DOCTORAL THESIS

Commutators of Calderón-Zygmund and generalized fractional integral operators with functions in generalized Campanato spaces on generalized Morrey spaces

一般化モリー空間における，一般化カンパナト空間に属する関数を伴うカルデロン・ジグムント作用素および一般化分数べき積分作用素の交換子

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Abstract

Let $\mathbb{R}^n$ be the n-dimensional Euclidean space. In this paper we consider the commutators $[b, T]$ and $[b, I_\rho]$, where T is a Calderón-Zygmund operator, I_ρ is a generalized fractional integral operator and b is a function in generalized Campanato spaces $\mathcal{L}^{(1,\phi)}(\mathbb{R}^n)$ with variable growth condition. We give necessary and sufficient conditions for the boundedness and compactness of the commutators on generalized Morrey spaces with variable growth condition.

It is well known that T is bounded on $L^p(\mathbb{R}^n)$ $(1 < p < \infty)$. Coifman, Rochberg and Weiss (1976) proved that, for $b \in \mathrm{BMO}(\mathbb{R}^n)$, the commutator $[b, T] = bT - Tb$ is also bounded on $L^p(\mathbb{R}^n)$ $(1 < p < \infty)$, that is,

$$\|[b, T]f\|_{L^p} = \|bTf - T(bf)\|_{L^p} \le C\|b\|_{\mathrm{BMO}}\|f\|_{L^p},$$

where C is a positive constant independent of b and f. They also gave a necessary condition for the boundedness, that is, $[b, T]$ is bounded on $L^p(\mathbb{R}^n)$ if and only if $b \in \mathrm{BMO}(\mathbb{R}^n)$. For the fractional integral operator I_α, Chanillo (1982) gave a necessary and sufficient condition for the L^p-L^q boundedness of $[b, I_\alpha]$. These results were extended to Morrey spaces by Di Fazio and Ragusa (1991).

On the other hand, Uchiyama (1978) gave a necessary and sufficient condition for the compactness of commutator $[b, T]$ on $L^p(\mathbb{R}^n)$. Namely, he proved that $[b, T]$ is compact on $L^p(\mathbb{R}^n)$ if and only if $b \in \mathrm{CMO}(\mathbb{R}^n)$, where $\mathrm{CMO}(\mathbb{R}^n)$ is the closure of $C^\infty_{\mathrm{comp}}(\mathbb{R}^n)$ in $\mathrm{BMO}(\mathbb{R}^n)$. This result was extended to Morrey spaces by Sawano and Shirai (2008) and Chen, Ding and Wang (2009, 2012).

In this paper we further extend all above resuts to generalized Morrey spaces with variable growth condition, by using Calderón-Zygmund operators T, generalized fractional integral operators I_ρ and functions $b \in \mathcal{L}^{(1,\phi)}(\mathbb{R}^n)$ generalized Campanato spaces with variable growth condition.

To prove the boundedness we show the norm estimates for the sharp maximal operators and the pointwise estimates for the sharp maximal operators of the commutators by the generalized fractional maximal operators. Then we use the boundedness of the generalized fractional maximal operators. Moreover, To prove the compactness we give relations between generalized Morrey spaces with variable growth condition and Musielak-Orlicz spaces. Then we give a criterion for the compactness of integral operators on generalized Morrey spaces with variable growth condition. We also extend the charactarization of $\mathrm{CMO}(\mathbb{R}^n)$ to the closure of $C^\infty_{\mathrm{comp}}(\mathbb{R}^n)$ in $\mathcal{L}^{(1,\phi)}(\mathbb{R}^n)$ generalized Campanato spaces with variable growth condition.

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Contents

Chapter 1

Introduction

1.1 Introduction

Let $\mathbb{R}^n$ be the n-dimensional Euclidean space. Let $b \in \mathrm{BMO}(\mathbb{R}^n)$ and T be a Calderón-Zygmund singular integral operator. In 1976 Coifman, Rochberg and Weiss [16] proved that the commutator $[b,T] = bT - Tb$ is bounded on $L^p(\mathbb{R}^n)$ $(1 < p < \infty)$, that is,

$$\|[b,T]f\|_{L^p} = \|bTf - T(bf)\|_{L^p} \le C\|b\|_{\mathrm{BMO}}\|f\|_{L^p},$$

where C is a positive constant independent of b and f, see Theorem 1.1.1. For the fractional integral operator I_α, Chanillo [8] proved the boundedness of $[b, I_\alpha]$ in 1982, see Theorem 1.1.2. Coifman, Rochberg and Weiss [16] and Chanillo [8] also gave the necessary conditions for the boundedness, that is, if the commutator $[b,T]$ or $[b, I_\alpha]$ is bounded, then b is in $\mathrm{BMO}(\mathbb{R}^n)$. These results were extended to Morrey and generalized Morrey spaces by Di Fazio and Ragusa [17] in 1991, and Mizuhara [36] in 1999, respectively. In this paper we further extend these results to generalized Morrey spaces $L^{(p,\varphi)}(\mathbb{R}^n)$ with variable growth condition. That is, under suitable assumptions, we have

$$\|[b,T]f\|_{L^{(q,\varphi)}(\mathbb{R}^n)} \le C\|b\|_{\mathcal{L}^{(1,\psi)}(\mathbb{R}^n)}\|f\|_{L^{(p,\varphi)}(\mathbb{R}^n)},$$
$$\|[b,I_\rho]f\|_{L^{(q,\varphi)}(\mathbb{R}^n)} \le C\|b\|_{\mathcal{L}^{(1,\psi)}(\mathbb{R}^n)}\|f\|_{L^{(p,\varphi)}(\mathbb{R}^n)},$$

where T is a Calderón-Zygmund operator, I_ρ is a generalized fractional integral operator and b is a function in Campanato spaces $\mathcal{L}^{(1,\psi)}(\mathbb{R}^n)$ with variable growth condition. We also give necessary conditions for the boundedness. For other extensions and generalization of [8, 16], see [21, 27, 53, 64, 65], etc.

Uchiyama [68] proved the compactness of commutator $[b,T]$ on $L^p(\mathbb{R}^n)$ in 1978. This result was extended to the compactness on Morrey spaces by Sawano and Shirai [61] in 2008. We also extend these results to generalized Morrey space $L^{(p,\varphi)}(\mathbb{R}^n)$ with variable growth condition by using $[b,T]$ and $[b,I_\rho]$ with $b\in\mathcal{L}^{(1,\psi)}(\mathbb{R}^n)$. To do this we extend the characterization of CMO to the closure of $C^\infty_{\mathrm{comp}}(\mathbb{R}^n)$ in $\mathcal{L}^{(1,\psi)}(\mathbb{R}^n)$ with variable growth condition.

This paper is a systematic reconstruction of all results in [2, 3, 4]. Related results are in [63, 5, 6].

Next we state the previous researches and the organization of this paper.

For the commutators $[b,T]$ and $[b,I_\alpha]$, the following theorems are known.

Theorem 1.1.1 (Coifman, Rochberg and Weiss [16])**.** *Let $p\in(1,\infty)$ and T be Calderón-Zygmund singular integral operator with smooth kernel. If $b\in\mathrm{BMO}(\mathbb{R}^n)$, then $[b,T]$ is bounded on $L^p(\mathbb{R}^n)$ and*

$$\|[b,T]f\|_{L^p}\le C\|b\|_{\mathrm{BMO}}\|f\|_{L^p},$$

where C is a positive constant independent of b and f. Conversely, if $[b,R_j]$ are bounded on $L^p(\mathbb{R}^n)$ for the Riesz transforms R_j, $j=1,\dots,n$, then $b\in\mathrm{BMO}(\mathbb{R}^n)$.

Theorem 1.1.2 (Chanillo [8])**.** *Let $\alpha\in(0,n)$, $p,q\in(1,\infty)$ and $-n/p+\alpha=-n/q$. If $b\in\mathrm{BMO}(\mathbb{R}^n)$, then $[b,I_\alpha]$ is bounded from $L^p(\mathbb{R}^n)$ to $L^q(\mathbb{R}^n)$ and*

$$\|[b,I_\alpha]f\|_{L^q}\le C\|b\|_{\mathrm{BMO}}\|f\|_{L^p},$$

where C is a positive constant independent of b and f. Conversely, if $n-\alpha$ is an even integer and $[b,I_\alpha]$ is bounded from $L^p(\mathbb{R}^n)$ to $L^q(\mathbb{R}^n)$, then $b\in\mathrm{BMO}(\mathbb{R}^n)$.

Theorems 1.1.1 and 1.1.2 were extended to Morrey spaces by Di Fazio and Ragusa [17] and Mizuhara [36]. In Chapter 2 we further extend these results to the boundedness of $[b,T]$ and $[b,I_\rho]$ on $L^{(p,\varphi)}(\mathbb{R}^n)$ for $b\in\mathcal{L}^{(1,\psi)}(\mathbb{R}^n)$ with $\varphi,\psi,\rho:\mathbb{R}^n\times(0,\infty)\to(0,\infty)$, where we assume the almost increasingness on ψ and use the equality $\mathcal{L}^{(1,\psi)}(\mathbb{R}^n)=\mathcal{L}^{(p,\psi^p)}(\mathbb{R}^n)$ with equivalent norms, see (1.2.6) for the definition of the almost increasingness and decreasingness. We also give the boundedness of $[b,T]$ and $[b,I_\rho]$ in case that ψ is almost decreasing in a different way.

To prove the results in Chapter 2 we use the sharp maximal operator $M^\sharp$ and generalized fractional maximal operators M_ρ. It is known that the usual fractional maximal operator M_α is dominated pointwise by the fractional integral operator I_α. That is, the boundedness of M_α follows from the boundedness of I_α. However, for generalized fractional maximal operators M_ρ, we need a better estimate than I_ρ. We prove the boundedness of M_ρ without the assumption (1.2.3) or (1.2.4). We also prove and use the boundedness of I_ρ from $L^p(\mathbb{R}^n)$ to Musielak-Orlicz spaces.

The organization of Chapter 2 is as follows. We state notation and theorems in Section 2.1. We give the boundednesss of the operators M_ρ and I_ρ in Section 2.2 and several lemmas in Section 2.3. Moreover, we investigate pointwise estimate by using the sharp maximal operator in Section 2.4 and Morrey norm estimate by the sharp maximal operator in Section 2.5. To estimate the Morrey norm by the sharp maximal operator we use the relation between Campanato spaces and Morrey spaces. Finally, using the results in Sections 2.2–2.5, we prove the theorems in Section 2.6.

In Chapter 3 we discuss the compactness of the commutators $[b, T]$ and $[b, I_\rho]$ on $L^{(p,\varphi)}(\mathbb{R}^n)$ for $b \in \mathcal{L}^{(1,\psi)}(\mathbb{R}^n)$ with $\varphi, \psi, \rho : \mathbb{R}^n \times (0, \infty) \to (0, \infty)$. Sawano and Shirai [61] treated commutators given by

$$[b, T]f(x) = \lim_{\epsilon \to +0} \int_{|x-y|>\epsilon} (b(x) - b(y))K(x, y)f(y)\, dy.$$

Using this expression, they proved the compactness of $[b, T]$ on Morrey spaces when b is a CMO function. We use their idea to prove our results. However, we need Musielak-Orlicz spaces to prove the compactness of $[b, T]$ and $[b, I_\rho]$ when b is in generalized Campanato spaces $\mathcal{L}^{(1,\psi)}(\mathbb{R}^n)$ with variable growth condition. We show the inclusion relation between Musielak-Orlicz spaces $L^\Phi(\mathbb{R}^n)$ and generalized Morrey spaces $L^{(p,\varphi)}(\mathbb{R}^n)$ with variable growth condition to prove the compactness of the commutators.

The organization of Chapter 3 is as follows. We state notation and theorems in Section 3.1. We give relations between generalized Morrey spaces $L^{(p,\varphi)}(\mathbb{R}^n)$ with variable growth condition and Musielak-Orlicz spaces $L^\Phi(\mathbb{R}^n)$ in Section 3.2. Next we give a criterion for the compactness of integral operators on generalized Morrey spaces with variable growth condition in Section 3.3 and prepare lemmas in Section 3.4. Then we prove the theorems in Section 3.5.

In Chapter 4 we extend the characterization of CMO to the closure of $C^\infty_{\mathrm{comp}}(\mathbb{R}^n)$ in $\mathcal{L}^{(1,\psi)}(\mathbb{R}^n)$. Uchiyama [68] considered the compactness of the commutator $[b,T]$ on $L^p(\mathbb{R}^n)$ in 1978, where T is a Calderón-Zygmund singular integral operator with convolution type of smooth kernel $K \not\equiv 0$. He proved that $[b,T]$ is compact on $L^p(\mathbb{R}^n)$ if and only if $b \in \mathrm{CMO}(\mathbb{R}^n)$, where $\mathrm{CMO}(\mathbb{R}^n)$ is the closure of $C^\infty_{\mathrm{comp}}(\mathbb{R}^n)$ in $\mathrm{BMO}(\mathbb{R}^n)$. In its proof he used the following characterization of $\mathrm{CMO}(\mathbb{R}^n)$, which was mentioned by Neri [54, Remark 2.6] without proof.

Theorem 1.1.3 ([68]). *Let $f \in \mathrm{BMO}(\mathbb{R}^n)$, and let $\mathrm{MO}(f, B(x,r))$ be the mean oscillation of f on the ball $B(x,r)$ centered at $x \in \mathbb{R}^n$ and of radius $r > 0$. Then $f \in \mathrm{CMO}(\mathbb{R}^n)$ if and only if f satisfies the following three conditions:*

(i) $\lim_{r\to+0} \sup_{x\in\mathbb{R}^n} \mathrm{MO}(f, B(x,r)) = 0$.

(ii) $\lim_{r\to\infty} \sup_{x\in\mathbb{R}^n} \mathrm{MO}(f, B(x,r)) = 0$.

(iii) $\lim_{|y|\to\infty} \mathrm{MO}(f, B(x+y,r)) = 0$ *for each ball $B(x,r)$.*

After that, using this characterization, many authors gave the characterization of various compact commutators on several function spaces. For example, Chen, Ding and Wang [10, 12] gave the characterization of the compact commutators $[b,T]$ and $[b,I_\alpha]$ on Morrey spaces. For the others, see [7, 9, 11, 13, 14, 35], etc.

In Chapter 4 we extend Theorem 1.1.3 to $\overline{C^\infty_{\mathrm{comp}}(\mathbb{R}^n)}^{\mathcal{L}_{1,\phi}(\mathbb{R}^n)}$ which is the closure of $C^\infty_{\mathrm{comp}}(\mathbb{R}^n)$ in the generalized Campanato space $\mathcal{L}_{1,\phi}(\mathbb{R}^n)$ with variable growth condition. To prove the extension of Theorem 1.1.3 we improve the proof of Uchiyama [68] by using the mollifier and a smooth cut-off method. As a corollary we give a characterization of the space $\overline{C^\infty_{\mathrm{comp}}(\mathbb{R}^n)}^{\mathrm{Lip}_\alpha(\mathbb{R}^n)}$ which is the closure of $C^\infty_{\mathrm{comp}}(\mathbb{R}^n)$ in the Lipschitz space $\mathrm{Lip}_\alpha(\mathbb{R}^n)$, $0 < \alpha < 1$. We state the theorem in Section 4.1 and prove it in Section 4.3.

In Chapter 5, as an application of the extension of Theorem 1.1.3, we prove that, if the commutator $[b,T]$ or $[b,I_\alpha]$ is compact, then b is in $\overline{C^\infty_{\mathrm{comp}}(\mathbb{R}^n)}^{\mathcal{L}_{1,\phi}(\mathbb{R}^n)}$. Actually, we show that, if b does not satisfy the characterization, then $[b,T]$ and $[b,I_\rho]$ are not compact. To do this we construct counterexamples. In Section 5.1, we state the theorems. Then we give lemmas in Section 5.2 and prove the theorems in Section 5.3.

At the end of this section, we make some conventions. Throughout this paper, we always use C to denote a positive constant that is independent of the main parameters involved but whose value may differ from line to line. Constants with subscripts, such as C_p, is dependent on the subscripts. If $f \le Cg$, we then write $f \lesssim g$ or $g \gtrsim f$; and if $f \lesssim g \lesssim f$, we then write $f \sim g$.

