $\mathbf{N}_0$.

In [8], the definition of stability of a semistar operation was given. We first extend the definition of stability in [8] to the presemistar operation case.

We set $X_1 = \mathcal{F}_f(D), X_2 = \mathcal{F}(D), X_3 = \mathcal{K}(D)$. Let $a, b, c \in \mathbf{N}_0$ such that $a+b+c \geq 2$ and let $\star \in \mathbf{PS}(D)$. Assume that $\star$ satisfies the following condition

(S) $(F_0 \cap \cdots \cap F_a \cap G_0 \cap \cdots \cap G_b \cap H_0 \cap \cdots \cap H_c)^\star = F_0^\star \cap \cdots \cap F_a^\star \cap G_0^\star \cap \cdots \cap G_b^\star \cap H_0^\star \cap \cdots \cap H_c^\star$

for all $F_i (0 \leq i \leq a), G_j (0 \leq j \leq b), H_k (0 \leq k \leq c)$ where $F_i \in X_1$ for $i \neq 0, G_j \in X_2$ for $j \neq 0, H_k \in X_3$ for $k \neq 0$ and $F_0 = G_0 = H_0 = K$.

If $\star$ satisfies condition (S), then $\star$ is called an $f_1^a f_2^b f_3^c$-stable presemistar operation on D or $\star$ is said to be $f_1^a f_2^b f_3^c$-stable.

Hereafter we set $f = f_1, g = f_2$, and $h = f_3$ and for simplicity, we also denote $f^a = f^a g^0 h^0, g^b = f^0 g^b h^0, h^c = f^0 g^0 h^c, f^a g^b = f^a g^b h^0, g^b h^c = f^0 g^b h^c, f^a h^c = f^a g^0 h^c$. For example, $\star$ is f^2-stable, if $(F_1 \cap F_2)^\star = F_1^\star \cap F_2^\star$ for every $F_1, F_2 \in X_1$, $\star$ is gh-stable, if $(G \cap H)^\star = G^\star \cap H^\star$ for every $G \in X_2, H \in X_3$ and $\star$ is h^2-stable, if $(H_1 \cap H_2)^\star = H_1^\star \cap H_2^\star$ for every $H_1, H_2 \in X_3$ and so on.

For the sake of completeness, we state the definition of stability of a star operation. Let $\star$ be a star operation on D and let $a, b \in \mathbf{N}_0$ such that $a + b \geq 2$. Then $\star$ is called an $f^a g^b$-stable star operation on D if $\star$ satisfies the following condition

$(S_0)\quad (F_0 \cap \cdots \cap F_a \cap G_0 \cap \cdots \cap G_b)^\star = F_0^\star \cap \cdots \cap F_a^\star \cap G_0^\star \cap \cdots \cap G_b^\star$

for all $F_i (0 \leq i \leq a), G_j (0 \leq j \leq b)$ where $F_i \in X_1$ for $i \neq 0, G_j \in X_2$ for $j \neq 0$ and $F_0 = G_0 = K$.

Note 2.1. Let $a, b \in \mathbf{N}_0$ such that $a + b \geq 2$. Then it is easy to see that the star operation v is $f^a g^b$-stable if and only if the semistar operation $\bar{v}$ is $f^a g^b$-stable.

If D is a Noetherian domain, then $X_1 = X_2$. Hence, for a Noetherian domain, we have $f_1 = f_2$ by definition and hence we have $f = g$.

Lemma 2.1. (cf.[8, Lemma 2.1 (3), (4)]) *Let $n \in \mathbf{N}$ and $a, b \in \mathbf{N}_0$ such that $a + b + n \geq 2$. Then we have the following implications :*

(1) *$f^a g^n h^b$-stable $\Longrightarrow f^{a+1} g^{n-1} h^b$-stable.*

(2) *$f^a g^b h^n$-stable $\Longrightarrow f^a g^{b+1} h^{n-1}$-stable.*

(3) *$f^a g^b h^n$-stable $\Longrightarrow f^{a+1} g^b h^{n-1}$-stable.*

Lemma 2.2. (cf.[8, Proposition 2.2 (1), (2), (3)]) *Let $n \in \mathbf{N}$ and $a, b \in \mathbf{N}_0$ such that $a + b + n \geq 2$. Then*

(1) *$f^{n+1} g^a h^b$-stable $\Longrightarrow f^n g^a h^b$-stable.*

(2) *$f^a g^{n+1} h^b$-stable $\Longrightarrow f^a g^n h^b$-stable.*

(3) *$f^a g^b h^{n+1}$-stable $\Longrightarrow f^a g^b h^n$-stable.*

If we take $a = b = 0$ in Lemma 2.2, then we have the following

Proposition 2.1. *Let $\star$ be a presemistar operation on D. If $\star$ is f^k-stable (resp. g^k-stable, h^k-stable) for some integer $k \geq 2$, then $\star$ is f^n-stable (resp. g^n-stable, h^n-stable) for every integer n such that $2 \leq n \leq k$.*

Proposition 2.2. (cf.[8, Lemma 2.1 (1), (2)]) *Let $\star$ be a presemistar operation on D. Then*

(1) *If $\star$ is g^n-stable for some integer $n \geq 2$, then $\star$ is g^{n+1}-stable.*

(2) *If $\star$ is h^n-stable for some integer $n \geq 2$, then $\star$ is h^{n+1}-stable.*

(3) *If $\star$ is g^n-stable for some integer $n \geq 2$, then $\star$ is $f^a g^b$-stable for all $a, b \in \mathbf{N}_0$ such that $a + b \geq n$.*

(4) *If $\star$ is h^n-stable for some integer $n \geq 2$, then $\star$ is $f^a g^b h^c$-stable for all $a, b, c \in \mathbf{N}_0$ such that $a + b + c \geq n$.*

Proof. The proofs of (1) and (2) are straightforward.
(3) This follows from Proposition 2.2 (1) and the definition of stability.
(4) This follows from Proposition 2.2 (2) and the definition of stability. □

The following implications are easily derived from the definition of stability and the inclusion relation $X_1 \subseteq X_2 \subseteq X_3$.

Note 2.2.

(1) h^2-stable $\Longrightarrow$ gh-stable $\Longrightarrow$ g^2-stable $\Longrightarrow$ fg-stable $\Longrightarrow$ f^2-stable.

(2) h^2-stable $\Longrightarrow$ gh-stable $\Longrightarrow$ fh-stable $\Longrightarrow$ fg-stable.

Corollary 2.1. *Let $\star$ be a presemistar operation on D.*
If $\star$ is g^k-stable (resp. h^k-stable) for some integer $k \geq 2$, then $\star$ is g^n-stable (resp. h^n-stable) for every integer $n \geq 2$.

Proof. This follows from Propositions 2.1 and 2.2. □

An integral domain D is called a *coherent domain* if every finitely generated ideal of D is finitely presented (or finitely related).