1.2 Definitions

We denote by $B(x,r)$ the open ball centered at $x \in \mathbb{R}^n$ and of radius r, that is,

$$B(x,r) = \{y \in \mathbb{R}^n : |y-x| < r\}.$$

For a measurable set $G \subset \mathbb{R}^n$, we denote by $|G|$ and χ_G the Lebesgue measure of G and the characteristic function of G, respectively. For a function $f \in L^1_{\mathrm{loc}}(\mathbb{R}^n)$ and a ball B, let

$$f_B = \fint_B f = \fint_B f(y)\,dy = \frac{1}{|B|}\int_B f(y)\,dy. \tag{1.2.1}$$

In this paper we consider generalized Morrey spaces $L^{(p,\varphi)}(\mathbb{R}^n)$ with $p \in [1,\infty)$ and variable growth function $\varphi : \mathbb{R}^n \times (0,\infty) \to (0,\infty)$. For a ball $B = B(x,r)$ we write $\varphi(B) = \varphi(x,r)$.

Definition 1.2.1. For $p \in [1,\infty)$ and $\varphi : \mathbb{R}^n \times (0,\infty) \to (0,\infty)$, let $L^{(p,\varphi)}(\mathbb{R}^n)$ be the set of all functions f such that the following functional is finite:

$$\|f\|_{L^{(p,\varphi)}(\mathbb{R}^n)} = \sup_B \left(\frac{1}{\varphi(B)}\fint_B |f(y)|^p\,dy\right)^{1/p},$$

where the supremum is taken over all balls B in $\mathbb{R}^n$.

Then $\|f\|_{L^{(p,\varphi)}(\mathbb{R}^n)}$ is a norm and $L^{(p,\varphi)}(\mathbb{R}^n)$ is a Banach space. Let $\varphi_\lambda(x,r) = r^\lambda$ for $\lambda \in [-n,0]$. Then $L^{(p,\varphi_\lambda)}(\mathbb{R}^n)$ is the classical Morrey space. That is,

$$\|f\|_{L^{(p,\varphi_\lambda)}(\mathbb{R}^n)} = \sup_B \left(\frac{1}{\varphi_\lambda(B)}\fint_B |f(y)|^p\,dy\right)^{1/p} = \sup_{B=B(x,r)} \left(\frac{1}{r^\lambda}\fint_B |f(y)|^p\,dy\right)^{1/p}.$$

If $\lambda = -n$, then $L^{(p,\varphi_{-n})}(\mathbb{R}^n) = L^p(\mathbb{R}^n)$. If $\lambda = 0$, then $L^{(p,\varphi_0)}(\mathbb{R}^n) = L^\infty(\mathbb{R}^n)$.

Generalized Morrey spaces $L^{(p,\varphi)}(\mathbb{R}^n)$ with variable growth function φ were introduced in [38] and studied in [39, 43, 47], etc.

We also consider generalized Campanato spaces with variable growth condition.

Definition 1.2.2. For $p \in [1,\infty)$ and $\varphi : \mathbb{R}^n \times (0,\infty) \to (0,\infty)$, let $\mathcal{L}^{(p,\varphi)}(\mathbb{R}^n)$ be the set of all functions f such that the following functional is finite:

$$\|f\|_{\mathcal{L}^{(p,\varphi)}(\mathbb{R}^n)} = \sup_B \left(\frac{1}{\varphi(B)} \fint_B |f(y) - f_B|^p \, dy \right)^{1/p},$$

where the supremum is taken over all balls B in $\mathbb{R}^n$.

Then $\|f\|_{\mathcal{L}^{(p,\varphi)}(\mathbb{R}^n)}$ is a norm modulo constant functions and thereby $\mathcal{L}^{(p,\varphi)}(\mathbb{R}^n)$ is a Banach space. If $p = 1$ and $\varphi \equiv 1$, then $\mathcal{L}^{(p,\varphi)}(\mathbb{R}^n) = \mathrm{BMO}(\mathbb{R}^n)$. If $p = 1$ and $\varphi(r) = r^\alpha$ $(0 < \alpha \le 1)$, then $\mathcal{L}^{(p,\varphi)}(\mathbb{R}^n)$ coincides with $\mathrm{Lip}_\alpha(\mathbb{R}^n)$.

Generalized Campanato spaces $\mathcal{L}^{(p,\varphi)}(\mathbb{R}^n)$ with variable growth function φ were introduced in [51] to characterize pointwise multipliers on $\mathrm{BMO}(\mathbb{R}^n)$ and studied in [37, 43, 46], etc. Moreover, it has been proved that $\mathcal{L}^{(p,\varphi)}(\mathbb{R}^n)$ is the dual space of the Hardy space $H^{p(\cdot)}(\mathbb{R}^n)$ with variable exponent in [48].

In Chapters 4 and 5 we also use the following space and norm.

Definition 1.2.3. For $p \in [1,\infty)$ and $\psi : \mathbb{R}^n \times (0,\infty) \to (0,\infty)$, let $\mathcal{L}_{p,\psi}(\mathbb{R}^n)$ be the set of all functions f such that the following functional is finite:

$$\|f\|_{\mathcal{L}_{p,\psi}(\mathbb{R}^n)} = \sup_B \frac{1}{\psi(B)} \left(\fint_B |f(y) - f_B|^p \, dy \right)^{1/p},$$

where the supremum is taken over all balls B in $\mathbb{R}^n$.

Then $\mathcal{L}_{1,\psi}(\mathbb{R}^n) = \mathcal{L}^{(1,\psi)}(\mathbb{R}^n)$ and $\|f\|_{\mathcal{L}_{1,\psi}(\mathbb{R}^n)} = \|f\|_{\mathcal{L}^{(1,\psi)}(\mathbb{R}^n)}$.

Next we recall Calderón-Zygmund operators and generalized fractional integral operators.

A linear operator T from $\mathcal{S}(\mathbb{R}^n)$ to $\mathcal{S}'(\mathbb{R}^n)$ is said to be a Calderón-Zygmund operator if T is bounded on $L^2(\mathbb{R}^n)$ and there exists a standard kernel K (see Definition 2.1.1) such that, for $f \in L^2_{\mathrm{comp}}(\mathbb{R}^n)$,

$$Tf(x) = \int_{\mathbb{R}^n} K(x,y) f(y) \, dy, \quad x \notin \operatorname{supp} f.$$

Observe that the Hilbert transform ($n = 1$, $K(x,y) = (x-y)/|x-y|^2$) and the Riesz transforms ($n \ge 2$, $K(x,y) = (x_j - y_j)/|x-y|^{n+1}$, $j = 1, \dots, n$) are Calderón-Zygmund operators. It is known that any Calderón-Zygmund operator T is bounded on $L^p(\mathbb{R}^n)$ for $1 < p < \infty$. This boundedness was extended to generalized Morrey spaces $L^{(p,\varphi)}(\mathbb{R}^n)$ with variable growth function φ by [38].

Let I_α be the fractional integral operator of order $\alpha \in (0, n)$, that is,

$$I_\alpha f(x) = \int_{\mathbb{R}^n} \frac{f(y)}{|x-y|^{n-\alpha}}\,dy.$$

Then it is known as the Hardy-Littlewood-Sobolev theorem that I_α is bounded from $L^p(\mathbb{R}^n)$ to $L^q(\mathbb{R}^n)$, if $\alpha \in (0, n)$, $p, q \in (1, \infty)$ and $-n/p + \alpha = -n/q$. This boundedness was extended to Morrey spaces by Adams [1] as follows: If $\alpha \in (0, n)$, $p, q \in (1, \infty)$, $\lambda \in [-n, 0)$ and $\lambda/p + \alpha = \lambda/q$, then I_α is bounded from $L^{(p,\varphi_\lambda)}(\mathbb{R}^n)$ to $L^{(q,\varphi_\lambda)}(\mathbb{R}^n)$. See also [58].

For a function $\rho : \mathbb{R}^n \times (0, \infty) \to (0, \infty)$, we consider the generalized fractional integral operator I_ρ defined by

$$I_\rho f(x) = \int_{\mathbb{R}^n} \frac{\rho(x, |x-y|)}{|x-y|^n} f(y)\,dy, \tag{1.2.2}$$

where we always assume that

$$\int_0^1 \frac{\rho(x,t)}{t}\,dt < \infty \quad \text{for each } x \in \mathbb{R}^n, \tag{1.2.3}$$

and that there exist positive constants C, K_1 and K_2 with $K_1 < K_2$ such that

$$\sup_{r \le t \le 2r} \rho(x,t) \le C \int_{K_1 r}^{K_2 r} \frac{\rho(x,t)}{t}\,dt \quad \text{for all } x \in \mathbb{R}^n \text{ and } r > 0. \tag{1.2.4}$$

The condition (1.2.3) is needed for the integral in (1.2.2) to converge for bounded functions f with compact support. The condition (1.2.4) was considered in [59].

If $\rho(x, r) = r^\alpha$, $0 < \alpha < n$, then I_ρ is the usual fractional integral operator I_α. If $\alpha(\cdot) : \mathbb{R}^n \to (0, n)$ and $\rho(x, r) = r^{\alpha(x)}$, then I_ρ is a generalized fractional integral operator $I_{\alpha(x)}$ with variable order defined by

$$I_{\alpha(x)} f(x) = \int_{\mathbb{R}^n} \frac{f(y)}{|x-y|^{n-\alpha(x)}}\,dy.$$

Let $0 < \alpha < n$ and

$$\rho(r) = \begin{cases} r^\alpha, & 0 < r < 1, \\ e^{-(r-1)}, & 1 \le r < \infty. \end{cases}$$

Then ρ satisfies (1.2.3) and (1.2.4). Other examples of more general ρ, see Corollaries 2.14 and 2.15 in [47]. The operators I_ρ with $\rho : (0,\infty) \to (0,\infty)$ are studied in [18, 19, 23, 24, 25, 40, 41, 42, 44, 62, 66], etc. The boundedness of I_ρ with $\rho : \mathbb{R}^n \times (0,\infty) \to (0,\infty)$ on generalized Morrey spaces $L^{(p,\varphi)}(\mathbb{R}^n)$ with $\varphi : \mathbb{R}^n \times (0,\infty) \to (0,\infty)$ was given by [47]. See also [31, 32].

In this paper we investigate the commutators

$$[b,T]f = bTf - T(bf) \quad \text{and} \quad [b,I_\rho]f = bI_\rho f - I_\rho(bf),$$

for $b \in \mathcal{L}^{(1,\psi)}(\mathbb{R}^n)$ and $f \in L^{(p,\varphi)}(\mathbb{R}^n)$ with $\varphi, \psi, \rho : \mathbb{R}^n \times (0,\infty) \to (0,\infty)$.

We say that a function $\theta : \mathbb{R}^n \times (0,\infty) \to (0,\infty)$ satisfies the doubling condition if there exists a positive constant C such that, for all $x \in \mathbb{R}^n$ and $r, s \in (0,\infty)$,

$$\frac{1}{C} \le \frac{\theta(x,r)}{\theta(x,s)} \le C, \quad \text{if } \frac{1}{2} \le \frac{r}{s} \le 2. \tag{1.2.5}$$

We say that θ is almost increasing (resp. almost decreasing) if there exists a positive constant C such that, for all $x \in \mathbb{R}^n$ and $r, s \in (0,\infty)$,

$$\theta(x,r) \le C\theta(x,s) \quad (\text{resp. } \theta(x,s) \le C\theta(x,r)), \quad \text{if } r < s. \tag{1.2.6}$$

We also consider the following condition; there exists a positive constant C such that, for all $x, y \in \mathbb{R}^n$ and $r \in (0,\infty)$,

$$\frac{1}{C} \le \frac{\theta(x,r)}{\theta(y,r)} \le C, \quad \text{if } |x-y| \le r. \tag{1.2.7}$$

For two functions $\theta, \kappa : \mathbb{R}^n \times (0,\infty) \to (0,\infty)$, we write $\theta \sim \kappa$ if there exists a positive constant C such that, for all $x \in \mathbb{R}^n$ and $r \in (0,\infty)$,

$$\frac{1}{C} \le \frac{\theta(x,r)}{\kappa(x,r)} \le C. \tag{1.2.8}$$

Let $1 \le p < \infty$ and $\varphi, \tilde{\varphi} : \mathbb{R}^n \times (0,\infty) \to (0,\infty)$. If $\varphi \sim \tilde{\varphi}$, then $L^{(p,\varphi)}(\mathbb{R}^n) = L^{(p,\tilde{\varphi})}(\mathbb{R}^n)$ with equivalent norms. If $\lim_{r\to\infty} \varphi(x,r)r^n = 0$ for some $x \in \mathbb{R}^n$, then $L^{(p,\varphi)}(\mathbb{R}^n) = \{0\}$, since

$$\int_{B(x,r)} |f(y)|^p \, dy \le \varphi(x,r)r^n \|f\|_{L^{(p,\varphi)}(\mathbb{R}^n)}^p \to 0 \quad \text{as} \quad r \to \infty.$$

If $\lim_{r\to 0}\varphi(x,r)=0$ on a subset $E\subset\mathbb{R}^n$, then, for all $f\in L^{(p,\varphi)}(\mathbb{R}^n)$, $f=0$ a.e. E, since

$$\frac{1}{|B(x,r)|}\int_{B(x,r)}|f(y)|\,dy\lesssim\varphi(x,r)\|f\|_{L^{(p,\varphi)}(\mathbb{R}^n)}\to 0\quad\text{as}\quad r\to 0.$$

In this paper we consider the following classes of φ:

Definition 1.2.4. (i) Let $\mathcal{G}^{\rm dec}$ be the set of all functions $\varphi:\mathbb{R}^n\times(0,\infty)\to(0,\infty)$ such that φ is almost decreasing and that $r\mapsto\varphi(x,r)r^n$ is almost increasing. That is, there exists a positive constant C such that, for all $x\in\mathbb{R}^n$ and $r,s\in(0,\infty)$,

$$C\varphi(x,r)\ge\varphi(x,s),\quad \varphi(x,r)r^n\le C\varphi(x,s)s^n,\quad\text{if } r<s.$$

(ii) Let $\mathcal{G}^{\rm inc}$ be the set of all functions $\varphi:\mathbb{R}^n\times(0,\infty)\to(0,\infty)$ such that φ is almost increasing and that $r\mapsto\varphi(x,r)/r$ is almost decreasing. That is, there exists a positive constant C such that, for all $x\in\mathbb{R}^n$ and $r,s\in(0,\infty)$,

$$\varphi(x,r)\le C\varphi(x,s),\quad C\varphi(x,r)/r\ge\varphi(x,s)/s,\quad\text{if } r<s.$$

If $\varphi\in\mathcal{G}^{\rm dec}$ or $\varphi\in\mathcal{G}^{\rm inc}$, then φ satisfies the doubling condition (1.2.5).

Remark 1.2.1. It is known by [45] that, if $\psi\in\mathcal{G}^{\rm inc}$ and ψ satisfies (1.2.7), then $\mathcal{L}^{(p,\psi^p)}(\mathbb{R}^n)=\mathcal{L}^{(1,\psi)}(\mathbb{R}^n)$ with equivalent norms for each $p\in[1,\infty)$, see Corollary 2.3.3. In particular, $\mathcal{L}^{(p,\psi^p)}(\mathbb{R}^n)=\mathrm{BMO}(\mathbb{R}^n)$ if $\psi\equiv 1$ and $\mathcal{L}^{(p,\psi^p)}(\mathbb{R}^n)=\mathrm{Lip}_\alpha(\mathbb{R}^n)$ if $\psi(x,r)=r^\alpha$, $0<\alpha\le 1$. For the relation between $\mathcal{L}^{(p,\psi^p)}(\mathbb{R}^n)$ and Hölder (Lipschitz) spaces $\Lambda_\psi(\mathbb{R}^n)$, see [43]. It is also known by [43] that, if $\psi\in\mathcal{G}^{\rm dec}$ and ψ satisfies (2.1.4) below, then, for every $f\in\mathcal{L}^{(p,\psi)}(\mathbb{R}^n)$, $f_{B(0,r)}$ converges as $r\to\infty$ and $\|f\|_{\mathcal{L}^{(p,\psi)}}\sim\|f-\lim_{r\to\infty}f_{B(0,r)}\|_{L^{(p,\psi)}}$, see Lemma 2.3.5.