Proposition 2.3. (cf.[3, Theorem 2.2]) *An integral domain D is a coherent domain if and only if the intersection of any two finitely generated ideals of D is again finitely generated. In particular, each Noetherian domain is a coherent domain.*

Proposition 2.4. *Let D be a coherent domain and let $\star$ be a presemistar operation on D. If $\star$ is f^k-stable for some integer $k \geq 2$, then $\star$ is also f^{k+1}-stable.*

Proof. Suppose that $\star$ is f^k-stable for some integer $k \geq 2$. Then $\star$ is f^n-stable for each integer n such that $2 \leq n \leq k$ by Proposition 2.1 and so $\star$ is f^2-stable. Now choose arbitrary elements $F_1, F_2, \cdots, F_k, F_{k+1} \in \mathcal{F}_f(D)$. Then, since $F_k \cap F_{k+1} \in \mathcal{F}_f(D)$ by Proposition 2.3, we have $(F_1 \cap F_2 \cap \cdots \cap F_k \cap F_{k+1})^\star = (F_1 \cap F_2 \cap \cdots \cap (F_k \cap F_{k+1}))^\star = {F_1}^\star \cap \cdots \cap {F_{k-1}}^\star \cap (F_k \cap F_{k+1})^\star = {F_1}^\star \cap \cdots \cap {F_{k-1}}^\star \cap {F_k}^\star \cap {F_{k+1}}^\star$. Therefore f^k-stable $\Longrightarrow$ f^{k+1}-stable for every integer $k \geq 2$. □

Corollary 2.2. *Let D be a coherent domain and let $\star$ be a presemistar operation on D. Then*
$\star$ is f^k-stable for some integer $k \geq 2 \Longleftrightarrow \star$ is f^n-stable for every integer $n \geq 2$.

Proof. This follows from Propositions 2.1 and 2.4. □

Proposition 2.5. *Every Prüfer domain is a coherent domain.*

Proof. This follows from [6, Proposition 25.4 (1)]. □

Corollary 2.3. *Let D be a Prüfer domain and let $\star$ be a presemistar operation on D. Then*
$\star$ is f^k-stable for some integer $k \geq 2 \Longleftrightarrow \star$ is f^n-stable for every integer $n \geq 2$.

Proof. This follows from Corollary 2.2 and Proposition 2.5. □

3. Stability properties (Semistar operation case)

We recall two types of integral domains. An integral domain D is called an *essential domain* if there exists a family of prime ideals $\{P_\lambda \mid \lambda \in \Lambda\}$ of D such that every D_{P_λ} is a valuation overring of D and $D = \cap_\lambda D_{P_\lambda}$. Next, an integral domain D is called a *v-domain* if each $F \in \mathcal{F}_f(D)$ is v-invertible, that is, $(AA^{-1})^v = D$.

Proposition 3.1. ([1, Theorem 7])

(1) *If D is an essential domain, then v is f^n-stable for every integer $n \geq 2$,*

(2) *If D is an integrally closed domain such that v is f^n-stable for every integer $n \geq 2$, then D is a v-domain.*

It follows from Proposition 3.1 that if D is an integrally closed domain which is not a v-domain, then v is not f^n-stable for some integer $n \geq 2$.

Proposition 3.2. ([9, Theorem 2]) *If D is a v-domain, then v is f^n-stable for every integer $n \geq 2$.*

Proposition 3.3. *Let D be a Prüfer domain. Then v is f^n-stable for every integer $n \geq 2$.*

Proof. By Proposition 2.5, D is a coherent domain and so by Proposition 2.4, it suffices to show that v is f^2-stable. To prove this, choose arbitrary elements $A, B \in \mathcal{F}_f(D)$. Then, by [6, Theorem 25.2 (g)], $(A \cap B)^{-1} = A^{-1} + B^{-1}$ and therefore $(A \cap B)^v = ((A \cap B)^{-1})^{-1} = (A^{-1} + B^{-1})^{-1} = D : (A^{-1} + B^{-1}) = (D : A^{-1}) \cap (D : B^{-1}) = A^v \cap B^v$ as desired. $\square$

Remark 3.1. The proof of Proposition 3.3 can be also derived from tha fact that every finitely generated fractional ideal of a Prüfer domain is invertible and so divisorial.

There exists a Noetherian domain D such that the star operation v on D is not f^2-stable as shown in the following example.

Example 3.1. ([5, Example 1.8]) Let $D = k[[X^3, X^4, X^5]]$ with a field k. Then D is a 1-dimensional Noetherian local domain with maximal ideal $M = (X^3, X^4, X^5)$. If we set $I = (X^3, X^4), J = (X^3, X^5)$, then $I^v = J^v = M$ and $I \cap J = (X^3)$ and so $(I \cap J)^v = (X^3) \subsetneqq I^v \cap J^v = M$. Thus v is not f^2-stable.

Here we must say that this example is due to W. Heinzer as noted in [1, p.4].

In general, f^2-stable does not imply g^2-stable even for a Prüfer domain as seen in the following example.

Example 3.2. If we choose a Prüfer domain D as constructed in [11, Example 3.1], then it is shown that the star operation v on D is not g^2-stable. But v is always f^2-stable by Proposition 3.3.

In [8, Example 3.16], Matsuda has provided a Prüfer domain with exactly two maximal ideals such that there exist semistar operations on D which are f^n-stable for every integer $n \geq 2$ and which are not g^2-stable (see also [8, Note, p.7]). It was shown in [7, Lemma 4] that every Prüfer domain with exactly two maximal ideals is a Bezout domain. Thus it follows that in general, f^2-stable does not imply g^2-stable even for a Bezout domain.

In general, g^2-stable does not imply h^2-stable as shown in the next example.

Example 3.3. ([8, Example 3.2 (2)]) Let $D = k[X,Y]$ with a field k. We set $E^\star = E$ for every $E \in \mathcal{F}(D)$ and set $E^\star = K(= k(X,Y))$ for every $E \in \mathcal{K}(D) \setminus \mathcal{F}(D)$. Then $\star$ is a semistar operation on D and is evidently g^2-stable. But if we set $E = k[X, Y, \frac{1}{Y}, \frac{1}{Y^2}, \cdots]$ and $F = k[Y, X, \frac{1}{X}, \frac{1}{X^2}, \cdots]$, then $(E \cap F)^\star = k[X,Y] \subsetneqq E^\star \cap F^\star = k(X,Y)$. Hence $\star$ is not h^2-stable.

Proposition 3.4. ([8, Propositions 3.8 and 3.12]) *Let $\star$ be a semistar operation on D. Then*

(1) *fg-stable $\Longrightarrow$ g^2-stable.*

(2) *fh-stable $\Longrightarrow$ h^2-stable.*

Corollary 3.1. *Let $\star$ be a semistar operation on D. Then*

(1) *fg-stable $\Longleftrightarrow$ g^2-stable.*

(2) *fh-stable $\Longleftrightarrow$ h^2-stable.*

Proof. This follows from Note 2.2 and Proposition 3.4. $\square$

4. Stability properties (Presemistar operation case)

For each nonzero integral ideal I of D, we set $E^{\lambda(I)} = E : I$ for each $E \in \mathcal{K}(D)$. Then, by [14, Proposition 3.1], the self-map $\lambda(I)$ of $\mathcal{K}(D)$ is a presemistar operation on D.

Proposition 4.1. *For each nonzero integral ideal $I, \lambda(I)$ is h^n-stable for every integer $n \geq 2$.*

Proof. By definition, $\lambda(I)$ is h^2-stable and hence, by Proposition 2.2 (2), $\lambda(I)$ is also h^n-stable for every integer $n \geq 2$. $\square$

Let $\mathcal{F}$ be a family of nonzero ideals of D with $D \in \mathcal{F}$. Then

(1) $\mathcal{F}$ is called a *semifilter* of D if, for all $I, J \in \mathcal{I}(D), I \supseteq J \in \mathcal{F}$ implies $I \in \mathcal{F}$.

(2) $\mathcal{F}$ is called a *filter* of D if it is a semifilter and $A, B \in \mathcal{F}$ implies $A \cap B \in \mathcal{F}$.

(3) $\mathcal{F}$ is said to be *monoidal* if $A, B \in \mathcal{F}$ implies $AB \in \mathcal{F}$.

As in [14], for each presemistar operation $\star$, we set $\mathcal{F}^\star = \{I \in \mathcal{I}(D) \mid I^\star = D^\star\}$. Then we have the following

Proposition 4.2. ([14, Lemma 4.1]) *Let $\star$ be a presemistar operation on D. If $\star$ is h^2-stable, then $\mathcal{F}^\star$ is a filter of D.*

For each filter $\mathcal{F}$ of D, we set $E^{\star_\mathcal{F}} = \cup\{E : J \mid J \in \mathcal{F}\}$ for every $E \in \mathcal{K}(D)$. Then we have the following

Proposition 4.3. ([14, Lemma 4.2]) *For each filter $\mathcal{F}$ of $D, \star_\mathcal{F}$ is an h^2-stable presemistar operation on D.*

A presemistar operation $\star$ on D is said to be *strong* if $E^\star F^\star \subseteq (EF)^\star$ for all $E, F \in \mathcal{K}(D)$ (see [14, Definition 4.1]) and $\star$ is said to be *proper* if $\star$ is not a semistar operation on D (see [14, p.34]).

Proposition 4.4. ([14, Theorem 4.1])

(1) *If $\mathcal{F}$ is a monoidal filter of D, then $\star_{\mathcal{F}}$ is h^2-stable and strong.*

(2) *If $\star$ is an h^2-stable and strong presemistar operation on D, then $\mathcal{F}^\star$ is a monoidal filter of D.*

Proposition 4.5. ([14, Proposition 4.4]) *Let $\star$ be a presemistar operation on D. If $\star$ is h^2-stable and strong, then $\star = \star_{\mathcal{F}^\star}$.*

Let I be an element of $\mathcal{K}(D)$ such that $D \subseteq I$. If we set $E^{\mu(I)} = EI$ for all $E \in \mathcal{K}(D)$, then $\mu(I)$ is a presemistar operation on D by [14, Proposition 3.2 (1)].

Example 4.1. Let k be a field and let $D = k[[X^5, X^6]]$. If we set $I = D + XD$, then $I^2 = D + XD + X^2D \neq I$ and so by [14, Proposition 3.2 (3)], $\mu(I)$ is a proper presemistar operation on D.

Proposition 4.6. *Let $0 \neq d$ be a nonunit of D.and let $I = \frac{1}{d}D$. Then $\mu(I)$ is a proper presemistar operation on D and is h^n-stable for every integer $n \geq 2$.*

Proof. Since $I = \frac{1}{d}D \supseteq \frac{1}{d}dD = D$ and $I \neq I^2$, it follows from [14, Proposition 3.2] that $\mu(I)$ is a proper presemistar operation on D. Moreover, for arbitrary two elements $E_1, E_2 \in \mathcal{K}(D)$, we have $(E_1 \cap E_2)^{\mu(I)} = (E_1 \cap E_2)\frac{1}{d}D = E_1\frac{1}{d}D \cap E_2\frac{1}{d}D = E_1^{\mu(I)} \cap E_2^{\mu(I)}$ and so $\mu(I)$ is h^2-stable. Then $\mu(I)$ is h^n-stable for every integer $n \geq 2$ by Corollary 2.1 as we wanted. $\square$

We shall now construct a presemistar operation of new type on D.

Theorem 4.1. *Let I be a nonzero integral ideal of D. We set*

$$E^{\mathrm{p(I)}} = \begin{cases} E : I & \text{for all } E \in \mathcal{F}(D) \\ K & \text{for all } E \in \mathcal{K}(D) \setminus \mathcal{F}(D) \end{cases}$$

Then

(1) $\mathrm{p(I)}$ *is a presemistar operation on D.*

(2) *If I is an invertible integral ideal of D, then* $\mathrm{p(I)}$ *is a proper presemistar operation on D.*

(3) $\mathrm{p(I)}$ *is always g^2-stable.*

(4) *If $I = I^2$, then* $\mathrm{p(I)}$ *is a semistar operation on D.*

Proof.

(1) Evidently $E \subseteq E^{\mathrm{p(I)}}$ for all $E \in \mathcal{K}(D)$ and so p(I) satisfies condition (E). Next we shall show taht p(I) satisfies condition (OP). Let $E \subseteq F$ in $\mathcal{K}(D)$. If $F \in \mathcal{F}(D)$, then $E \in \mathcal{F}(D)$ and so $E^{\mathrm{p(I)}} = E : I \subseteq F : I = F^{\mathrm{p(I)}}$. Next assume that $E \in \mathcal{F}(D)$ and $F \notin \mathcal{F}(D)$. Then $E^{\mathrm{p(I)}} = E : I \subseteq K = F^{\mathrm{p(I)}}$. Lastly assume that $E \notin \mathcal{F}(D)$ and $F \notin \mathcal{F}(D)$. Then $E^{\mathrm{p(I)}} = K = F^{\mathrm{p(I)}}$. Thus condition (OP) holds. Choose an element $0 \neq x \in K$. If $E \in \mathcal{F}(D)$, then $xE \in \mathcal{F}(D)$ and therefore $(xE)^{\mathrm{p(I)}} = (xE) : I = x(E : I) = xE^{\mathrm{p(I)}}$. If $E \notin \mathcal{F}(D)$, then $xE \notin \mathcal{F}(D)$ and hence $(xE)^{\mathrm{p(I)}} = K = xK = xE^{\mathrm{p(I)}}$. Thus p(I) satisfies condition (T). Hence p(I) is a presemistar operation on D.

(2) Assume that I is an invertible integral ideal of D. Then $I \neq I^2$ and so we have $(D^{\mathrm{p(I)}})^{\mathrm{p(I)}} = D : I^2 \neq D : I = D^{\mathrm{p(I)}}$. Hence p(I) is not a semistar operation on D.