Remark 1.2.2. Let $\varphi\in\mathcal{G}^{\rm dec}$. If φ satisfies

$$\lim_{r\to 0}\varphi(x,r)=\infty,\quad \lim_{r\to\infty}\varphi(x,r)=0,\tag{1.2.9}$$

then there exists $\tilde\varphi\in\mathcal{G}^{\rm dec}$ such that $\varphi\sim\tilde\varphi$ and that $\varphi(x,\cdot)$ is continuous, strictly decreasing and bijective from $(0,\infty)$ to itself for each x. This fact follows from [44, Proposition 3.4].

Remark 1.2.3. We say that a function $\alpha(\cdot):\mathbb{R}^n\to(-\infty,\infty)$ is log-Hölder continuous if there exists a positive constant $C_{\alpha(\cdot)}$ such that

$$|\alpha(x)-\alpha(y)|\le\frac{C_{\alpha(\cdot)}}{\log(e/|x-y|)}\quad\text{for}\quad 0<|x-y|<1.$$

Let $\alpha(\cdot) : \mathbb{R}^n \to (-\infty, \infty)$ be log-Hölder continuous and satisfy

$$-\infty < \inf_{x \in \mathbb{R}^n} \alpha(x) \le \sup_{x \in \mathbb{R}^n} \alpha(x) < \infty, \quad -\infty < \alpha_* < \infty,$$

and let

$$\varphi(x, r) = \begin{cases} r^{\alpha(x)}, & 0 < r < 1, \\ r^{\alpha_*}, & 1 \le r < \infty. \end{cases}$$

Then φ satisfies (1.2.5) and (1.2.7), see [46, Proposition 3.3].

Chapter 2

Boundedness

2.1 Theorems and examples

First we recall the definition of Calderón-Zygmund operators following [69]. Let Ω be the set of all nonnegative nondecreasing functions ω on $(0, \infty)$ such that $\int_0^1 \frac{\omega(t)}{t} dt < \infty$.

Definition 2.1.1 (standard kernel)**.** Let $\omega \in \Omega$. A continuous function $K(x, y)$ on $\mathbb{R}^n \times \mathbb{R}^n \setminus \{(x, x) \in \mathbb{R}^{2n}\}$ is said to be a standard kernel of type ω if the following conditions are satisfied;

$$|K(x,y)| \le \frac{C}{|x-y|^n} \quad \text{for} \quad x \neq y, \tag{2.1.1}$$

$$|K(x,y) - K(x,z)| + |K(y,x) - K(z,x)| \le \frac{C}{|x-y|^n}\,\omega\left(\frac{|y-z|}{|x-y|}\right) \quad \text{for} \quad 2|y-z| \le |x-y|. \tag{2.1.2}$$

Definition 2.1.2 (Calderón-Zygmund operator)**.** Let $\omega \in \Omega$. A linear operator T from $\mathcal{S}(\mathbb{R}^n)$ to $\mathcal{S}'(\mathbb{R}^n)$ is said to be a Calderón-Zygmund operator of type ω, if T is bounded on $L^2(\mathbb{R}^n)$ and there exists a standard kernel K of type ω such that, for $f \in L^2_{\mathrm{comp}}(\mathbb{R}^n)$,

$$Tf(x) = \int_{\mathbb{R}^n} K(x,y) f(y)\, dy, \quad x \notin \operatorname{supp} f. \tag{2.1.3}$$

Remark 2.1.1. If $x \notin \operatorname{supp} f$, then $K(x, y)$ is continuous on $\operatorname{supp} f$ with respect to y. Therefore, if (2.1.3) holds for $f \in L^2_{\mathrm{comp}}(\mathbb{R}^n)$, then (2.1.3) holds for $f \in L^1_{\mathrm{comp}}(\mathbb{R}^n)$.

It is known by [69, Theorem 2.4] that any Calderón-Zygmund operator of type $\omega \in \Omega$ is bounded on $L^p(\mathbb{R}^n)$ for $1 < p < \infty$.

This result was extended to generalized Morrey spaces $L^{(p,\varphi)}(\mathbb{R}^n)$ with variable growth function φ by [38] as the following: Assume that $\varphi \in \mathcal{G}^{\mathrm{dec}}$ and that there exists a positive constant C such that, for all $x \in \mathbb{R}^n$ and $r \in (0,\infty)$,

$$\int_r^\infty \frac{\varphi(x,t)}{t}\,dt \le C\varphi(x,r). \tag{2.1.4}$$

For $f \in L^{(p,\varphi)}(\mathbb{R}^n)$, $1 < p < \infty$, we define Tf on each ball B by

$$Tf(x) = T(f\chi_{2B})(x) + \int_{\mathbb{R}^n\setminus 2B} K(x,y)f(y)\,dy, \quad x \in B. \tag{2.1.5}$$

Then the first term in the right hand side is well defined, since $f\chi_{2B} \in L^p(\mathbb{R}^n)$, and the integral of the second term converges absolutely. Moreover, $Tf(x)$ is independent of the choice of the ball containing x. By this definition we can show that T is a bounded operator on $L^{(p,\varphi)}(\mathbb{R}^n)$. For the definition of Tf, see also [49, Section 5] and [60].

For functions f in Morrey spaces, we define $[b,T]f$ on each ball B by

$$[b,T]f(x) = [b,T](f\chi_{2B})(x) + \int_{\mathbb{R}^n\setminus 2B} (b(x)-b(y))K(x,y)f(y)\,dy, \quad x \in B, \tag{2.1.6}$$

see Remark 2.3.2 for its well-definedness. Then we have the following theorem.

Theorem 2.1.1. *Let* $1 < p \le q < \infty$ *and* $\varphi, \psi : \mathbb{R}^n \times (0,\infty) \to (0,\infty)$*. Assume that* $\varphi \in \mathcal{G}^{\mathrm{dec}}$ *and* $\psi \in \mathcal{G}^{\mathrm{inc}}$*. Let* T *be a Calderón-Zygmund operator of type* $\omega \in \Omega$*.*

(i) *Assume that* ψ *satisfies* (1.2.7)*, that* φ *satisfies* (2.1.4)*, that* $\int_0^1 \frac{\omega(t)\log(1/t)}{t}\,dt < \infty$ *and that there exists a positive constant* C_0 *such that, for all* $x \in \mathbb{R}^n$ *and* $r \in (0,\infty)$*,*

$$\psi(x,r)\varphi(x,r)^{1/p} \le C_0\varphi(x,r)^{1/q}. \tag{2.1.7}$$

If $b \in \mathcal{L}^{(1,\psi)}(\mathbb{R}^n)$*, then* $[b,T]f$ *in* (2.1.6) *is well defined for all* $f \in L^{(p,\varphi)}(\mathbb{R}^n)$ *and there exists a positive constant* C*, independent of* b *and* f*, such that*

$$\|[b,T]f\|_{L^{(q,\varphi)}} \le C\|b\|_{\mathcal{L}^{(1,\psi)}}\|f\|_{L^{(p,\varphi)}}.$$

(ii) *Conversely, assume that* φ *satisfies* (1.2.7) *and that there exists a positive constant* C_0 *such that, for all* $x \in \mathbb{R}^n$ *and* $r \in (0, \infty)$,

$$C_0 \psi(x,r)\varphi(x,r)^{1/p} \geq \varphi(x,r)^{1/q}. \tag{2.1.8}$$

If T *is a convolution type such that*

$$Tf(x) = p.v. \int_{\mathbb{R}^n} K(x-y)f(y)\,dy \tag{2.1.9}$$

with homogeneous kernel K *satisfying* $K(x) = |x|^{-n}K(x/|x|)$, $\int_{S^{n-1}} K = 0$, $K \in C^\infty(S^{n-1})$ *and* $K \not\equiv 0$, *and if* $[b,T]$ *is bounded from* $L^{(p,\varphi)}(\mathbb{R}^n)$ *to* $L^{(q,\varphi)}(\mathbb{R}^n)$, *then* $b \in \mathcal{L}^{(1,\psi)}(\mathbb{R}^n)$ *and there exists a positive constant* C, *independent of* b, *such that*

$$\|b\|_{\mathcal{L}^{(1,\psi)}} \leq C\|[b,T]\|_{L^{(p,\varphi)}\to L^{(q,\varphi)}},$$

where $\|[b,T]\|_{L^{(p,\varphi)}\to L^{(q,\varphi)}}$ *is the operator norm of* $[b,T]$ *from* $L^{(p,\varphi)}(\mathbb{R}^n)$ *to* $L^{(q,\varphi)}(\mathbb{R}^n)$.

In the above theorem, if $\psi \equiv 1$ and $\varphi(x,r) = r^{-n}$, then $\mathcal{L}^{(1,\psi)}(\mathbb{R}^n) = \mathrm{BMO}(\mathbb{R}^n)$ and $L^{(p,\varphi)}(\mathbb{R}^n) = L^p(\mathbb{R}^n)$ with $p = q$. This is the case of Theorem 1.1.1.

If $\psi(x,r) = r^\alpha$, $0 < \alpha \leq 1$, and $\varphi(x,r) = r^{-n}$, then $\mathcal{L}^{(1,\psi)}(\mathbb{R}^n) = \mathrm{Lip}_\alpha(\mathbb{R}^n)$, $L^{(p,\varphi)}(\mathbb{R}^n) = L^p(\mathbb{R}^n)$ and $L^{(q,\varphi)}(\mathbb{R}^n) = L^q(\mathbb{R}^n)$ with $-n/p + \alpha = -n/q$. That is,

$$\|[b,T]f\|_{L^q} \lesssim \|b\|_{\mathrm{Lip}_\alpha}\|f\|_{L^p}.$$

This is the case of Janson [28, Lemma 12].

Example 2.1.1. Let $1 < p \leq q < \infty$ and $\beta(\cdot), \lambda(\cdot) : \mathbb{R}^n \to (-\infty, \infty)$. Assume that

$$\begin{aligned} 0 \leq \inf_{x\in\mathbb{R}^n} \beta(x) \leq \sup_{x\in\mathbb{R}^n} \beta(x) \leq 1, \quad 0 \leq \beta_* \leq 1, \\ -n \leq \inf_{x\in\mathbb{R}^n} \lambda(x) \leq \sup_{x\in\mathbb{R}^n} \lambda(x) < 0, \quad -n \leq \lambda_* < 0. \end{aligned}$$

Let

$$\psi(x,r) = \begin{cases} r^{\beta(x)}, \\ r^{\beta_*}, \end{cases} \quad \varphi(x,r) = \begin{cases} r^{\lambda(x)}, & 0 < r < 1, \\ r^{\lambda_*}, & 1 \leq r < \infty. \end{cases}$$

Let T be a Calderón-Zygmund operator of type $\omega \in \Omega$ with $\int_0^1 \frac{\omega(t)\log(1/t)}{t}\,dt < \infty$. If $\beta(\cdot)$ is log-Hölder continuous and

$$\beta(x) + \lambda(x)/p \geq \lambda(x)/q, \quad \beta_* + \lambda_*/p \leq \lambda_*/q,$$

then

$$\|[b,T]f\|_{L^{(q,\varphi)}} \leq C\|b\|_{\mathcal{L}^{(1,\psi)}}\|f\|_{L^{(p,\varphi)}},$$

since ψ satisfies (1.2.7) (see Remark 1.2.3), φ satisfies (2.1.4), and (2.1.7) holds. Conversely, if $\lambda(\cdot)$ is log-Hölder continuous and

$$\beta(x) + \lambda(x)/p \leq \lambda(x)/q, \quad \beta_* + \lambda_*/p \geq \lambda_*/q,$$

and if T is a convolution type with homogeneous kernel K satisfying $K(x) = |x|^{-n}K(x/|x|)$, $\int_{S^{n-1}} K = 0$, $K \in C^\infty(S^{n-1})$ and $K \not\equiv 0$, then

$$\|b\|_{\mathcal{L}^{(1,\psi)}} \leq C\|[b,T]\|_{L^{(p,\varphi)} \to L^{(q,\varphi)}}.$$

We also take the cases

$$\psi(x,r) = \begin{cases} r^{\beta(x)}(1/\log(e/r))^{\beta_1(x)}, & 0 < r < 1, \\ r^{\beta_*}(\log(er))^{\beta_{**}}, & 1 \leq r < \infty, \end{cases}$$

etc.

Next, we consider generalized fractional integral operators I_ρ. Assume that ρ satisfies (1.2.3) and (1.2.4). Let $1 < p < \infty$ and $\varphi \in \mathcal{G}^{\mathrm{dec}}$. Then, for $f \in L^{(p,\varphi)}(\mathbb{R}^n)$, under some suitable condition, the integral in (1.2.2) converges absolutely and we can show that I_ρ is a bounded operator from $L^{(p,\varphi)}(\mathbb{R}^n)$ to $L^{(q,\varphi)}(\mathbb{R}^n)$, see Theorem 2.2.2 for details.

For the commutator $[b, I_\rho]$ we have the following theorem.

Theorem 2.1.2. *Let $1 < p < q < \infty$ and $\rho, \varphi, \psi : \mathbb{R}^n \times (0,\infty) \to (0,\infty)$. Assume that $\varphi \in \mathcal{G}^{\mathrm{dec}}$ and $\psi \in \mathcal{G}^{\mathrm{inc}}$. Assume also that ρ satisfies* (1.2.3) *and* (1.2.4)*. Let $\rho^*(x,r) = \int_0^r \frac{\rho(x,t)}{t}\,dt$.*

(i) *Assume that ρ, ρ^* and ψ satisfy* (1.2.7)*, that φ satisfies* (2.1.4) *and that there exist positive constants ϵ, C_ρ, C_0, C_1 and an exponent $\tilde{p} \in (p,q]$ such that, for* *all* $x, y \in \mathbb{R}^n$ *and* $r, s \in (0, \infty)$*,*

$$(2.1.10) \quad C_\rho \frac{\rho(x,r)}{r^{n-\epsilon}} \ge \frac{\rho(x,s)}{s^{n-\epsilon}}, \quad \text{if } r < s,$$

$$(2.1.11) \quad \left| \frac{\rho(x,r)}{r^n} - \frac{\rho(y,s)}{s^n} \right| \le C_\rho \, (|r-s| + |x-y|) \frac{\rho^*(x,r)}{r^{n+1}},$$

$$\text{if } \frac{1}{2} \le \frac{r}{s} \le 2 \text{ and } |x-y| < r/2,$$

$$(2.1.12) \quad \int_0^r \frac{\rho(x,t)}{t} \, dt \, \varphi(x,r)^{1/p} + \int_r^\infty \frac{\rho(x,t)\varphi(x,t)^{1/p}}{t} \, dt \le C_0 \varphi(x,r)^{1/\tilde{p}},$$

$$(2.1.13) \quad \psi(x,r)\varphi(x,r)^{1/\tilde{p}} \le C_1 \varphi(x,r)^{1/q}.$$

If $b \in \mathcal{L}^{(1,\psi)}(\mathbb{R}^n)$*, then* $[b, I_\rho]f$ *is well defined for all* $f \in L^{(p,\varphi)}(\mathbb{R}^n)$ *and there exists a positive constant* C*, independent of* b *and* f*, such that*

$$\|[b, I_\rho]f\|_{L^{(q,\varphi)}} \le C \|b\|_{\mathcal{L}^{(1,\psi)}} \|f\|_{L^{(p,\varphi)}}.$$

(ii) *Conversely, assume that* φ *satisfies* (1.2.7)*, that* $\rho(x,r) = r^\alpha$*,* $0 < \alpha < n$*, and that*

$$(2.1.14) \qquad C_0 \psi(x,r) r^\alpha \, \varphi(x,r)^{1/p} \ge \varphi(x,r)^{1/q}.$$

If $[b, I_\alpha]$ *is bounded from* $L^{(p,\varphi)}(\mathbb{R}^n)$ *to* $L^{(q,\varphi)}(\mathbb{R}^n)$*, then* $b \in \mathcal{L}^{(1,\psi)}(\mathbb{R}^n)$ *and there exists a positive constant* C*, independent of* b*, such that*

$$\|b\|_{\mathcal{L}^{(1,\psi)}} \le C \|[b, I_\alpha]\|_{L^{(p,\varphi)} \to L^{(q,\varphi)}},$$

where $\|[b, I_\alpha]\|_{L^{(p,\varphi)} \to L^{(q,\varphi)}}$ *is the operator norm of* $[b, I_\alpha]$ *from* $L^{(p,\varphi)}(\mathbb{R}^n)$ *to* $L^{(q,\varphi)}(\mathbb{R}^n)$*.*

In the above theorem, if $\rho(x,r) = r^\alpha$, $0 < \alpha < n$, $\psi \equiv 1$ and $\varphi(x,r) = r^{-n}$, then $I_\rho = I_\alpha$, $\mathcal{L}^{(1,\psi)}(\mathbb{R}^n) = \mathrm{BMO}(\mathbb{R}^n)$, $L^{(p,\varphi)}(\mathbb{R}^n) = L^p(\mathbb{R}^n)$ and $L^{(q,\varphi)}(\mathbb{R}^n) = L^q(\mathbb{R}^n)$. This is the case of Theorem 1.1.2. See also [34]. For the well-definedness of $[b, I_\rho]f$ under the assumption in Theorem 2.1.2 (i), see Remark 2.3.3.

Example 2.1.2. Let $1 < p < \tilde{p} \le q < \infty$ and $\alpha(\cdot), \beta(\cdot), \lambda(\cdot) : \mathbb{R}^n \to (-\infty, \infty)$. Assume that

$$\begin{aligned} 0 < \inf_{x\in\mathbb{R}^n} \alpha(x) \le \sup_{x\in\mathbb{R}^n} \alpha(x) < n, &\quad 0 < \alpha_* < n, \\ 0 \le \inf_{x\in\mathbb{R}^n} \beta(x) \le \sup_{x\in\mathbb{R}^n} \beta(x) \le 1, &\quad 0 \le \beta_* \le 1, \\ -n \le \inf_{x\in\mathbb{R}^n} \lambda(x) \le \sup_{x\in\mathbb{R}^n} \lambda(x) < 0, &\quad -n \le \lambda_* < 0. \end{aligned}$$

Let

$$\rho(x,r) = \begin{cases} r^{\alpha(x)}, \\ r^{\alpha_*}, \end{cases} \quad \psi(x,r) = \begin{cases} r^{\beta(x)}, \\ r^{\beta_*}, \end{cases} \quad \varphi(x,r) = \begin{cases} r^{\lambda(x)}, & 0 < r < 1, \\ r^{\lambda_*}, & 1 \le r < \infty. \end{cases}$$

If $\alpha(\cdot)$ is Lipschitz continuous, $\beta(\cdot)$ is log-Hölder continuous and

$$\begin{aligned} &\sup_{x\in\mathbb{R}^n} (\alpha(x) + \lambda(x)/p) < 0, \\ &\alpha(x) + \lambda(x)/p \ge \lambda(x)/\tilde{p}, \quad \alpha_* + \lambda_*/p \le \lambda_*/\tilde{p}, \\ &\beta(x) + \lambda(x)/\tilde{p} \ge \lambda(x)/q, \quad \beta_* + \lambda_*/\tilde{p} \le \lambda_*/q, \end{aligned}$$

then

$$\|[b, I_\rho]f\|_{L^{(q,\varphi)}} \le C\|b\|_{\mathcal{L}^{(1,\psi)}}\|f\|_{L^{(p,\varphi)}}.$$

To confirm it we only check on (2.1.11). First we note that ρ satisfies (1.2.5) and (1.2.7) by Remark 1.2.3 and that $\rho \sim \rho^*$. If $0 < s/2 < r \le s < 1$ and $|x - y| \le s$, then $r^{\alpha(x)} \sim s^{\alpha(x)} \sim s^{\alpha(y)}$. Hence

$$\begin{aligned} \left|\frac{\rho(x,r)}{r^n} - \frac{\rho(y,s)}{s^n}\right| &\le \left|r^{\alpha(x)-n} - s^{\alpha(x)-n}\right| + \left|s^{\alpha(x)-n} - s^{\alpha(y)-n}\right| \\ &\lesssim |r-s|r^{\alpha(x)-n-1} + |\alpha(x) - \alpha(y)||\log s|s^{\alpha(x)-n} \\ &\le |r-s|r^{\alpha(x)-n-1} + \|\alpha\|_{\mathrm{Lip}}|x-y|s^{\alpha(x)-n-1} \\ &\lesssim (|r-s| + |x-y|)r^{\alpha(x)-n-1}. \end{aligned}$$

If $1/2 \le r < 1 \le s < 2$, then $r^{\alpha(x)} \sim s^{\alpha_*} \sim 1$. Hence

$$\begin{aligned} \left|\frac{\rho(x,r)}{r^n} - \frac{\rho(y,s)}{s^n}\right| &\le \left|r^{\alpha(x)-n} - 1^{\alpha(x)-n}\right| + \left|1^{\alpha_*-n} - s^{\alpha_*-n}\right| \\ &\lesssim |r-1| + |1-s| \lesssim |r-s|r^{\alpha(x)-n-1}. \end{aligned}$$

If $1 \le r \le s \le 2r$, then $r^{\alpha_*} \sim s^{\alpha_*}$. Hence

$$\left|\frac{\rho(x,r)}{r^n} - \frac{\rho(y,s)}{s^n}\right| = \left|r^{\alpha_*-n} - s^{\alpha_*-n}\right| \lesssim |r-s| r^{\alpha_*-n-1}.$$

Therefore, (2.1.11) holds. Conversely, if $\lambda(\cdot)$ is log-Hölder continuous, α is constant and

$$\alpha + \beta(x) + \lambda(x)/p \le \lambda(x)/q, \quad \alpha + \beta_* + \lambda_*/p \ge \lambda_*/q,$$

then

$$\|b\|_{\mathcal{L}^{(1,\psi)}} \le C\|[b, I_\alpha]\|_{L^{(p,\varphi)} \to L^{(q,\varphi)}}.$$

We also take the cases

$$\rho(x,r) = \begin{cases} r^{\alpha(x)}(1/\log(e/r))^{\alpha_1(x)}, & 0 < r < 1, \\ r^{\alpha_*}(\log(er))^{\alpha_{**}}, & 1 \le r < \infty, \end{cases}$$

$$\rho(x,r) = \begin{cases} r^{\alpha(x)}, & 0 < r < 1, \\ e^{-(r-1)}, & 1 \le r < \infty, \end{cases}$$

etc.

At the end of this section we give the result in case $\psi \in \mathcal{G}^{\mathrm{dec}}$.

Theorem 2.1.3. *Let $p, p_0, q \in (1,\infty)$ and $\varphi, \psi : \mathbb{R}^n \times (0,\infty) \to (0,\infty)$. Assume that $1/p + 1/p_0 = 1/q$ and that both φ and ψ are in $\mathcal{G}^{\mathrm{dec}}$ and satisfy* (2.1.4)*. Let T be a Calderón-Zygmund operator of type $\omega \in \Omega$. If $b \in \mathcal{L}^{(p_0,\psi)}(\mathbb{R}^n)$, then $[b,T]f$ is well defined for all $f \in L^{(p,\varphi)}(\mathbb{R}^n)$ and there exists a positive constant C, independent of b and f, such that*

$$\|[b,T]f\|_{L^{(q,\theta)}} \le C\|b\|_{\mathcal{L}^{(p_0,\psi)}}\|f\|_{L^{(p,\varphi)}},$$

where $\theta^{1/q} = \psi^{1/p_0}\varphi^{1/p}$.

Theorem 2.1.4. *Let $p, p_0, \tilde{p}, q \in (1,\infty)$ and $\rho, \varphi : \mathbb{R}^n \times (0,\infty) \to (0,\infty)$. Assume that $p < \tilde{p}$, $1/p_0 + 1/\tilde{p} = 1/q$ and that φ is in $\mathcal{G}^{\mathrm{dec}}$ and satisfies* (2.1.4)*. Assume also* (2.1.12)*. If $b \in \mathcal{L}^{(p_0,\varphi)}(\mathbb{R}^n)$, then $[b, I_\rho]f$ is well defined for all $f \in L^{(p,\varphi)}(\mathbb{R}^n)$ and there exists a positive constant C, independent of b and f, such that*

$$\|[b, I_\rho]f\|_{L^{(q,\varphi)}} \le C\|b\|_{\mathcal{L}^{(p_0,\varphi)}}\|f\|_{L^{(p,\varphi)}}.$$

As a related result, we have the boundedness of $[b, I_\rho]$ on Orlicz spaces. A function $\Phi : [0, \infty) \to [0, \infty)$ is called a Young function if Φ is increasing, continuous, convex and bijective from $[0, \infty)$ to $[0, \infty)$. A Young function Φ is said to satisfy the Δ_2-condition, denoted by $\Phi \in \Delta_2$, if there exists a constant $C > 0$ such that $\Phi(2t) \le C\Phi(t)$ for all $t > 0$. A Young function Φ is said to satisfy the ∇_2-condition, denoted by $\Phi \in \nabla_2$, if there exists a constant $k > 1$ such that $\Phi(t) \le \frac{1}{2k}\Phi(kt)$ for all $t > 0$. Then we have the following theorem, whose proof is in [63]

Theorem 2.1.5 ([63, Theorem 3.13]). *Let $\rho, \psi : (0, \infty) \to (0, \infty)$, and let Φ and Ψ be Young functions. Assume that ρ satisfies (1.2.3). Let $b \in L^1_{loc}(\mathbb{R}^n)$.*

(i) *Let $\Phi, \Psi \in \Delta_2 \cap \nabla_2$. Assume that ψ is almost increasing and that $r \mapsto \rho(r)/r^{n-\epsilon}$ is almost decreasing for some $\epsilon \in (0, n)$. Assume also that there exists a positive constant A and $\Theta \in \nabla_2$ such that, for all $r \in (0, \infty)$,*

$$\int_0^r \frac{\rho(t)}{t}\,dt\;\Phi^{-1}(1/r^n) + \int_r^\infty \frac{\rho(t)\,\Phi^{-1}(1/t^n)}{t}\,dt \le A\Theta^{-1}(1/r^n),$$
$$\psi(r)\Theta^{-1}(1/r^n) \le A\Psi^{-1}(1/r^n),$$

and that there exist a positive constant C_ρ such that, for all $r, s \in (0, \infty)$,

$$\left|\frac{\rho(r)}{r^n} - \frac{\rho(s)}{s^n}\right| \le C_\rho\,|r - s|\frac{1}{r^{n+1}}\int_0^r \frac{\rho(t)}{t}\,dt, \quad \text{if } \frac{1}{2} \le \frac{r}{s} \le 2.$$

If $b \in \mathcal{L}_{1,\psi}(\mathbb{R}^n)$, then $[b, I_\rho]$ is bounded from $L^\Phi(\mathbb{R}^n)$ to $L^\Psi(\mathbb{R}^n)$ and there exists a positive constant C such that, for all $f \in L^\Phi(\mathbb{R}^n)$,

$$\|[b, I_\rho]f\|_{L^\Psi} \le C\|b\|_{\mathcal{L}_{1,\psi}}\|f\|_{L^\Phi}.$$

(ii) *Conversely, assume that there exists a positive constant A such that, for all $r \in (0, \infty)$,*

$$\Psi^{-1}(1/r^n) \le Ar^\alpha\psi(r)\Phi^{-1}(1/r^n).$$

If $[b, I_\alpha]$ is well defined and bounded from $L^\Phi(\mathbb{R}^n)$ to $L^\Psi(\mathbb{R}^n)$, then b is in $\mathcal{L}_{1,\psi}(\mathbb{R}^n)$ and there exists a positive constant C, independent of b, such that

$$\|b\|_{\mathcal{L}_{1,\psi}} \le C\|[b, I_\alpha]\|_{L^\Phi \to L^\Psi},$$

where $\|[b, I_\alpha]\|_{L^\Phi \to L^\Psi}$ is the operator norm of $[b, I_\alpha]$ from $L^\Phi(\mathbb{R}^n)$ to $L^\Psi(\mathbb{R}^n)$.

2.2 Boundedness of generalized fractional maximal and integral operators

For a function $\rho : \mathbb{R}^n \times (0,\infty) \to (0,\infty)$, let

$$M_\rho f(x) = \sup_{B \ni x} \rho(B) \fint_B |f(y)|\, dy, \tag{2.2.1}$$

where the supremum is taken over all balls B containing x.

If $\rho(B) = |B|^{\alpha/n}$, then M_ρ is the usual fractional maximal operator M_α defined by

$$M_\alpha f(x) = \sup_{B \ni x} \frac{1}{|B|^{1-\alpha/n}} \int_B |f(y)|\, dy.$$

If $\rho \equiv 1$, then M_ρ is the the Hardy-Littlewood maximal operator M, that is,

$$Mf(x) = \sup_{B \ni x} \fint_B |f(y)|\, dy. \tag{2.2.2}$$

It is well known that M is bounded from $L^p(\mathbb{R}^n)$ to itself if $p \in (1,\infty]$. This boundedness is extended to Morrey spaces $L^{(p,\varphi_\lambda)}(\mathbb{R}^n)$ by Chiarenza and Frasca [15] in 1987 as the following: If $p \in (1,\infty)$ and $\lambda \in [-n,0]$, then the operator M is bounded from $L^{(p,\varphi_\lambda)}(\mathbb{R}^n)$ to itself.

For generalized Morrey spaces $L^{(p,\varphi)}(\mathbb{R}^n)$ with variable growth function φ, the following theorem is known.

Theorem 2.2.1 ([47, Theorem 2.3]). *Let $1 < p < \infty$ and $\varphi : \mathbb{R}^n \times (0,\infty) \to (0,\infty)$. Assume that φ is almost decreasing, that is, there exists a positive constant C such that*

$$C\varphi(x,r) \ge \varphi(x,s) \quad \text{for} \quad x \in \mathbb{R}^n,\ 0 < r < s. \tag{2.2.3}$$

Then the operator M is bounded from $L^{(p,\varphi)}(\mathbb{R}^n)$ to itself.

On the boundedness of I_ρ the following theorem is known.

Theorem 2.2.2 ([47, Corollary 2.13]). *Let $1 < p < q < \infty$ and $\rho, \varphi : \mathbb{R}^n \times (0,\infty) \to (0,\infty)$. Assume that ρ satisfies* (1.2.3) *and* (1.2.4) *and that φ is in $\mathcal{G}^{\rm dec}$ and satisfies*

(1.2.9). *Assume also that there exists a positive constant* C *such that, for all* $x \in \mathbb{R}^n$ *and* $r \in (0,\infty)$,

$$\int_0^r \frac{\rho(x,t)}{t}\,dt\,\varphi(x,r)^{1/p} + \int_r^\infty \frac{\rho(x,t)\varphi(x,t)^{1/p}}{t}\,dt \le C\varphi(x,r)^{1/q}. \tag{2.2.4}$$

Then I_ρ *is bounded from* $L^{(p,\varphi)}(\mathbb{R}^n)$ *to* $L^{(q,\varphi)}(\mathbb{R}^n)$.

If $\rho(x,r)/r^n \le C\rho(x,s)/s^n$ for $0 < s < r < \infty$, then

$$M_\rho f(x) \le C I_\rho |f|(x), \quad x \in \mathbb{R}^n. \tag{2.2.5}$$

Hence, the boundedness of M_ρ follows from the boundedness of I_ρ. For example, the Hardy-Littlewood-Sobolev theorem yields that M_α is bounded from $L^p(\mathbb{R}^n)$ to $L^q(\mathbb{R}^n)$, if $\alpha \in (0,n)$, $p,q \in (1,\infty)$ and $-n/p+\alpha = -n/q$. However, if $\rho(x,r) = (\log(e+1/r))^{-\beta}$, $\beta > 1$, for example, then it turns out that the boundedness of M_ρ is better than the boundedness of I_ρ by the following theorem. Actually, (2.2.4) cannot be replaced by (2.2.6), see [19, Theorem 1.1].

Theorem 2.2.3. *Let* $1 < p < q < \infty$ *and* $\rho, \varphi : \mathbb{R}^n \times (0,\infty) \to (0,\infty)$. *Assume that* φ *is in* $\mathcal{G}^{\mathrm{dec}}$ *and satisfies* (1.2.9). *Assume also that there exists a positive constant* C_0 *such that, for all* $x \in \mathbb{R}^n$ *and* $r \in (0,\infty)$,

$$\rho(x,r)\varphi(x,r)^{1/p} \le C_0\varphi(x,r)^{1/q}. \tag{2.2.6}$$

Then M_ρ *is bounded from* $L^{(p,\varphi)}(\mathbb{R}^n)$ *to* $L^{(q,\varphi)}(\mathbb{R}^n)$.