(3) For every $E, F \in \mathcal{F}(D)$, we have $(E \cap F)^{\mathrm{p(I)}} = (E \cap F) : I = (E : I) \cap (F : I) = E^{\mathrm{p(I)}} \cap F^{\mathrm{p(I)}}$ and therefore p(I) is g^2-stable.

(4) If $E \in \mathcal{F}(D)$, then $E^{\mathrm{p(I)}} = E : I \in \mathcal{F}(D)$ and so, by hypothesis, $(E^{\mathrm{p(I)}})^{\mathrm{p(I)}} = (E : I) : I = E : I^2 = E : I = E^{\mathrm{p(I)}}$. Next, if $E \notin \mathcal{F}(D)$, then $E^{\mathrm{p(I)}} = K \notin \mathcal{F}(D)$ and hence $(E^{\mathrm{p(I)}})^{\mathrm{p(I)}} = K = E^{\mathrm{p(I)}}$. Thus p(I) satisfies condition (I).

□

Now we shall consider the following condition:

Definition 4.1.

(NI) There exist $F \in \mathcal{F}(D)$ and $E \in \mathcal{K}(D) \setminus \mathcal{F}(D)$ such that $F \nsubseteq E$ and $E \neq K$.

We say that an integral domain D has *Noninclusion Property* if D satisfies the above condition (NI).

Theorem 4.2. *Assume that D satisfies condition* (NI)*. Then* p(I) *is not gh-stable for each invertible integral ideal I of D.*

Proof. By hypothesis, there exist $F \in \mathcal{F}(D)$ and $E \in \mathcal{K}(D) \setminus \mathcal{F}(D)$ such that $F \nsubseteq E$ and $E \neq K$. Then $F^{\mathrm{p(I)}} = F : I = FI^{-1}$ and $E^{\mathrm{p(I)}} = K$ and therefore $F^{\mathrm{p(I)}} \cap E^{\mathrm{p(I)}} = FI^{-1}$. Now, since $F \cap E \subseteq F$, we have $F \cap E \in \mathcal{F}(D)$ and so $(F \cap E)^{\mathrm{p(I)}} = (F \cap E)I^{-1}$. Suppose that $(F \cap E)^{\mathrm{p(I)}} = F^{\mathrm{p(I)}} \cap E^{\mathrm{p(I)}}$. Then $(F \cap E)I^{-1} = FI^{-1}$ and hence $F = F \cap E$ which implies $F \subseteq E$, a contradiction. Hence we have $(F \cap E)^{\mathrm{p(I)}} \neq F^{\mathrm{p(I)}} \cap E^{\mathrm{p(I)}}$. Thus p(I) is not gh-stable as we wanted. □

Proposition 4.7. *Assume that an integral domain D satisfies condition* (NI)*. Then there exists a proper presemistar operation $\star$ which is g^2-stable but is not gh-stable.*

Proof. It follows from Theorems 4.1 and 4.2 that $\star = $ p(I) is g^2-stable but is not gh-stable for every invertible integral ideal I of D. □

Corollary 4.1. *Let D be a Noetherian domain which satisfies condition* (NI)*. Then there exists a proper presemistar operation $\star$ which is f^2-stable but is not fh-stable.*

Proof. Since D is a Noetherian domain, we get $f = g$ and so our assertion follows from Proposition 4.7. □

Now we shall show that there exists an integral domain D which satisfies condition (NI) in Definition 4.1.

Example 4.2. Let k be a field and let $D = k[\mathrm{X}, \mathrm{Y}_1, \mathrm{Y}_2, \cdots]$. Set $I = D + \frac{1}{\mathrm{X}}D \in \mathcal{F}(D)$ and set $J = \sum_{n=1}^{\infty} D\frac{1}{\mathrm{X}+\mathrm{Y}_n} \in \mathcal{K}(D) \setminus \mathcal{F}(D)$. Then $I \nsubseteq J$ and $J \neq K = k(\mathrm{X}, \mathrm{Y}_1, \mathrm{Y}_2, \cdots)$, where K is the quotient field of D. Therefore D satisfies condition (NI). Evidently D is not a Noetherian domain. If we take $J' = \sum_{n=1}^{\infty} D\frac{1}{\mathrm{Y}_n} \in \mathcal{K}(D) \setminus \mathcal{F}(D)$, then we also have $I \nsubseteq J'$ and $J' \neq K = k(\mathrm{X}, \mathrm{Y}_1, \mathrm{Y}_2, \cdots)$.

The integral domain D constructed in Example 4.2 is not a Noetherian domain. But we can show that there exists a Noetherian domain D which satisfies condition (NI) in Definition 4.1.

Example 4.3. Let k be a field and let $D = k[\mathrm{X}, \mathrm{Y}]$. Set $I = D + \frac{1}{\mathrm{X}}D \in \mathcal{F}(D)$ and $J = \sum_{m=1}^{\infty} D\frac{1}{\mathrm{X}+\mathrm{Y}^m} \in \mathcal{K}(D) \setminus \mathcal{F}(D)$. Then $I \nsubseteq J$ and $J \neq K = k(X, Y)$, where $K = k(X, Y)$ is the quotient field of $D = k[X, Y]$. Hence D satisfies condition (NI). Evidently D is a Noetherian domain.

In [4], an integral domain D is called a *conducive domain* if $D : R = \{x \in K \mid xR \subseteq D\} \neq (0)$ for each overring R of D other than K. It is well known that D is a conducive domain if and only if $\mathcal{K}(D) = \mathcal{F}(D) \cup \{K\}$ (see [13, Proposition 43]).

Note 4.1. If an integral domain D satisfies condition (NI), then D is not a conducive domain.

Proposition 4.8. *Let D be a non-conducive integral domain and let $I \in \mathcal{K}(D) \setminus \mathcal{F}(D)$ such that $D \subsetneq I \subsetneq K$. We set*

$$E^{\mathrm{q(I)}} = \begin{cases} EI & \text{for all } E \in \mathcal{F}(D) \\ K & \text{for all } E \in \mathcal{K}(D) \setminus \mathcal{F}(D) \end{cases}$$

Then

(1) $\mathrm{q(I)}$ *is a proper presemistar operation on D.*

(2) *If $D \subsetneq I \subsetneq J \subsetneq K$ with $J \in \mathcal{K}(D)$, then* $\mathrm{q(I)} \lneq \mathrm{q(J)}$.

Proof.