Proof. We may assume that $\varphi(x,\cdot)$ is continuous, strictly decreasing and bijective from $(0,\infty)$ to itself for each $x \in \mathbb{R}^n$, see Remark 1.2.2.

We prove that, for $f \in L^{(p,\varphi)}(\mathbb{R}^n)$ with $\|f\|_{L^{(p,\varphi)}(\mathbb{R}^n)} = 1$,

$$M_\rho f(x) \le C Mf(x)^{p/q}, \quad x \in \mathbb{R}^n, \tag{2.2.7}$$

for some positive constant C independent of f and x. Then we have the conclusion by using Theorem 2.2.1. To prove (2.2.7) we show that, for any ball $B = B(x,r)$,

$$\rho(B) \fint_B |f| \le C_0 Mf(x)^{p/q}. \tag{2.2.8}$$

Choose $u > 0$ such that $\varphi(x, u) = Mf(x)^p$. If $r \le u$, then $\varphi(B) = \varphi(x, r) \ge Mf(x)^p$ and $\varphi(B)^{1/q-1/p} \le Mf(x)^{p/q-1}$. By (2.2.6) we have

$$\rho(B) \fint_B |f| \le C_0 \varphi(B)^{1/q-1/p} \fint_B |f| \le C_0 Mf(x)^{p/q}.$$

If $r > u$, then $\varphi(B) = \varphi(x, r) < Mf(x)^p$ and $\varphi(B)^{1/q} < Mf(x)^{p/q}$. By (2.2.6) we have

$$\rho(B) \fint_B |f| \le \rho(B) \left(\fint_B |f|^p \right)^{1/p} \le \rho(B)\varphi(B)^{1/p} \le C_0 \varphi(B)^{1/q} \le C_0 Mf(x)^{p/q}.$$

Then we have (2.2.8) and the conclusion. □

Next we recall N-functions and Musielak-Orlicz spaces. A function $\Phi : [0, \infty) \to [0, \infty)$ is called an N-function if Φ is increasing, convex and bijective from $[0, \infty)$ to itself, and if

$$\lim_{t \to +0} \frac{\Phi(t)}{t} = 0, \quad \lim_{t \to \infty} \frac{\Phi(t)}{t} = \infty.$$

Then the function $\Psi : [0, \infty) \to [0, \infty)$ defined by

$$\Psi(t) = \sup\{ts - \Phi(s) : s \ge 0\}$$

is also an N-function, and (Φ, Ψ) is called a complementary pair.

Let $\Phi : \mathbb{R}^n \times [0, \infty) \to [0, \infty)$. In this paper we also call Φ an N-function if $\Phi(x, t)$ is an N-function with respect to t for each x and it is a measurable function with respect to x for each t. We define a function $\Psi : \mathbb{R}^n \times [0, \infty) \to [0, \infty)$ by

$$\Psi(x, t) = \sup\{ts - \Phi(x, s) : s \ge 0\}.$$

Then Ψ is also an N-function and we have Young's inequality

$$st \le \Phi(x, s) + \Psi(x, t). \tag{2.2.9}$$

For an N-function $\Phi : \mathbb{R}^n \times [0, \infty) \to [0, \infty)$, let

$$L^\Phi(\mathbb{R}^n) = \left\{ f : \int_{\mathbb{R}^n} \Phi(x, \varepsilon |f(x)|)\, dx < \infty \ \text{ for some } \ \varepsilon > 0 \right\},$$
$$\|f\|_{L^\Phi} = \inf \left\{ \lambda > 0 : \int_{\mathbb{R}^n} \Phi\left(x, \frac{|f(x)|}{\lambda}\right) dx \le 1 \right\}.$$

Then $\|\cdot\|_{L^\Phi}$ is a norm and thereby $L^\Phi(\mathbb{R}^n)$ is a Banach space. We note that N-functions are special cases of Young functions and Orlicz and Musielak-Orlicz spaces are usually defined by Young functions. In this paper, however, we need only N-functions.

Let (Φ, Ψ) be a complementary pair of N-functions from $\mathbb{R}^n \times [0,\infty)$ to $[0,\infty)$. Then it is known that

$$t \le \Phi^{-1}(x,t)\Psi^{-1}(x,t) \le 2t, \quad t \ge 0, \tag{2.2.10}$$

where Φ^{-1} and Ψ^{-1} are the inverse functions of Φ and Ψ with respect to t, respectively. It is also known that

$$\int_{\mathbb{R}^n} |f(x)g(x)|\, dx \le 2\|f\|_{L^\Phi}\|g\|_{L^\Psi}. \tag{2.2.11}$$

This generalized Hölder's inequality follows from Young's inequality (2.2.9).

Lemma 2.2.4. *Let $k > 0$ and $\rho : \mathbb{R}^n \times (0,\infty) \to (0,\infty)$. Assume that ρ satisfies* (1.2.3). *Let*

$$\rho^*(x,r) = \int_0^r \frac{\rho(x,t)}{t}\, dt. \tag{2.2.12}$$

If $r \mapsto \rho(x,r)/r^k$ is almost decreasing, then $r \mapsto \rho^*(x,r)/r^k$ is also almost decreasing.

Proof. If $r < s$, then $\rho(x,(s/r)t) \lesssim (s/r)^k \rho(x,t)$. Hence,

$$\int_0^s \frac{\rho(x,t)}{t}\, dt = \int_0^r \frac{\rho(x,(s/r)t)}{t}\, dt \lesssim \left(\frac{s}{r}\right)^k \int_0^r \frac{\rho(x,t)}{t}\, dt.$$

In the above implicit constants are independent of x and r. This shows the conclusion. □

Remark 2.2.1. Since ρ^* is increasing with respect to r, we see that ρ^* satisfies the doubling condition (1.2.5) if $r \mapsto \rho(x,r)/r^k$ is almost decreasing for some $k > 0$. Moreover we see that $\rho \lesssim \rho^*$ if ρ satisfies (1.2.4).

The following theorem is a generalization of [50, Theorem 1.3].

Theorem 2.2.5. *Let $1 < s < \infty$ and $\rho : \mathbb{R}^n \times (0,\infty) \to (0,\infty)$. Assume that ρ satisfies* (1.2.3) *and* (1.2.4) *and that $r \mapsto \rho(x,r)r^{n/s-\epsilon}$ is almost decreasing for some positive constant ϵ. Then there exist an N-function Φ and a positive constant C such that, for all $x \in \mathbb{R}^n$ and $r > 0$,*

$$C^{-1}\Phi^{-1}\left(x, \frac{1}{r^n}\right) \le \frac{1}{r^{n/s}} \int_0^r \frac{\rho(x,t)}{t}\, dt \le C\Phi^{-1}\left(x, \frac{1}{r^n}\right). \tag{2.2.13}$$

Moreover, I_ρ is bounded from $L^s(\mathbb{R}^n)$ to $L^\Phi(\mathbb{R}^n)$.

Proof. Part 1. We first show (2.2.13) for some N-function Φ. Let ρ^* be as in (2.2.12). Then ρ^* satisfies the doubling condition (1.2.5), see Remark 2.2.1. Hence we have

$$\frac{\rho^*(x,r)}{r^{n/s}} \sim \int_r^{2r} \frac{\rho^*(x,t)}{t^{n/s+1}}\, dt \lesssim \int_r^\infty \frac{\rho^*(x,t)}{t^{n/s+1}}\, dt \lesssim \frac{\rho^*(x,r)}{r^{n/s-\epsilon}} \int_r^\infty \frac{dt}{t^{1+\epsilon}} \sim \frac{\rho^*(x,r)}{r^{n/s}}.$$

Let

$$h(x,r) = \int_r^\infty \frac{\rho^*(x,t)}{t^{n/s+1}}\, dt \quad \textit{and} \quad H(x,r) = \int_r^\infty \frac{h(x,t)}{t}\, dt.$$

Then $H(x,r) \sim h(x,r) \sim \frac{\rho^*(x,r)}{r^{n/s}}$, where implicit constants are independent of x and r. In this case $H(x,\cdot)$ is in C^2-class with respect to r and bijective from $(0,\infty)$ to itself for each x, since $H(x,r) \to 0$ as $r \to \infty$ and $r^{-\epsilon} \times \frac{\rho^*(x,r)}{r^{n/s-\epsilon}} \to \infty$ as $r \to +0$. Let H^{-1} be the inverse function of H with respect to r and let

$$\Phi(x,u) = \begin{cases} 0, & u = 0, \\ 1/H^{-1}(x,u)^n & u > 0. \end{cases}$$

Then

$$\Phi^{-1}\left(x, \frac{1}{r^n}\right) = H(x,r), \quad r > 0,$$

and (2.2.13) holds. We show that Φ is an N-function in the following.

Let $u = H(x,r)$ and $v = 1/r^n$. Then $v = \Phi(x,u)$ and

$$\begin{aligned} \partial_u \Phi(x,u) &= \frac{\partial v}{\partial u} = \frac{dv}{dr} \Big/ \frac{\partial u}{\partial r} = \left(-\frac{n}{r^{n+1}}\right) \Big/ \left(-\frac{h(x,r)}{r}\right) \\ &= \frac{n}{r^n h(x,r)} \sim \frac{1}{r^{n-n/s}\rho^*(x,r)}. \end{aligned}$$

If $u \to +0$, then $r \to \infty$ and $\partial_u \Phi(x,u) \to 0$. If $u \to \infty$, then $r \to +0$ and $\partial_u \Phi(x,u) \to \infty$. Since

$$\begin{aligned}\partial_r(r^n h(x,r)) &= nr^{n-1}\int_r^\infty \frac{\rho^*(x,t)}{t^{n/s+1}}\,dt - r^n \frac{\rho^*(x,r)}{r^{n/s+1}} \\ &\ge r^{n-1}\rho^*(x,r)\left(n\int_r^\infty \frac{1}{t^{n/s+1}}\,dt - \frac{1}{r^{n/s}}\right) > 0,\end{aligned}$$

we see that $\partial v/\partial u$ is decreasing with respect to r, that is, $\partial(\partial v/\partial u)/\partial r \le 0$. Hence

$$\frac{\partial^2 v}{\partial u^2} = \left(\frac{\partial}{\partial r}\frac{\partial v}{\partial u}\right) \Big/ \frac{\partial u}{\partial r} \ge 0.$$

Then Φ is an N-function.

Part 2. We show the boundedness of I_ρ from $L^s(\mathbb{R}^n)$ to $L^\Phi(\mathbb{R}^n)$. We use the method by Hedberg [26]. Let $f \in L^s(\mathbb{R}^n)$ and write

$$J_1 = \int_{B(x,r)} \frac{\rho(x,|x-y|)}{|x-y|^n} f(y)\,dy, \quad J_2 = \int_{\mathbb{R}^n \setminus B(x,r)} \frac{\rho(x,|x-y|)}{|x-y|^n} f(y)\,dy.$$

Let M be the Hardy-Littlewood maximal operator. Then

$$\begin{aligned}(2.2.14) \qquad |J_1| &\le \sum_{j=0}^\infty \int_{B(x,2^{-j}r)\setminus B(x,2^{-j-1}r)} \frac{\rho(x,|x-y|)}{|x-y|^n}|f(y)|\,dy \\ &\lesssim \sum_{j=0}^\infty \left(\sup_{2^{-j-1}r \le t \le 2^{-j}r} \rho(x,t)\right) \fint_{B(x,2^{-j}r)} |f(y)|\,dy \\ &\lesssim \sum_{j=0}^\infty Mf(x) \int_{K_1 2^{-j-1}r}^{K_2 2^{-j-1}r} \frac{\rho(x,t)}{t}\,dt = Mf(x)\int_0^{K_2 r/2} \frac{\rho(x,t)}{t}\,dt,\end{aligned}$$

and

$$|J_2| \le \left(\int_{\mathbb{R}^n\setminus B(x,r)} \left(\frac{\rho(x,|x-y|)}{|x-y|^n}\right)^{s'} dy\right)^{1/s'} \|f\|_{L^s},$$

where $1/s + 1/s' = 1$. Since

$$\begin{aligned}&\left(\int_{B(x,2^{j+1}r)\setminus B(x,2^j r)} \left(\frac{\rho(x,|x-y|)}{|x-y|^n}\right)^{s'} dy\right)^{1/s'} \\ &\lesssim \left(\sup_{2^j r \le t \le 2^{j+1}r} \rho(x,t)\right) \left(\int_{B(x,2^{j+1}r)\setminus B(x,2^j r)} \frac{1}{|x-y|^{ns'}}\,dy\right)^{1/s'} \\ &\lesssim (2^j r)^{-n/s} \int_{K_1 2^j r}^{K_2 2^j r} \frac{\rho(x,t)}{t}\,dt \sim \int_{K_1 2^j r}^{K_2 2^j r} \frac{\rho(x,t)}{t^{n/s+1}}\,dt,\end{aligned}$$

we have

$$(2.2.15)\quad |J_2| \le \int_{K_1 r}^{\infty} \frac{\rho(x,t)}{t^{n/s+1}}\,dt\,\|f\|_{L^s} \lesssim \frac{\rho(x,K_1 r)}{(K_1 r)^{n/s-\epsilon}} \int_{K_1 r}^{\infty} \frac{1}{t^{1+\epsilon}}\,dt\,\|f\|_{L^s}$$
$$\sim \frac{\rho(x,K_1 r)}{r^{n/s}}\,\|f\|_{L^s} \lesssim \frac{1}{r^{n/s}} \int_{(K_1)^2 r/2}^{K_1 K_2 r/2} \frac{\rho(x,t)}{t}\,dt\,\|f\|_{L^s}.$$

By (2.2.14) and (2.2.15) we have

$$|I_\rho f(x)| \le C' \left(Mf(x) + \|f\|_s \frac{1}{r^{n/s}} \right) \int_0^{K_3 r} \frac{\rho(x,t)}{t}\,dt,$$

where $K_3 = \max(1, K_2/2, K_1K_2/2)$ and C' is independent of x, r and f. By the boundedness of M on $L^s(\mathbb{R}^n)$ there exists a positive constant C_s such that

$$\|Mf\|_{L^s} \le C_s \|f\|_{L^s}.$$

Set $r = (1/\sigma)^{s/n}$ and $\sigma = Mf(x)/(C_s\|f\|_{L^s})$. Then

$$Mf(x) + \|f\|_{L^s} \frac{1}{r^{n/s}} = \left(1 + \frac{1}{C_s}\right) Mf(x),$$

and, by (2.2.13),

$$\int_0^{K_3 r} \frac{\rho(x,t)}{t}\,dt \le C(K_3 r)^{n/s} \Phi^{-1}\left(x, \frac{1}{(K_3 r)^n}\right) \lesssim r^{n/s} \Phi^{-1}\left(x, \frac{1}{r^n}\right) = \frac{\Phi^{-1}(x,\sigma^s)}{\sigma}.$$

Therefore

$$|I_\rho f(x)| \lesssim Mf(x) \frac{\Phi^{-1}(x,\sigma^s)}{\sigma} = C_s \Phi^{-1}\left(x, \left(\frac{Mf(x)}{C_s\|f\|_{L^s}}\right)^s\right) \|f\|_{L^s},$$

that is,

$$\Phi\left(x, \frac{|I_\rho f(x)|}{C''\|f\|_{L^s}}\right) \le \left(\frac{Mf(x)}{C_s\|f\|_{L^s}}\right)^s,$$

where C'' is independent of x and f. This shows

$$\int_{\mathbb{R}^n} \Phi\left(x, \frac{|I_\rho f(x)|}{C''\|f\|_{L^s}}\right) dx \le 1,$$

and

$$\|I_\rho f\|_{L^\Phi} \le C''\|f\|_{L^s}.$$

The proof is complete. □

Remark 2.2.2. We cannot replace $\int_0^r \frac{\rho(x,t)}{t}\,dt$ by $\rho(x,r)$ in (2.2.13), see [50, Section 5].

At the end of this section we give the Musielak-Orlicz norm of the characteristic functions of balls B.

Lemma 2.2.6. *Let* $1 < s < \infty$ *and* $\rho : \mathbb{R}^n \times (0,\infty) \to (0,\infty)$. *Assume the same condition as Theorem 2.2.5. Let* Φ *be the N-function in Theorem 2.2.5 and* Ψ *be its complementary N-function. If* ρ^* *satisfies* (1.2.7), *then, for all balls* B, *its characteristic function* χ_B *is in* $L^\Psi(\mathbb{R}^n)$ *and*

$$\|\chi_B\|_{L^\Psi(\mathbb{R}^n)} \le C_{s,\rho}\,|B|^{1-1/s}\rho^*(B),$$

where $C_{s,\rho}$ *is a positive constant independent of* B.