(1) By definition, $E \subseteq E^{\mathrm{q(I)}}$ for all $E \in \mathcal{K}(D)$. Next let $E \subseteq F$ in $\mathcal{K}(D)$. If $F \in \mathcal{F}(D)$, then $E \in \mathcal{F}(D)$ and so $E^{\mathrm{q(I)}} = EI \subseteq FI = F^{\mathrm{q(I)}}$. If $E \in \mathcal{F}(D), F \notin \mathcal{F}(D)$, then $E^{\mathrm{q(I)}} = EI \subseteq K = F^{\mathrm{q(I)}}$. If $E \notin \mathcal{F}(D), F \notin \mathcal{F}(D)$, then $E^{\mathrm{q(I)}} = K = F^{\mathrm{q(I)}}$. Thus condition (OP) holds. Now, let $E \in \mathcal{K}(D)$ and $x \neq 0 \in K$. If

$E \in \mathcal{F}(D)$, then $xE \in \mathcal{F}(D)$ and hence $(xE)^{\mathrm{q(I)}} = xEI = xE^{\mathrm{q(I)}}$. If $E \notin \mathcal{F}(D)$, then $xE \notin \mathcal{F}(D)$ and so $(xE)^{\mathrm{q(I)}} = K = xK = xE^{\mathrm{q(I)}}$. Thus condition (T) also holds. Therefore q(I) is a presemistar operation on D. Now we shall show that q(I) is not a semistar operation on D. By definition, we have $D^{\mathrm{q(I)}} = I$ and $I^{\mathrm{q(I)}} = K$, because $I \notin \mathcal{F}(D)$. Hence $(D^{\mathrm{q(I)}})^{\mathrm{q(I)}} = I^{\mathrm{q(I)}} = K$ and so $D^{\mathrm{q(I)}} \neq (D^{\mathrm{q(I)}})^{\mathrm{q(I)}}$ which implies that q(I) is not a semistar operation on D.

(2) Since $J \in \mathcal{K}(D) \setminus \mathcal{F}(D), \mathrm{q(J)}$ is also a proper presemistar operation on D by (1) and $E^{\mathrm{q(I)}} \subseteq E^{\mathrm{q(J)}}$ for all $E \in \mathcal{K}(D)$. Hence $\mathrm{q(I)} \leq \mathrm{q(J)}$ and furthermore $D^{\mathrm{q(I)}} = I \subsetneq J = D^{\mathrm{q(J)}}$. Thus we have $\mathrm{q(I)} \lneq \mathrm{q(J)}$ as we wanted.

□

Proposition 4.9. *Let D be a non-conducive integral domain and let T be a flat overring of D such that $T \neq K$ and $T \notin \mathcal{F}(D)$. We set*

$$E^{\mathrm{q(T)}} = \begin{cases} ET & \text{for all } E \in \mathcal{F}(D) \\ K & \text{for all } E \in \mathcal{K}(D) \setminus \mathcal{F}(D) \end{cases}$$

Then

(1) q(T) *is a proper presemistar operation on D.*

(2) q(T) *is a g^2-stable presemistar operation on D.*

Proof.

(1) This easily follows from (1) of Proposition 4.8.

(2) It follows from [10, (3.H) (1)] that $E_1T \cap E_2T = (E_1 \cap E_2)T$ holds for all $E_1, E_2 \in \mathcal{F}(D)$ and hence q(T) is g^2-stable.

□

In the following example, we shall show that there exists an integral domain D that has a proper presemistar operation $\star$ on D which is g^2-stable but is not h^2-stable.

Example 4.4. Let k be a field and let $D = k[X, Y]$. Set $S_1 = \{X^n \mid n = 0, 1, 2, \cdots\}$ and $S_2 = \{Y^m \mid m = 0, 1, 2, \cdots\}$. Then D_{S_1} and D_{S_2} are flat overrings of D such that $D_{S_1}, D_{S_2} \in \mathcal{K}(D) \setminus \mathcal{F}(D)$. Furthermore D is not a conducive domain. To see this, let J be a K-fractional ideal of D defined in Example 4.3, then $J \notin \mathcal{F}(D)$ and $J \neq K = k(X, Y)$ and so D is not a conducive domain by [13, Proposition 43]. Thus there exists a non-conducive integral domain D such that D has a flat overring T which satisfes the conditions in Proposition 4.9. Hence $\mathrm{q(D_{S_i})}$ is g^2-stable for each $i \in \{1, 2\}$ by Proposition 4.9. Now we set $E_1 = \sum_{n=1}^{\infty} D\frac{1}{X^n}$ and $E_2 = \sum_{n=1}^{\infty} D\frac{1}{Y^n}$. Then it is easily seen that $E_1, E_2 \in \mathcal{K}(D) \setminus \mathcal{F}(D)$ and $E_1 \cap E_2 = D$. If we set $\star = \mathrm{q(D_{S_i})}$ for some $i \in \{1, 2\}$, then, by definition, $(E_1 \cap E_2)^\star = D^\star = D_{S_i} \neq K$ but $(E_1)^\star \cap (E_2)^\star = K \cap K = K$ and so $(E_1 \cap E_2)^\star \neq (E_1)^\star \cap (E_2)^\star$. Thus $\star$ is not h^2-stable but is g^2-stable by Proposition 4.9.

Note 4.2. It will be seen that in general, f^2-stable does not imply h^2-stable. Let $D = k[X, Y]$ with a field k and let S_i be a multiplicatively closed subset of D constructed in Example 4.4 for each $i \in \{1, 2\}$. Then $\mathrm{q(D_{S_i})}$ is a proper presemistar operation on D which is f^2-stable but not h^2-stable for each $i \in \{1, 2\}$ as shown in Example 4.4, because D is a Noetherian domain.

We shall show that there exists a proper presemistar operation which is not f^2-stable.

Example 4.5. Let k be a field and let $D = k[[X^3, X^4, X^5]]$. Then D is a Noetherian local domain with maximal ideal $M = (X^3, X^4, X^5)$. Choose a nonunit $a \neq 0$ of D. Then the presemistar operation $v[aD]$ defined in [15, Proposition 3.1] is a proper presemistar operation on D by [15, Proposition 3.2 (1)]. If we take $I = (X^3, X^4)$ and $J = (X^3, X^5)$, then $I^{\bar{v}} = J^{\bar{v}} = M$ and $(I \cap J)^{\bar{v}} = (X^3)$. Hence, by [15, Lemma 3.1], we have $I^{v[aD]} = \frac{1}{a} I^{\bar{v}} = \frac{1}{a} M$, $J^{v[aD]} = \frac{1}{a} J^{\bar{v}} = \frac{1}{a} M$ and $(I \cap J)^{v[aD]} = \frac{1}{a}(I \cap J)^{\bar{v}} = \frac{1}{a}(X^3)$ and so $(I \cap J)^{v[aD]} = \frac{1}{a}(X^3) \subsetneq \frac{1}{a} M = I^{v[aD]} \cap J^{v[aD]}$. Thus $v[aD]$ is not f^2-stable.

We can also construct a proper presemistar operation which is not fh-stable.

Example 4.6. Let k be a field and let $D = k[X, Y]$. Then D is a non-conducive domain as shown in Example 4.4. We set $I = D + D\frac{1}{X} \in \mathcal{F}_f(D)$ and $J = D + \sum_{m=1}^{\infty} D\frac{1}{X+Y^m} \in \mathcal{K}(D) \setminus \mathcal{F}(D)$. Then, since $D \subsetneq J \subsetneq K$, $\mathrm{q(J)}$ is defined and is a proper presemistar operation on D. It is easy to see that $I \cap J = D$ and hence we get $I^{\mathrm{q(J)}} \cap J^{\mathrm{q(J)}} = IJ \cap K = IJ \neq J = D^{\mathrm{q(J)}} = (I \cap J)^{\mathrm{q(J)}}$. Therefore $\mathrm{q(J)}$ is not fh-stable.