Proof. Let $B = B(x,r)$. Since ρ^* satisfies (1.2.7), from (2.2.13) it follows that

$$C_\Phi^{-1} \le \frac{\Phi^{-1}(x,1/|B|)}{\Phi^{-1}(y,1/|B|)} \le C_\Phi \quad \text{for } y \in B,$$

where C_Φ is a positive constant independent of $x, y \in \mathbb{R}^n$ and $r > 0$. Using the condition (2.2.10), we have

$$\begin{aligned}
\int_{\mathbb{R}^n} \Psi\left(y, \frac{\chi_B(y)}{C_\Phi|B|\Phi^{-1}(x,1/|B|)}\right) dy &= \int_B \Psi\left(y, \frac{1}{C_\Phi|B|\Phi^{-1}(x,1/|B|)}\right) dy \\
&\le \int_B \Psi\left(y, \frac{1}{|B|\Phi^{-1}(y,1/|B|)}\right) dy \\
&\le \int_B \Psi\left(y, \Psi^{-1}(y,1/|B|)\right) dy \\
&= \int_B \frac{1}{|B|}\,dy = 1.
\end{aligned}$$

Then

$$\|\chi_B\|_{L^\Psi(\mathbb{R}^n)} \le C_\Phi|B|\Phi^{-1}(x,1/|B|) \lesssim |B|^{1-1/s}\rho^*(B).$$

This is the conclusion. □

2.3 Lemmas

In this section we give several lemmas to prove theorems.

Let

$$\mathrm{MO}(f,B) = ⨍_B |f(x) - f_B|\,dx.$$

Lemma 2.3.1 ([37, Corollary 2.4]). *There exists a positive constant c_n dependent only on n such that, for all $x \in \mathbb{R}^n$ and $r, s \in (0,\infty)$,*

$$|f_{B(x,r)} - f_{B(x,s)}| \le c_n \int_r^{2s} \frac{\mathrm{MO}(f, B(x,t))}{t}\, dt, \quad \text{if } r < s.$$

The following lemma is a corollary of the John-Nirenberg inequality ([29]).

Lemma 2.3.2. *Let $p \in (1,\infty)$. Then there exists a positive constant $c(n,p)$ such that for all cubes Q in $\mathbb{R}^n$ and all functions f in $L^1_{\mathrm{loc}}(\mathbb{R}^n)$, we have*

$$\left(\fint_Q |f(x) - f_Q|^p\, dx \right)^{1/p} \le c(n,p) \sup_{R \subset Q} \fint_R |f(x) - f_R|\, dx, \tag{2.3.1}$$

where the supremum is taken over all cubes R in Q.

By Hölder's inequality and Lemma 2.3.2 we have the following corollary, which is known by [45, Theorem 3.1] for spaces of homogeneous type. We give a proof for readers' convenience.

Corollary 2.3.3. *Let $p \in (1,\infty)$ and $\psi \in \mathcal{G}^{\mathrm{inc}}$. Assume that ψ satisfies* (1.2.7). *Then $\mathcal{L}^{(p,\psi^p)}(\mathbb{R}^n) = \mathcal{L}^{(1,\psi)}(\mathbb{R}^n)$ with equivalent norms.*

Proof. By Hölder's inequality we have $\|f\|_{\mathcal{L}^{(1,\psi)}} \le \|f\|_{\mathcal{L}^{(p,\psi^p)}}$. Next we show the reverse inequality. For any balls B_1 and B_2, if $B_1 \subset B_2$, then $\psi(B_1) \lesssim \psi(B_2)$, since ψ is almost increasing and satisfies (1.2.7). For any ball B, take the cube Q such that $B \subset Q \subset \sqrt{n}B$. If $f \in \mathcal{L}^{(1,\psi)}(\mathbb{R}^n)$, then, using Lemma 2.3.2, we have

$$\begin{aligned}\left(\fint_Q |f(x) - f_Q|^p\, dx \right)^{1/p} &\lesssim \sup_{R \subset Q} \fint_R |f(x) - f_R|\, dx \\ &\lesssim \sup_{B' \subset \sqrt{n}B} \psi(B') \|f\|_{\mathcal{L}^{(1,\psi)}} \lesssim \psi(\sqrt{n}B) \|f\|_{\mathcal{L}^{(1,\psi)}}.\end{aligned}$$

Therefore, by the doubling condition of ψ we have the conclusion. □

Lemma 2.3.4 ([45, Lemma 7.1]). *Let φ satisfy the doubling condition* (1.2.5) *and* (2.1.4), *that is,*

$$\int_r^\infty \frac{\varphi(x,t)}{t}\, dt \le C\, \varphi(x,r).$$

Then, for all $p \in (0,\infty)$, there exists a positive constant C_p such that, for all $x \in \mathbb{R}^n$ and $r > 0$,

$$\int_r^\infty \frac{\varphi(x,t)^{1/p}}{t}\, dt \le C_p\, \varphi(x,r)^{1/p}. \tag{2.3.2}$$

Lemma 2.3.5 ([43, Theorem 2.1 and Remark 2.1])**.** *Let $p \in [1, \infty)$ and φ satisfy the doubling condition* (1.2.5) *and* (2.1.4)*. Then, for every $f \in \mathcal{L}^{(p,\varphi)}(\mathbb{R}^n)$, $f_{B(0,r)}$ converges as $r \to \infty$ and*

$$2^{-1}\|f\|_{\mathcal{L}^{(p,\varphi)}} \le \|f - \lim_{r\to\infty} f_{B(0,r)}\|_{L^{(p,\varphi)}} \le (1 + 2(\log 2)^{-1}C_p)\|f\|_{\mathcal{L}^{(p,\varphi)}},$$

where C_p is the constant in (2.3.2)*.*

Lemma 2.3.6 ([39, Lemma 4.1])**.** *Let $p_i \in [1, \infty)$ and $\varphi_i : \mathbb{R}^n \times (0, \infty) \to (0, \infty)$, $i = 1, 2, 3$. If $1/p_1 + 1/p_3 = 1/p_2$ and $\varphi_1^{1/p_1}\varphi_3^{1/p_3} = \varphi_2^{1/p_2}$, then*

$$\|fg\|_{L^{(p_2,\varphi_2)}} \le \|f\|_{L^{(p_1,\varphi_1)}}\|g\|_{L^{(p_3,\varphi_3)}}.$$

Lemma 2.3.7. *Let $p \in (1, \infty)$ and $\psi \in \mathcal{G}^{\rm inc}$. Assume that ψ satisfies* (1.2.7)*. Then there exists a positive constant C dependent only on n, p and ψ such that, for all $f \in \mathcal{L}^{(1,\psi)}(\mathbb{R}^n)$ and for all $x \in \mathbb{R}^n$ and $r, s \in (0, \infty)$,*

$$\left(⨍_{B(x,s)} |f(y) - f_{B(x,r)}|^p \, dy\right)^{1/p} \le C \int_r^s \frac{\psi(x,t)}{t} \, dt \, \|f\|_{\mathcal{L}^{(1,\psi)}}, \quad \textit{if} \;\; 2r < s.$$

Proof. By Lemma 2.3.1 and Corollary 2.3.3 we have

$$\begin{aligned}
&\left(⨍_{B(x,s)} |f(y) - f_{B(x,r)}|^p \, dy\right)^{1/p} \\
&\le \left(⨍_{B(x,s)} |f(y) - f_{B(x,s)}|^p \, dy\right)^{1/p} + |f_{B(x,s)} - f_{B(x,r)}| \\
&\lesssim \psi(x,s)\|f\|_{\mathcal{L}^{(1,\psi)}} + \int_r^{2s} \frac{\psi(x,t)}{t} \, dt \, \|f\|_{\mathcal{L}^{(1,\psi)}}.
\end{aligned}$$

From the doubling condition (1.2.5) of ψ it follows that

$$\psi(x,s) \sim \int_s^{2s} \frac{\psi(x,t)}{t} \, dt \sim \int_{s/2}^{s} \frac{\psi(x,t)}{t} \, dt \le \int_r^{s} \frac{\psi(x,t)}{t} \, dt.$$

Then we have the conclusion. □

Remark 2.3.1. In Lemma 2.3.7 we also have

$$\left(⨍_{B(x,s)} |f(y) - f_{B(x,r)}|^p \, dy\right)^{1/p} \le C \left(\log_2 \frac{s}{r}\right) \psi(x,s) \, \|f\|_{\mathcal{L}^{(1,\psi)}}, \quad \text{if} \;\; 2r < s,$$

since

$$\int_{2^j r}^{2^{j+1}r} \frac{\psi(x,t)}{t}\,dt \lesssim \psi(x,s),$$

for $j = 0, 1, \ldots, [\log_2 \frac{s}{r}] + 1$.

Lemma 2.3.8. *Let $p \in (1,\infty)$ and $\varphi \in \mathcal{G}^{\rm dec}$. Let K be a standard kernel satisfying* (2.1.1)*. Then there exists a positive constant C such that, for all $f \in L^{(p,\varphi)}(\mathbb{R}^n)$ and all balls $B = B(z,r)$,*

$$\int_{\mathbb{R}^n \setminus 2B} |K(x,y)f(y)|\,dy \le C \int_{2r}^{\infty} \frac{\varphi(z,t)^{1/p}}{t}\,dt\,\|f\|_{L^{(p,\varphi)}}, \quad x \in B.$$

Proof. If $x \in B$ and $y \notin 2B$, then $|z-y|/2 \le |x-y| \le (3/2)|z-y|$. From (2.1.1) it follows that $|K(x,y)| \lesssim |x-y|^{-n} \sim |z-y|^{-n}$. Then

$$\int_{\mathbb{R}^n \setminus 2B} |K(x,y)||f(y)|\,dy \lesssim \int_{\mathbb{R}^n \setminus 2B} \frac{|f(y)|}{|z-y|^n}\,dy = \sum_{j=1}^{\infty} \int_{2^{j+1}B \setminus 2^j B} \frac{|f(y)|}{|z-y|^n}\,dy.$$

By Hölder's inequality and the doubling condition of φ we have

$$\begin{aligned}\int_{2^{j+1}B \setminus 2^j B} \frac{|f(y)|}{|z-y|^n}\,dy &\lesssim \frac{1}{(2^{j+1}r)^n} \int_{2^{j+1}B \setminus 2^j B} |f(y)|\,dy \\ &\lesssim \left(\fint_{2^{j+2}B} |f(y)|^p\,dy \right)^{1/p} \lesssim \int_{2^j r}^{2^{j+1}r} \frac{\varphi(z,t)^{1/p}}{t}\,dt\,\|f\|_{L^{(p,\varphi)}}.\end{aligned}$$

Therefore, we have the conclusion. □

Lemma 2.3.9. *Let $p \in (1,\infty)$, $\varphi \in \mathcal{G}^{\rm dec}$ and $\psi \in \mathcal{G}^{\rm inc}$. Assume that ψ satisfies* (1.2.7)*. Let K be a standard kernel satisfying* (2.1.1)*. Then there exists a positive constant C such that, for all $b \in \mathcal{L}^{(1,\psi)}(\mathbb{R}^n)$, all $f \in L^{(p,\varphi)}(\mathbb{R}^n)$ and all balls $B = B(z,r)$,*

$$\begin{aligned}&\int_{\mathbb{R}^n \setminus 2B} |(b(y) - b_B)K(x,y)f(y)|\,dy \\ &\qquad \le C \int_r^{\infty} \frac{\psi(z,t)}{t} \left(\int_t^{\infty} \frac{\varphi(z,u)^{1/p}}{u}\,du \right) dt\,\|b\|_{\mathcal{L}^{(1,\psi)}} \|f\|_{L^{(p,\varphi)}}, \quad x \in B.\end{aligned}$$

Proof. If $x \in B$ and $y \notin 2B$, then $|z-y|/2 \le |x-y| \le (3/2)|z-y|$. From (2.1.1) it follows that $|K(x-y)| \lesssim |x-y|^{-n} \sim |z-y|^{-n}$. Then

$$\int_{\mathbb{R}^n \setminus 2B} |(b(y)-b_B)K(x,y)f(y)|\,dy \lesssim \int_{\mathbb{R}^n \setminus 2B} \frac{|(b(y)-b_B)f(y)|}{|z-y|^n}\,dy$$
$$= \sum_{j=1}^{\infty} \int_{2^{j+1}B \setminus 2^j B} \frac{|(b(y)-b_B)f(y)|}{|z-y|^n}\,dy.$$

By Hölder's inequality, Lemma 2.3.7 and the doubling condition of ψ and φ we have

$$\int_{2^{j+1}B \setminus 2^j B} \frac{|(b(y)-b_B)f(y)|}{|z-y|^n}\,dy$$
$$\lesssim \frac{1}{(2^{j+1}r)^n} \int_{2^{j+1}B \setminus 2^j B} |(b(y)-b_B)f(y)|\,dy$$
$$\lesssim \left(⨍_{2^{j+1}B} |b-b_B|^{p'}\,dy \right)^{1/p'} \left(⨍_{2^{j+1}B} |f(y)|^p\,dy \right)^{1/p}$$
$$\lesssim \int_r^{2^{j+1}r} \frac{\psi(z,t)}{t}\,dt\, \varphi(z,2^{j+1}r)^{1/p} \,\|b\|_{\mathcal{L}^{(1,\psi)}} \|f\|_{L^{(p,\varphi)}}$$
$$\lesssim \int_{2^j r}^{2^{j+1}r} \left(\int_r^u \frac{\psi(z,t)}{t}\,dt \right) \frac{\varphi(z,u)^{1/p}}{u}\,du\, \|b\|_{\mathcal{L}^{(1,\psi)}} \|f\|_{L^{(p,\varphi)}}.$$

Therefore,

$$\int_{\mathbb{R}^n \setminus 2B} |(b(y)-b_B)K(x,y)f(y)|\,dy$$
$$\lesssim \int_r^{\infty} \left(\int_r^u \frac{\psi(z,t)}{t}\,dt \right) \frac{\varphi(z,u)^{1/p}}{u}\,du\, \|b\|_{\mathcal{L}^{(1,\psi)}} \|f\|_{L^{(p,\varphi)}}$$
$$= \int_r^{\infty} \frac{\psi(z,t)}{t} \left(\int_t^{\infty} \frac{\varphi(z,u)^{1/p}}{u}\,du \right) dt\, \|b\|_{\mathcal{L}^{(1,\psi)}} \|f\|_{L^{(p,\varphi)}}.$$

This is the conclusion. □

Remark 2.3.2. Under the assumption in Theorem 2.1.1 (i), let $b \in \mathcal{L}^{(1,\psi)}(\mathbb{R}^n)$ and $f \in L^{(p,\varphi)}(\mathbb{R}^n)$. Then f is in $L^p_{\mathrm{loc}}(\mathbb{R}^n)$ and bf is in $L^{p_1}_{\mathrm{loc}}(\mathbb{R}^n)$ for all $p_1 < p$ by Corollary 2.3.3. Hence, $T(f\chi_{2B})$ and $T(bf\chi_{2B})$ are well defined for any ball $B = B(z,r)$. By (2.1.4), (2.1.7) and Lemma 2.3.4 we have

$$\int_r^{\infty} \frac{\varphi(z,t)^{1/p}}{t}\,dt \lesssim \varphi(z,r)^{1/p}, \tag{2.3.3}$$

and

$$\text{(2.3.4)}\quad \int_r^\infty \frac{\psi(z,t)}{t}\left(\int_t^\infty \frac{\varphi(z,u)^{1/p}}{u}\,du\right)dt \lesssim \int_r^\infty \frac{\psi(z,t)\varphi(z,t)^{1/p}}{t}\,dt \lesssim \int_r^\infty \frac{\varphi(z,t)^{1/q}}{t}\,dt \lesssim \varphi(z,r)^{1/q}.$$

Then, by Lemmas 2.3.8 and 2.3.9, the integrals

$$\int_{\mathbb{R}^n\setminus 2B} |K(x,y)f(y)|\,dy \quad \text{and} \quad \int_{\mathbb{R}^n\setminus 2B} |K(x,y)b(y)f(y)|\,dy$$

converge. That is, we can write

$$[b,T]f(x) = [b,T](f\chi_{2B})(x) + \int_{\mathbb{R}^n\setminus 2B} (b(x)-b(y))K(x,y)f(y)\,dy, \quad x \in B.$$

Moreover, if $x \in B_1 \cap B_2$, then, taking B_3 such that $B_1 \cup B_2 \subset B_3$, we have

$$\begin{aligned}
&\left([b,T](f\chi_{2B_i})(x) + \int_{\mathbb{R}^n\setminus 2B_i} (b(x)-b(y))K(x,y)f(y)\,dy\right) \\
&- \left([b,T](f\chi_{2B_3})(x) + \int_{\mathbb{R}^n\setminus 2B_3} (b(x)-b(y))K(x,y)f(y)\,dy\right) \\
&= -[b,T](f\chi_{2B_3\setminus 2B_i})(x) + \int_{2B_3\setminus 2B_i} (b(x)-b(y))K(x,y)f(y)\,dy = 0,
\end{aligned}$$

by (2.1.3). That is,

$$\begin{aligned}
&[b,T](f\chi_{2B_1})(x) + \int_{\mathbb{R}^n\setminus 2B_1} (b(x)-b(y))K(x,y)f(y)\,dy \\
&= [b,T](f\chi_{2B_2})(x) + \int_{\mathbb{R}^n\setminus 2B_2} (b(x)-b(y))K(x,y)f(y)\,dy, \quad x \in B_1 \cap B_2.
\end{aligned}$$

This shows that $[b,T]f(x)$ in (2.1.6) is independent of the choice of the ball B containing x.