Let D be an integral domain and let $\star$ be a presemistar operation on D. Let S_1 (resp. S_2, S_3) be the set of properties $\{f^n\text{-stable} \mid n \geq 2\}$ (resp. the set of properties $\{g^n\text{-stable} \mid n \geq 2\}$, the set of properties $\{h^n\text{-stable} \mid n \geq 2\}$), let S_4 (resp. S_5, S_6) be the set of properties $\{f^n g^m\text{-stable} \mid n \geq 1, m \geq 1\}$ (resp. the set of properties $\{f^n h^m\text{-stable} \mid n \geq 1, m \geq 1\}$, the set of properties $\{g^n h^m\text{-stable} \mid n \geq 1, m \geq 1\}$) and let S_7 be the set of properties $\{f^a g^b h^n\text{-stable} \mid a + b \geq 1, n \geq 1\}$. We shall study implications of these properties in $\{S_i \mid i = 1, 2, \cdots, 7\}$.

First, the following implications are derived from results in Section 2 and the definition of stability.

Theorem 4.3. *Let D be an integral domain and let $\star$ be a presemistar operation on D. Then*

(i) *If D is a coherent domain, then every two properties in S_1 are equivalent.*

(ii) *Every two properties in S_2 are equivalent.*

(iii) *Every two properties in S_3 are equivalent.*

(iv) *If $\star$ satisfies some property in S_2, then $\star$ satisfies every property in S_4.*

(v) *If $\star$ satisfies some property in S_3, then $\star$ satisfies every property in S_5, S_6 and S_7.*

(vi) *If $\star$ satisfies every (resp., some) property in S_6, then $\star$ satisfies every (resp., some) property in S_5.*

(vii) *If $\star$ satisfies every (resp., some) property in S_5, then $\star$ satisfies every (resp., some) property in S_4.*

(viii) *If $\star$ satisfies every (resp., some) property in S_6, then $\star$ satisfies every (resp., some) property in S_7.*

Proof.

(i) This follows from Propositions 2.1 and 2.4.

(ii) This follows from Corollary 2.1.

(iii) This follows from Corollary 2.1.

(iv) This follows from Propositions 2.1 and 2.2.

(v) This follows from Propositions 2.1 and 2.2.

(vi) This follows from the fact that $g^n h^m$-stable $\Longrightarrow f^n h^m$-stable for all $n, m \in \mathbf{N}$.

(vii) This follows from the fact that $f^n h^m$-stable $\Longrightarrow f^n g^m$-stable for all $n, m \in \mathbf{N}$.

(viii) This follows from the fact that $g^n h^m$-stable $\Longrightarrow f^a g^b h^m$-stable for all $n, m \in \mathbf{N}$ and all $a, b \in \mathbf{N}_0$ such that $a + b = n$.

□

Now we shall give a presemistar operation of new type which is g^2-stable.

Theorem 4.4. *Let D be an integral domain and let $I \in \mathcal{K}(D) \setminus \mathcal{F}(D)$ such that $D \subsetneq I \subsetneq K$. We set*

$$E^{\mathrm{r(I)}} = \begin{cases} E & \textit{for all } E \in \mathcal{F}(D) \\ EI & \textit{for all } E \in \mathcal{K}(D) \setminus \mathcal{F}(D) \end{cases}$$

Then

(1) $\mathrm{r(I)}$ *is a g^2-stable presemistar operation on D.*

(2) *If $I \subsetneq I^2 \subsetneq I^3$, then* $\mathrm{r(I)}$ *is a proper presemistar operation on D.*

Proof.

(1) First, $E \subseteq E^{\mathrm{r(I)}}$ for all $E \in \mathcal{K}(D)$. Next, let $E_1 \subseteq E_2$ with $E_1, E_2 \in \mathcal{K}(D)$. If $E_2 \in \mathcal{F}(D)$, then $E_1 \in \mathcal{F}(D)$ and so $E_1^{\mathrm{r(I)}} = E_1 \subseteq E_2 = E_2^{\mathrm{r(I)}}$. If $E_1 \in \mathcal{F}(D)$ and $E_2 \notin \mathcal{F}(D)$, then $E_1^{\mathrm{r(I)}} = E_1 \subseteq E_2 \subseteq E_2 I = E_2^{\mathrm{r(I)}}$. If $E_1, E_2 \notin \mathcal{F}(D)$, then $E_1^{\mathrm{r(I)}} = E_1 I \subseteq E_2 I = E_2^{\mathrm{r(I)}}$. Hence $E_1^{\mathrm{r(I)}} \subseteq E_2^{\mathrm{r(I)}}$ for all $E_1 \subseteq E_2$ in $\mathcal{K}(D)$. Let $0 \neq x \in K$ and $E \in \mathcal{K}(D)$. If $E \in \mathcal{F}(D)$, then $xE \in \mathcal{F}(D)$. Hence $(xE)^{\mathrm{r(I)}} = xE = xE^{\mathrm{r(I)}}$. Next, if $E \notin \mathcal{F}(D)$, then $xE \notin \mathcal{F}(D)$ and then $(xE)^{\mathrm{r(I)}} = xEI = xE^{\mathrm{r(I)}}$. Thus $(xE)^{\mathrm{r(I)}} = xE^{\mathrm{r(I)}}$ for all $0 \neq x \in K$ and all $E \in \mathcal{K}(D)$. Therefore the map $\mathrm{r(I)}$ is a presemistar operation on D. It is evident that $\mathrm{r(I)}$ is g^2-stable.

(2) We have $I^{\mathrm{r(I)}} = I^2$ and $(I^{\mathrm{r(I)}})^{\mathrm{r(I)}} = I^2 I = I^3 \neq I^2 = I^{\mathrm{r(I)}}$ and so r(I) is a proper presemistar operation on D.

$\square$

Proposition 4.10.

(1) *Let k be a field and let $D = k[X_1, X_2, \cdots]$ be a polynomial ring with infinite variables $\{X_n\}_{n \in \mathbf{N}}$. If we set $I = D + D\frac{1}{X_1} + D\frac{1}{X_2} + D\frac{1}{X_3} + D\frac{1}{X_4} + \cdots$, then* r(I) *is proper and is not h^2-stable.*

(2) *Let k be a field and let $D = k[X, X_1, X_2, X_3, \cdots]$ with infinite variables $\{X\} \cup \{X_n\}_{n \in \mathbf{N}}$. If we set $I = D + D\frac{1}{X} + D\frac{1}{X_1} + D\frac{1}{X_2} + \cdots$, then* r(I) *is proper and is not fh-stable.*

Proof.

(1) We set $J_1 = D + D\frac{1}{X_1} + D\frac{1}{X_3} + D\frac{1}{X_5} + \cdots$ and $J_2 = D + D\frac{1}{X_2} + D\frac{1}{X_4} + D\frac{1}{X_6} + \cdots$. Then $J_1, J_2 \notin \mathcal{F}(D), J_1 \subsetneq I, J_2 \subsetneq I$, and $D \subsetneq I \subsetneq I^2 \subsetneq I^3$. Now it is easy to see that $D = J_1 \cap J_2$ and $\frac{1}{X_1 X_2} \notin D$. But $\frac{1}{X_1 X_2} \in J_1 J_2 \subseteq J_1 I \cap J_2 I$. Then $(J_1 \cap J_2)^{\mathrm{r(I)}} = D^{\mathrm{r(I)}} = D \subsetneq J_1 J_2 \subseteq J_1 I \cap J_2 I = J_1^{\mathrm{r(I)}} \cap J_2^{\mathrm{r(I)}}$ and therefore r(I) is not h^2-stable. Moreover it follows from Theorem 4.4 (2) that r(I) is proper.

(2) We set $A = D + D\frac{1}{XX_1} \in \mathcal{F}_f(D)$ and $J = D + D\frac{1}{X_1} + D\frac{1}{X_2} + \cdots$. Then it is easy to see that $D = A \cap J$, $\frac{1}{XX_1} \notin D$ and $\frac{1}{XX_1} \in JI \cap A$.. Hence we have $(A \cap J)^{\mathrm{r(I)}} = D^{\mathrm{r(I)}} = D \subsetneq A \cap JI = A^{\mathrm{r(I)}} \cap J^{\mathrm{r(I)}}$ which implies that r(I) is not fh-stable. It also follows from Theorem 4.4 (2) that r(I) is proper.

$\square$

Furthermore we can also show that there exist infinitely many proper presemistar operations which are not fh-stable.

Proposition 4.11. *Let k be a field and let $D = k[X, Y]$. We set $J_k = D + \sum_{m=1}^{\infty} D\frac{1}{X^k + Y^m} \in \mathcal{K}(D) \setminus \mathcal{F}(D)$ for each integer $k \geq 1$. Then* $\mathrm{q(J_k)}$ *is a proper presemistar operation on D and is not fh-stable for each integer $k \geq 1$.*

Proof. If we set $I_k = D + D\frac{1}{X^k} \in \mathcal{F}_f(D)$ for each integer $k \geq 1$, then $(I_k \cap J_k)^{\mathrm{q(J_k)}} = J_k \neq I_k J_k = (I_k)^{\mathrm{q(J_k)}} \cap (J_k)^{\mathrm{q(J_k)}}$ for each integer $k \geq 1$ as in Example 4.6. Thus $\mathrm{q(J_k)}$ is not fh-stable for each integer $k \geq 1$. $\square$

Theorem 4.5. *Let D be an integral domain and let $\star$ be a presemistar operation on D. Then*

(i) *If $\star$ satisfies every property in S_2, then $\star$ need not satisfy a property in S_6.*

(ii) *If $\star$ satisfies every property in S_1, then $\star$ need not satisfy a property in S_5.*

(iii) *If $\star$ satisfies every property in S_2, then $\star$ need not satisfy a property in S_3.*

(iv) $\star$ *need not satisfy a property in* S_1.

(v) $\star$ *need not satisfy a property in* S_5.

Proof.

(i) Assume that D satisfies condition (NI). If we choose an invertible integral ideal I of D, then $\star = \mathrm{p(I)}$ is g^2-stable by Theorem 4.1 but is not gh-stable by Theorem 4.2 and therefore our assertion is valid.

(ii) Assume that D is a Noetherian domain which satisfies condition (NI). Then, for each invertible integral ideal $I, \star = \mathrm{p(I)}$ is f^2-stable but is not fh-stable by Proposition 4.7.

(iii) In Example 4.4, it was shown that there exists a proper presemistar operation $\star$ which is g^2-stable but is not h^2-stable.

(iv) This was shown in Example 4.5.

(v) This was shown in Example 4.6 for a Noetherian domain D and in Proposition 4.10 (2) for a non-Noetherian domain D.

□

Note 4.3.

(1) If we take D and I as in Proposition 4.10 (2), then we obtain that $\mathrm{r(I)}$ is f^n-stable for each integer $n \geq 2$ by Theorem 4.4 but is not fh-stable. Hence this presemistar operation $\mathrm{r(I)}$ would be a concrete example which satisfies Theorem 4.5 (ii).

(2) If we take D and I as in Proposition 4.10 (1), then we obtain that $\mathrm{r(I)}$ is g^n-stable for each integer $n \geq 2$ by Theorem 4.4 but is not h^2-stable. Hence this presemistar operation $\mathrm{r(I)}$ would be a concrete example which satisfies Theorem 4.5 (iii).

We shall now show that every presemistar operation on a valuation domain V is h^2-stable. First, we shall recall that each valuation domain is a *conducive domain* and so we have $\mathcal{K}(V) = \mathcal{F}(V) \cup \{K\}$ by [13, Proposition 43], where K is the quotient field of V.

Proposition 4.12. *Let V be a valuation domain. Then every presemistar operation on V is h^2-stable.*

Proof. Let E_1 and E_2 be arbitrary elements in $\mathcal{F}(V)$. Then $dE_1 \subseteq V$ and $dE_2 \subseteq V$ for some $0 \neq d \in V$ and so $dE_1 \subseteq dE_2$ or $dE_2 \subseteq dE_1$. Hence we have $E_1 \subseteq E_2$ or $E_2 \subseteq E_1$. Thus $\mathcal{K}(V)$ is linearly ordered with respect to the inclusion relation and therefore our assertion is valid. □

As in [14], we denote the set of all presemistar operations on D by $\mathbf{PS}(D)$.

Note 4.4. Suppose that V is a DVR with maximal ideal M. Then we can give a direct proof of Proposition 4.12. First, we recall that we proved $\mathbf{PS}(V) = \{d, e\} \cup \{\lambda(M^n) \mid n \in \mathbf{N}\}$ in [14, Theorem 3.1]. Next, by Proposition 4.1, $\lambda(M^n)$ is h^2-stable for each $n \in \mathbf{N}$. Hence every presemistar operation on a DVR V is h^2-stable.

Proposition 4.13. *Let D be a conducive domain and let $\star$ be a presemistar operation on D. Then*

(1) *If $\star$ is g^2-stable, then $\star$ is h^2-stable.*

(2) *If $\star$ satisfies some property in S_2, then $\star$ satisfies every property in S_3.*

Proof.

(1) For each $E \in \mathcal{F}(D)$, we have $(E \cap K)^\star = E^\star = E^\star \cap K = E^\star \cap K^\star$ and therefore, if $\star$ is g^2-stable, then $\star$ is evidently h^2-stable, because $\mathcal{K}(D) = \mathcal{F}(D) \cup \{K\}$ holds.

(2) This follows from (1) and Theorem 4.3 (ii) and (iii).

□

5. The ascent and the descent of stability

We recall from [14] the definition of both the ascent and the descent of a presemistar operation.

Definition 5.1. Let T be an overring of D. Then

(1) For each $\star \in \mathbf{PS}(D)$. we set $E^{\alpha_T(\star)} = E^\star$ for all $E \in \mathcal{K}(T)$. Then $\alpha_T(\star)$ is a presemistar operation on T.