Lemma 2.3.10. *Under the assumption of Theorem 2.1.1 (i), there exists a positive constant C such that, for all $b \in \mathcal{L}^{(1,\psi)}(\mathbb{R}^n)$, all $f \in L^{(p,\varphi)}(\mathbb{R}^n)$ and all balls $B = B(z,r)$,*

$$\left| \fint_B \left(\int_{\mathbb{R}^n\setminus 2B} (b(x)-b(y))K(x,y)f(y)\,dy \right) dx \right| \le C\varphi(B)^{1/q}\, \|b\|_{\mathcal{L}^{(1,\psi)}} \|f\|_{L^{(p,\varphi)}}.$$

Proof. For $x \in B$, let

$$G_1(x) = |b(x) - b_B| \int_{\mathbb{R}^n \setminus 2B} |K(x,y)f(y)|\,dy,$$
$$G_2(x) = \int_{\mathbb{R}^n \setminus 2B} |(b(y) - b_B)K(x,y)f(y)|\,dy.$$

Then

$$\left| \int_{\mathbb{R}^n \setminus 2B} (b(x) - b(y))K(x,y)f(y)\,dy \right| \le G_1(x) + G_2(x).$$

Using Lemmas 2.3.8 and 2.3.9, we have

$$\int_{\mathbb{R}^n \setminus 2B} |K(x,y)||f(y)|\,dy \lesssim \int_{2r}^{\infty} \frac{\varphi(z,t)^{1/p}}{t}\,dt\,\|f\|_{L^{(p,\varphi)}}, \quad x \in B, \tag{2.3.5}$$

and

$$\begin{aligned} &\int_{\mathbb{R}^n \setminus 2B} |b(y) - b_B||K(x,y)||f(y)|\,dy \\ &\quad \lesssim \int_r^{\infty} \frac{\psi(z,t)}{t} \left(\int_t^{\infty} \frac{\varphi(z,u)^{1/p}}{u}\,du \right) dt\,\|b\|_{\mathcal{L}^{(1,\psi)}} \|f\|_{L^{(p,\varphi)}}, \quad x \in B. \end{aligned} \tag{2.3.6}$$

Then, using (2.3.5), (2.3.3) and (2.1.7), we have

$$\begin{aligned} ⨍_B G_1(x)\,dx &\lesssim ⨍_B |b(x) - b_B|\,dx\,\varphi(z,r)^{1/p}\|f\|_{L^{(p,\varphi)}} \\ &\lesssim \psi(z,r)\varphi(z,r)^{1/p}\|b\|_{\mathcal{L}^{(1,\psi)}}\|f\|_{L^{(p,\varphi)}} \\ &\lesssim \varphi(z,r)^{1/q}\|b\|_{\mathcal{L}^{(1,\psi)}}\|f\|_{L^{(p,\varphi)}}. \end{aligned}$$

Using (2.3.6) and (2.3.4), we also have

$$⨍_B G_2(x)\,dx \lesssim \varphi(z,r)^{1/q}\|b\|_{\mathcal{L}^{(1,\psi)}}\|f\|_{L^{(p,\varphi)}}.$$

Then we have the conclusion. □

Lemma 2.3.11. *Let $p \in (1,\infty)$ and $\varphi \in \mathcal{G}^{\mathrm{dec}}$. Assume that ρ satisfies* (1.2.3) *and* (1.2.4)*. Then there exists a positive constant C such that, for all $f \in L^{(p,\varphi)}(\mathbb{R}^n)$ and all balls $B(x,r)$,*

$$\int_{\mathbb{R}^n \setminus B(x,r)} \frac{\rho(x,|x-y|)}{|x-y|^n}|f(y)|\,dy \le C \int_{K_1 r}^{\infty} \frac{\rho(x,t)\varphi(x,t)^{1/p}}{t}\,dt\,\|f\|_{L^{(p,\varphi)}},$$

where K_1 is the constant in (1.2.4).

Proof. Let $B = B(x, r)$. Then

$$\int_{\mathbb{R}^n \setminus B(x,r)} \frac{\rho(x, |x-y|)}{|x-y|^n} |f(y)|\, dy = \sum_{j=0}^{\infty} \int_{2^{j+1}B \setminus 2^j B} \frac{\rho(x, |x-y|)}{|x-y|^n} |f(y)|\, dy.$$

By (1.2.4), Hölder's inequality and the doubling condition of φ we have

$$\begin{aligned} &\int_{2^{j+1}B \setminus 2^j B} \frac{\rho(x, |x-y|)}{|x-y|^n} |f(y)|\, dy \lesssim \frac{\sup_{2^j r \le t \le 2^{j+1} r} \rho(x,t)}{(2^{j+1} r)^n} \int_{2^{j+1}B \setminus 2^j B} |f(y)|\, dy \\ &\lesssim \int_{K_1 2^j r}^{K_2 2^j r} \frac{\rho(x,t)}{t}\, dt \left(⨍_{2^{j+2}B} |f(y)|^p\, dy \right)^{1/p} \lesssim \int_{K_1 2^j r}^{K_2 2^j r} \frac{\rho(x,t)\varphi(x,t)^{1/p}}{t}\, dt\, \|f\|_{L^{(p,\varphi)}}. \end{aligned}$$

Therefore, we have the conclusion. □

Lemma 2.3.12. *Let $p \in (1, \infty)$, $\varphi \in \mathcal{G}^{\rm dec}$ and $\psi \in \mathcal{G}^{\rm inc}$. Assume that ψ satisfies* (1.2.7). *Assume also that ρ satisfies* (1.2.3) *and* (1.2.4). *Then there exists a positive constant C such that, for all $b \in \mathcal{L}^{(1,\psi)}(\mathbb{R}^n)$, all $f \in L^{(p,\varphi)}(\mathbb{R}^n)$ and all balls $B(x, r)$,*

$$\begin{aligned} &\int_{\mathbb{R}^n \setminus B(x,r)} |b(y) - b_{B(x,r)}| \frac{\rho(x, |x-y|)}{|x-y|^n} |f(y)|\, dy \\ &\qquad \le C \int_{K_1 r}^{\infty} \frac{\psi(x,t)}{t} \left(\int_t^{\infty} \frac{\rho(x,u)\varphi(x,u)^{1/p}}{u}\, du \right) dt\, \|b\|_{\mathcal{L}^{(1,\psi)}} \|f\|_{L^{(p,\varphi)}}, \end{aligned}$$

where K_1 is the constant in (1.2.4).

Proof. Let $B = B(x, r)$. Then

$$\begin{aligned} &\int_{\mathbb{R}^n \setminus B(x,r)} |b(y) - b_B| \frac{\rho(x, |x-y|)}{|x-y|^n} |f(y)|\, dy \\ &\qquad = \sum_{j=0}^{\infty} \int_{2^{j+1}B \setminus 2^j B} |b(y) - b_B| \frac{\rho(x, |x-y|)}{|x-y|^n} |f(y)|\, dy. \end{aligned}$$

By (1.2.4), Hölder's inequality, Lemma 2.3.7 and the doubling condition of ψ and

φ we have

$$
\begin{aligned}
&\int_{2^{j+1}B\setminus 2^jB} |b(y)-b_B|\frac{\rho(x,|x-y|)}{|x-y|^n}|f(y)|\,dy \\
&\lesssim \frac{\sup_{2^jr\le u\le 2^{j+1}r}\rho(x,u)}{(2^{j+1}r)^n}\int_{2^{j+1}B\setminus 2^jB} |b(y)-b_B||f(y)|\,dy \\
&\lesssim \int_{K_1 2^j r}^{K_2 2^j r} \frac{\rho(x,u)}{u}\,du \left(⨍_{2^{j+1}B} |b-b_B|^{p'}\,dy\right)^{1/p'}\left(⨍_{2^{j+1}B} |f(y)|^p\,dy\right)^{1/p} \\
&\lesssim \int_{K_1 2^j r}^{K_2 2^j r} \frac{\rho(x,u)}{u}\,du \int_r^{2^{j+1}r}\frac{\psi(x,t)}{t}\,dt\,\varphi(x,2^{j+1}r)^{1/p}\,\|b\|_{\mathcal{L}^{(1,\psi)}}\|f\|_{L^{(p,\varphi)}} \\
&\lesssim \int_{K_1 2^j r}^{K_2 2^j r} \left(\int_{K_1 r}^{u}\frac{\psi(x,t)}{t}\,dt\right)\frac{\rho(x,u)\varphi(x,u)^{1/p}}{u}\,du\,\|b\|_{\mathcal{L}^{(1,\psi)}}\|f\|_{L^{(p,\varphi)}}.
\end{aligned}
$$

Therefore,

$$
\begin{aligned}
&\int_{\mathbb{R}^n\setminus B} |b(y)-b_B|\frac{\rho(x,|x-y|)}{|x-y|^n}|f(y)|\,dy \\
&\qquad\lesssim \int_{K_1 r}^{\infty}\left(\int_{K_1 r}^{u}\frac{\psi(x,t)}{t}\,dt\right)\frac{\rho(x,u)\varphi(x,u)^{1/p}}{u}\,du\,\|b\|_{\mathcal{L}^{(1,\psi)}}\|f\|_{L^{(p,\varphi)}} \\
&\qquad\qquad= \int_{K_1 r}^{\infty}\frac{\psi(x,t)}{t}\left(\int_t^{\infty}\frac{\rho(x,u)\varphi(x,u)^{1/p}}{u}\,du\right)dt\,\|b\|_{\mathcal{L}^{(1,\psi)}}\|f\|_{L^{(p,\varphi)}}.
\end{aligned}
$$

This is the conclusion. □

Remark 2.3.3. Under the assumption in Theorem 2.1.2 (i), let $b \in \mathcal{L}^{(1,\psi)}(\mathbb{R}^n)$ and $f \in L^{(p,\varphi)}(\mathbb{R}^n)$. Then f is in $L^p_{\mathrm{loc}}(\mathbb{R}^n)$ and bf is in $L^{p_1}_{\mathrm{loc}}(\mathbb{R}^n)$ for all $p_1 < p$ by Corollary 2.3.3. Since $\frac{\rho(x,|y|)}{|y|^n}$ is integrable near the origin with respect to y, $I_\rho(|f|\chi_{2B})$ and $I_\rho(|bf|\chi_{2B})$ are well defined for any ball $B = B(x,r)$. By (2.1.12) and (2.1.13) we have

$$
\int_{K_1 r}^{\infty}\frac{\rho(x,t)\varphi(x,t)^{1/p}}{t}\,dt \lesssim \varphi(x,K_1 r)^{1/\tilde{p}} \lesssim \varphi(x,r)^{1/\tilde{p}}, \tag{2.3.7}
$$

and

$$
\begin{aligned}
&\int_{K_1 r}^{\infty}\frac{\psi(x,t)}{t}\left(\int_t^{\infty}\frac{\rho(x,u)\varphi(x,u)^{1/p}}{u}\,du\right)dt \\
&\qquad\lesssim \int_{K_1 r}^{\infty}\frac{\psi(x,t)\varphi(x,t)^{1/\tilde{p}}}{t}\,dt \lesssim \int_{K_1 r}^{\infty}\frac{\varphi(x,t)^{1/q}}{t}\,dt \lesssim \varphi(x,r)^{1/q}.
\end{aligned} \tag{2.3.8}
$$

Then, by Lemmas 2.3.11 and 2.3.12, the integrals

$$\int_{\mathbb{R}^n\setminus 2B} \frac{\rho(x,|x-y|)}{|x-y|^n}|f(y)|\,dy \quad \text{and} \quad \int_{\mathbb{R}^n\setminus 2B} \frac{\rho(x,|x-y|)}{|x-y|^n}|b(y)f(y)|\,dy$$

converge. That is, the integrals

$$\int_{\mathbb{R}^n} \frac{\rho(x,|x-y|)}{|x-y|^n}f(y)\,dy \quad \text{and} \quad \int_{\mathbb{R}^n} \frac{\rho(x,|x-y|)}{|x-y|^n}b(y)f(y)\,dy$$

converge absolutely a.e. x and we can write

$$[b,I_\rho]f(x) = \int_{\mathbb{R}^n} (b(x)-b(y))\frac{\rho(x,|x-y|)}{|x-y|^n}f(y)\,dy, \quad \text{a.e. } x.$$

Lemma 2.3.13. *Under the assumption of Theorem 2.1.2 (i), there exists a positive constant C such that, for all $b \in \mathcal{L}^{(1,\psi)}(\mathbb{R}^n)$, all $f \in L^{(p,\varphi)}(\mathbb{R}^n)$ and all balls $B = B(z,r)$,*

$$\left| \fint_B \left(\int_{\mathbb{R}^n\setminus 2B} (b(x)-b(y))\frac{\rho(x,|x-y|)}{|x-y|^n}f(y)\,dy \right) dx \right| \le C\varphi(B)^{1/q}\,\|b\|_{\mathcal{L}^{(1,\psi)}}\|f\|_{L^{(p,\varphi)}}.$$

Proof. For $x \in B$, let

$$G_1(x) = |b(x)-b_B| \int_{\mathbb{R}^n\setminus 2B} \frac{\rho(x,|x-y|)}{|x-y|^n}|f(y)|\,dy,$$
$$G_2(x) = \int_{\mathbb{R}^n\setminus 2B} |b(y)-b_B|\frac{\rho(x,|x-y|)}{|x-y|^n}|f(y)|\,dy.$$

Then

$$\left| \int_{\mathbb{R}^n\setminus 2B} (b(x)-b(y))\frac{\rho(x,|x-y|)}{|x-y|^n}f(y)\,dy \right| \le G_1(x)+G_2(x).$$

If $x \in B$ and $y \notin 2B$, then $|x-z| < |x-y|$ and $|z-y|/2 \le |x-y| \le (3/2)|z-y|$. By the properties (1.2.7) and (1.2.4) of ρ we have

$$\rho(x,|x-y|) \sim \rho(z,|x-y|) \le \sup_{|z-y|/2 \le t \le (3/2)|z-y|} \rho(z,t),$$

and

$$\begin{aligned}
&\int_{2^{j+2}B\setminus 2^{j+1}B} \frac{\rho(x,|x-y|)}{|x-y|^n}|f(y)|\,dy \\
&\lesssim \int_{2^{j+2}B\setminus 2^{j+1}B} \frac{\sup_{|z-y|/2\le t\le (3/2)|z-y|}\rho(z,t)}{|z-y|^n}|f(y)|\,dy \\
&\lesssim \frac{\sup_{2^jr\le t\le 3\cdot 2^{j+1}r}\rho(z,t)}{(2^{j+2}r)^n}\int_{2^{j+2}B\setminus 2^{j+1}B}|f(y)|\,dy \\
&\lesssim \int_{2^jK_1r}^{3\cdot 2^jK_2r}\frac{\rho(z,t)}{t}\,dt\left(⨍_{2^{j+2}B}|f(y)|^p\,dy\right)^{1/p}.
\end{aligned}$$

Using this estimate and a similar way to Lemmas 2.3.11 and 2.3.12, we have that, for all $x\in B$,

$$\begin{aligned}
G_1(x) &\lesssim |b(x)-b_B|\int_{K_1r}^{\infty}\frac{\rho(z,t)\varphi(z,t)^{1/p}}{t}\,dt\,\|f\|_{L^{(p,\varphi)}}, \\
G_2(x) &\lesssim C\int_{K_1r}^{\infty}\frac{\psi(z,t)}{t}\left(\int_t^{\infty}\frac{\rho(z,u)\varphi(z,u)^{1/p}}{u}\,du\right)dt\,\|b\|_{\mathcal{L}^{(1,\psi)}}\|f\|_{L^{(p,\varphi)}}.
\end{aligned}$$

Then, using (2.3.7) and (2.3.8) also, we have

$$\begin{aligned}
⨍_B G_1(x)\,dx &\lesssim ⨍_B |b(x)-b_B|\,dx\,\varphi(z,r)^{1/\tilde{p}}\|f\|_{L^{(p,\varphi)}} \\
&\lesssim \psi(z,r)\varphi(z,r)^{1/\tilde{p}}\|b\|_{\mathcal{L}^{(1,\psi)}}\|f\|_{L^{(p,\varphi)}} \\
&\lesssim \varphi(z,r)^{1/q}\|b\|_{\mathcal{L}^{(1,\psi)}}\|f\|_{L^{(p,\varphi)}},
\end{aligned}$$

and

$$⨍_B G_2(x)\,dx \lesssim \varphi(z,r)^{1/q}\|b\|_{\mathcal{L}^{(1,\psi)}}\|f\|_{L^{(p,\varphi)}}.$$

Then we have the conclusion. □

2.4 Sharp maximal operator and pointwise estimate

For $f\in L^1_{\mathrm{loc}}(\mathbb{R}^n)$, let

(2.4.1) $$M^\sharp f(x)=\sup_{B\ni x}⨍_B|f(y)-f_B|\,dy,\quad x\in\mathbb{R}^n,$$

where the supremum is taken over all balls B containing x.