(2) For each $\star \in \mathbf{PS}(T)$. we set $E^{\delta_T(\star)} = (ET)^\star$ for all $E \in \mathcal{K}(D)$. Then $\delta_T(\star)$ is a presemistar operation on D.

The presemistar operation $\alpha_T(\star)$ (resp. $\delta_T(\star)$) in Definition 5.1 is called *the ascent of* $\star$ (from D to T) (resp. *the descent of* $\star$ (from T to D)).

For each overring T of D, we set $X_1(T) = \mathcal{F}_f(T), X_2(T) = \mathcal{F}(T)$ and $X_3(T) = \mathcal{K}(T)$.

Remark 5.1. Let T be an overring of D. Then it easily follows that $X_3(T) \subseteq X_3, ET \in X_2(T)$ for all $E \in X_2$, and $ET \in X_1(T)$ for all $E \in X_1$.

A presemistar operation $\star$ on T is called an ${f_T}^a {g_T}^b {h_T}^c$-stable presemistar operation on T or $\star$ is said to be ${f_T}^a {g_T}^b {h_T}^c$-stable, if $\star$ satisfies the following condition

($\mathrm{S_T}$) $(F_0 \cap \cdots \cap F_a \cap G_0 \cap \cdots \cap G_b \cap H_0 \cap \cdots \cap H_c)^\star = F_0^\star \cap \cdots \cap F_a^\star \cap G_0^\star \cap \cdots \cap G_b^\star \cap H_0^\star \cap \cdots \cap H_c^\star$

for all $F_i(0 \leq i \leq a), G_j(0 \leq j \leq b), H_k(0 \leq k \leq c)$ where $F_i \in X_1(T)$ for $i \neq 0, G_j \in X_2(T)$ for $j \neq 0, H_k \in X_3(T)$ for $k \neq 0$ and $F_0 = G_0 = H_0 = K$.

For simplicity, we shall also denote ${f_T}^a = {f_T}^a {g_T}^0 {h_T}^0$, ${g_T}^b = {f_T}^0 {g_T}^b {h_T}^0$, $\cdots$, ${f_T}^a {h_T}^c = {f_T}^a {g_T}^0 {h_T}^c$, ${g_T}^b {h_T}^c = {f_T}^0 {g_T}^b {h_T}^c$ as in Section 2.

Remark 5.2. Let T be an overring of D. If $T \in \mathcal{F}(D)$, then $X_2(T) \subseteq X_2$ and if $T \in \mathcal{F}_f(D)$, then $X_1(T) \subseteq X_1$.

Note 5.1.

(1) If T is an overring of D such that $T \in \mathcal{F}_f(D)$, then T is integral over D.

(2) Let $u_1, u_2, \cdots, u_n$ be elements of K which are integral over D. If we set $T = D[u_1, u_2, \cdots, u_n]$, then T is integral over D and $T \in \mathcal{F}_f(D)$.

Proof. For the proof of (1), see [16, Proposition 13.20] and for the proof of (2), see [16, Corollary 13.21]. □

Proposition 5.1. ([14, Proposition 2.3 (A) (2) and (B) (2)]) *Let T be an overring of D.*

(1) *If $\star \in \mathbf{PS}(D)$ is h^2-stable, then $\alpha_T(\star)$ is ${h_T}^2$-stable.*

(2) *Assume that T is faithfully flat over D. If $\star \in \mathbf{PS}(T)$ is ${h_T}^2$-stable, then $\delta_T(\star)$ is h^2-stable.*

Proposition 5.2. *Let T be an overring of D.*

(1) *Assume that $T \in \mathcal{F}(D)$. If $\star \in \mathbf{PS}(D)$ is g^2-stable, then $\alpha_T(\star)$ is ${g_T}^2$-stable.*

(2) *Assume that T is flat over D. If $\star \in \mathbf{PS}(T)$ is ${g_T}^2$-stable, then $\delta_T(\star)$ is g^2-stable.*

Proof.

(1) Let $E, F \in \mathcal{F}(T) = X_2(T)$. Then by Remark 5.2, $E, F \in X_2 = \mathcal{F}(D)$ and then, by assumption, $(E \cap F)^{\alpha_T(\star)} = (E \cap F)^\star = E^\star \cap F^\star = E^{\alpha_T(\star)} \cap F^{\alpha_T(\star)}$ and so $\alpha_T(\star)$ is ${g_T}^2$-stable.

(2) Let $E, F \in \mathcal{F}(D) = X_2$. Then $ET, FT \in X_2(T)$ as stated in Remark 5.1. Since T is flat over $D, (E \cap F)T = ET \cap FT$ by [10, (3.H) (1)] and hence $(E \cap F)^{\delta_T(\star)} = ((E \cap F)T)^\star = (ET \cap FT)^\star = (ET)^\star \cap (FT)^\star = E^{\delta_T(\star)} \cap F^{\delta_T(\star)}$. Thus $\delta_T(\star)$ is g^2-stable.

□

Proposition 5.3. *Let T be an overring of D.*

(1) *Assume that $T \in \mathcal{F}_f(D)$. If $\star \in \mathbf{PS}(D)$ is f^2-stable, then $\alpha_T(\star)$ is ${f_T}^2$-stable.*

(2) *Assume that T is flat over D. If $\star \in \mathbf{PS}(T)$ is ${f_T}^2$-stable, then $\delta_T(\star)$ is f^2-stable.*

Proof. The proof is similar to that of Proposition 5.2. □

Proposition 5.4. *Let T be an overring of D.*

(1) *Assume that $T \in \mathcal{F}_f(D)$. If $\star \in \mathbf{PS}(D)$ is fg-stable, then $\alpha_T(\star)$ is $f_T g_T$-stable.*

(2) *Assume that T is flat over D. If $\star \in \mathbf{PS}(T)$ is $f_T g_T$-stable, then $\delta_T(\star)$ is fg-stable.*

Proof.

(1) Let $E \in \mathcal{F}_f(T)$ and $F \in \mathcal{F}(T)$. Then, since $T \in \mathcal{F}_f(D)$, $E \in \mathcal{F}_f(D)$ and $F \in \mathcal{F}(D)$. Hence, by assumption, we have $(E \cap F)^{\alpha_T(\star)} = (E \cap F)^\star = E^\star \cap F^\star = E^{\alpha_T(\star)} \cap F^{\alpha_T(\star)}$. Thus $\alpha_T(\star)$ is $f_T g_T$-stable.

(2) Let $E \in X_1$ and $F \in X_2$. Then $ET \in X_1(T)$ and $FT \in X_2(T)$ by Remark 5.1. Then, by assumption, $(E \cap F)^{\delta_T(\star)} = ((E \cap F)T)^\star = (ET \cap FT)^\star = (ET)^\star \cap (FT)^\star = E^{\delta_T(\star)} \cap F^{\delta_T(\star)}$. Hence $\delta_T(\star)$ is fg-stable.

□

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