Proposition 2.4.1. *Let $p,\eta \in (1,\infty)$, $\varphi \in \mathcal{G}^{\rm dec}$ and $\psi \in \mathcal{G}^{\rm inc}$. Let T be a Calderón-Zygmund operator of type ω. Assume that ψ satisfies* (1.2.7)*, that φ satisfies* (2.1.4)*, that $\int_0^1 \frac{\omega(t)\log(1/t)}{t}\,dt < \infty$ and that $\int_r^\infty \frac{\psi(x,t)\varphi(x,t)^{1/p}}{t}\,dt < \infty$ for each $x \in \mathbb{R}^n$ and $r > 0$. Then there exists a positive constant C such that, for all $b \in \mathcal{L}^{(1,\psi)}(\mathbb{R}^n)$, $f \in L^{(p,\varphi)}(\mathbb{R}^n)$ and $x \in \mathbb{R}^n$,*

$$M^\sharp[b,T]f(x) \le C\|b\|_{\mathcal{L}^{(1,\psi)}}\Big(\big(M_{\psi^\eta}(|Tf|^\eta)(x)\big)^{1/\eta} + \big(M_{\psi^\eta}(|f|^\eta)(x)\big)^{1/\eta}\Big). \tag{2.4.2}$$

Proof of Proposition 2.4.1. We first note that Tf is well defined as mentioned after Remark 2.1.1 and that $[b,T]f$ is well defined by (2.1.6) as seen in Remark 2.3.2 under the assumption that $\int_r^\infty \frac{\psi(x,t)\varphi(x,t)^{1/p}}{t}\,dt < \infty$ for each $x \in \mathbb{R}^n$ and $r > 0$.

For any ball $B = B(x,r)$, let $f = f_1 + f_2$ with $f_1 = f\chi_{2B}$, and let

$$\begin{aligned}
F_1(y) &= (b(y) - b_{2B})Tf(y),\\
F_2(y) &= T((b - b_{2B})f_1)(y),\\
F_3(y) &= T((b - b_{2B})f_2)(y) - C_B,
\end{aligned}$$

for $y \in B$, where $C_B = T((b - b_{2B})f_2)(x)$ and

$$T((b - b_{2B})f_2)(y) = \int_{\mathbb{R}^n} K(y,z)(b(z) - b_{2B})f_2(z)\,dz, \quad y \in B.$$

Then, observing Remark 2.3.2, we have

$$[b,T]f + C_B = [b - b_{2B}, T]f + C_B = F_1 - F_2 - F_3.$$

We show that

$$\begin{aligned}
&⨍_B |F_i(y)|\,dy \\
&\quad \le C\|b\|_{\mathcal{L}^{(1,\psi)}}\Big(\big(M_{\psi^\eta}(|Tf|^\eta)(x)\big)^{1/\eta} + \big(M_{\psi^\eta}(|f|^\eta)(x)\big)^{1/\eta}\Big), \quad i = 1,2,3.
\end{aligned} \tag{2.4.3}$$

Then we have the conclusion.

Now, by Hölder's inequality and Corollary 2.3.3 we have

$$\begin{aligned}
⨍_B |F_1(y)|\,dy &\le \frac{1}{\psi(B)}\left(⨍_B |b(y) - b_{2B}|^{\eta'}\,dy\right)^{1/\eta'}\left(\psi(B)^\eta ⨍_B |Tf(y)|^\eta\,dy\right)^{1/\eta}\\
&\lesssim \|b\|_{\mathcal{L}^{(1,\psi)}}\big(M_{\psi^\eta}(|Tf|^\eta)(x)\big)^{1/\eta}.
\end{aligned}$$

Choose $v \in (1, \eta)$, and let $1/v = 1/u + 1/\eta$. Then by the boundedness of T on $L^v(\mathbb{R}^n)$ and Hölder's inequality we have

$$\begin{aligned}
\fint_B |F_2(y)|\,dy &\le \left(\fint_B |F_2(y)|^v\,dy\right)^{1/v} \\
&\lesssim \left(\frac{1}{|B|}\int_{\mathbb{R}^n} |(b(y)-b_{2B})f_1(y)|^v\,dy\right)^{1/v} \\
&\sim \left(\fint_{2B} |(b(y)-b_{2B})f(y)|^v\,dy\right)^{1/v} \\
&\lesssim \frac{1}{\psi(2B)}\left(\fint_{2B} |b(y)-b_{2B}|^u\,dy\right)^{1/u}\left(\psi(2B)^\eta \fint_{2B} |f(y)|^\eta\,dy\right)^{1/\eta} \\
&\lesssim \|b\|_{\mathcal{L}^{(1,\psi)}}\left(M_{\psi^\eta}(|f|^\eta)(x)\right)^{1/\eta}.
\end{aligned}$$

Finally, for $y \in B$,

$$\begin{aligned}
|F_3(y)| &= |T((b-b_{2B})f_2)(y) - T((b-b_{2B})f_2)(x)| \\
&= \left|\int_{\mathbb{R}^n} (K(y,z)-K(x,z))\,(b(z)-b_{2B})f_2(z)\,dz\right| \\
&\lesssim \int_{\mathbb{R}^n\setminus 2B} \frac{1}{|x-z|^n}\,\omega\!\left(\frac{|x-y|}{|x-z|}\right)|b(z)-b_{2B}||f(z)|\,dz \\
&= \sum_{j=0}^{\infty}\int_{2^{j+2}B\setminus 2^{j+1}B} \frac{1}{|x-z|^n}\,\omega\!\left(\frac{|x-y|}{|x-z|}\right)|b(z)-b_{2B}||f(z)|\,dz.
\end{aligned}$$

If $z \in 2^{j+2}B \setminus 2^{j+1}B$, then

$$\frac{1}{|x-z|^n}\,\omega\!\left(\frac{|x-y|}{|x-z|}\right) \le \frac{\omega(1/2^{j+1})}{|2^{j+1}B|}.$$

Hence

$$\begin{aligned}
|F_3(y)| &\lesssim \sum_{j=0}^{\infty} \frac{\omega(1/2^{j+1})}{|2^{j+1}B|}\int_{2^{j+2}B\setminus 2^{j+1}B} |b(z)-b_{2B}||f(z)|\,dz \\
&\lesssim \sum_{j=0}^{\infty} \omega(1/2^{j+2})\left(\fint_{2^{j+2}B} |b(z)-b_{2B}|^{\eta'}\,dz\right)^{1/\eta'}\left(\fint_{2^{j+2}B} |f(z)|^\eta\,dz\right)^{1/\eta}.
\end{aligned}$$

By Lemma 2.3.7 and Remark 2.3.1 we have

$$
\begin{aligned}
|F_3(y)| &\lesssim \sum_{j=0}^{\infty}(j+2)\omega(1/2^{j+2})\psi(2^{j+2}B)\left(⨍_{2^{j+2}B}|f(z)|^\eta\,dz\right)^{1/\eta}\|b\|_{\mathcal{L}^{(1,\psi)}} \\
&\lesssim \int_0^1\left(\log\frac{1}{t}\right)\frac{\omega(t)}{t}\,dt\,\big(M_{\psi^\eta}(|f|^\eta)(x)\big)^{1/\eta}\|b\|_{\mathcal{L}^{(1,\psi)}} \\
&\lesssim \big(M_{\psi^\eta}(|f|^\eta)(x)\big)^{1/\eta}\|b\|_{\mathcal{L}^{(1,\psi)}}.
\end{aligned}
$$

Therefore,

$$
⨍_B|F_3(y)|\,dy \lesssim \big(M_{\psi^\eta}(|f|^\eta)(x)\big)^{1/\eta}\|b\|_{\mathcal{L}^{(1,\psi)}}.
$$

Then we have (2.4.3) and the conclusion. □

Proposition 2.4.2. *Let* $p,\eta\in(1,\infty)$, $\varphi\in\mathcal{G}^{\rm dec}$ *and* $\psi\in\mathcal{G}^{\rm inc}$. *Assume that* $\rho:\mathbb{R}^n\times(0,\infty)\to(0,\infty)$ *satisfies* (1.2.3) *and* (1.2.4). *Let* $\rho^*(x,r)=\int_0^r\frac{\rho(x,t)}{t}\,dt$. *Assume that* ρ, ρ^* *and* ψ *satisfy* (1.2.7), *that* φ *satisfies* (2.1.4) *and that there exist positive constants* ϵ, C_ρ *such that* (2.1.10) *and* (2.1.11) *hold. Assume also that*

(2.4.4)

$$
\int_r^\infty\frac{\rho(x,t)\varphi(x,t)^{1/p}}{t}\,dt<\infty,\qquad\int_r^\infty\frac{\psi(x,t)}{t}\left(\int_t^\infty\frac{\rho(x,u)\varphi(x,u)^{1/p}}{u}\,du\right)dt<\infty,
$$

for each $x\in\mathbb{R}^n$ *and* $r>0$. *Then there exists a positive constant* C *such that, for all* $b\in\mathcal{L}^{(1,\psi)}(\mathbb{R}^n)$, $f\in L^{(p,\varphi)}(\mathbb{R}^n)$ *and* $x\in\mathbb{R}^n$,

$$
M^\sharp([b,I_\rho]f)(x)\le C\|b\|_{\mathcal{L}^{(1,\psi)}}\left(\big(M_{\psi^\eta}(|I_\rho f|^\eta)(x)\big)^{1/\eta}+\big(M_{(\rho^*\psi)^\eta}(|f|^\eta)(x)\big)^{1/\eta}\right). \tag{2.4.5}
$$

Proof of Proposition 2.4.2. We first note that $I_\rho f$ and $[b,I_\rho]f$ are well defined as seen in Remark 2.3.3 under the assumption (2.4.4). For any ball $B=B(x,r)$, let $f=f_1+f_2$ with $f_1=f\chi_{2B}$, and let

$$
\begin{aligned}
F_1(y)&=(b(y)-b_{2B})I_\rho f(y),\\
F_2(y)&=I_\rho((b-b_{2B})f_1)(y),\\
F_3(y)&=I_\rho((b-b_{2B})f_2)(y)-C_B,
\end{aligned}
$$

for $y\in B$, where $C_B=I_\rho((b-b_{2B})f_2)(x)$ and

$$
I_\rho((b-b_{2B})f_2)(y)=\int_{\mathbb{R}^n}\frac{\rho(y,|y-z|)}{|y-z|^n}(b(z)-b_{2B})f_2(z)\,dz,\quad y\in B.
$$

Then, observing Remark 2.3.3, we have

$$[b, I_\rho]f + C_B = [b - b_{2B}, I_\rho]f + C_B = F_1 - F_2 - F_3.$$

We show that

$$(2.4.6)\quad \fint_B |F_i(y)|\,dy$$
$$\le C\|b\|_{\mathcal{L}^{(1,\psi)}} \left(\left(M_{\psi^\eta}(|I_\rho f|^\eta)(x)\right)^{1/\eta} + \left(M_{(\rho^*\psi)^\eta}(|f|^\eta)(x)\right)^{1/\eta} \right), \quad i = 1, 2, 3.$$

Then we have the conclusion.

Now, by Hölder's inequality we have

$$\fint_B |F_1(y)|\,dy \le \frac{1}{\psi(B)} \left(\fint_B |b(y) - b_{2B}|^{\eta'}\,dy \right)^{1/\eta'} \left(\psi(B)^\eta \fint_B |I_\rho f(y)|^\eta\,dy \right)^{1/\eta}$$
$$\lesssim \|b\|_{\mathcal{L}^{(1,\psi)}} \left(M_{\psi^\eta}(|I_\rho f|^\eta)(x)\right)^{1/\eta}.$$

Choose $v \in (1, \eta)$ such that $n/v - \epsilon/2 \ge n - \epsilon$. Then by (2.1.10) we have

$$C_\rho \frac{\rho(x,t)}{t^{n/v-\epsilon/2}} \ge \frac{\rho(x,s)}{s^{n/v-\epsilon/2}}, \quad \text{if } t < s.$$

Hence, from Theorem 2.2.5 it follows that there exists an N-function Φ such that I_ρ is bounded from $L^v(\mathbb{R}^n)$ to $L^\Phi(\mathbb{R}^n)$. Let Ψ be the complementary function of Φ. Then by the generalized Hölder's inequality (2.2.11), Lemma 2.2.6 and the boundedness of I_ρ we have

$$\fint_B |F_2(y)|\,dy \le \frac{2}{|B|} \|\chi_B\|_{L^\Psi(\mathbb{R}^n)} \|F_2\|_{L^\Phi(\mathbb{R}^n)}$$
$$\lesssim \frac{\rho^*(B)}{|B|^{1/v}} \|(b - b_{2B})f\|_{L^v(2B)}.$$

Let $1/v = 1/u + 1/\eta$. Then by Hölder's inequality we have

$$\fint_B |F_2(y)|\,dy$$
$$\lesssim \rho^*(B) \left(\fint_{2B} |b(y) - b_{2B}|^u\,dy \right)^{1/u} \left(\fint_{2B} |f(y)|^\eta\,dy \right)^{1/\eta}$$
$$\lesssim \frac{1}{\psi(2B)} \left(\fint_{2B} |b(y) - b_{2B}|^u\,dy \right)^{1/u} \left((\rho^*(2B)\psi(2B))^\eta \fint_{2B} |f(y)|^\eta\,dy \right)^{1/\eta}$$
$$\lesssim \|b\|_{\mathcal{L}^{(1,\psi)}} \left(M_{(\rho^*\psi)^\eta}(|f|^\eta)(x)\right)^{1/\eta}.$$

Finally, for $y \in B$, using (2.1.11), we have

$$
\begin{aligned}
|F_3(y)| &= |I_\rho((b-b_{2B})f_2)(y) - I_\rho((b-b_{2B})f_2)(x)| \\
&= \left| \int_{\mathbb{R}^n} \left(\frac{\rho(y,|y-z|)}{|y-z|^n} - \frac{\rho(x,|x-z|)}{|x-z|^n} \right) (b(z)-b_{2B}) f_2(z)\, dz \right| \\
&\lesssim \int_{\mathbb{R}^n \setminus 2B} \frac{|x-y|\rho^*(x,|x-z|)}{|x-z|^{n+1}} |b(z)-b_{2B}||f(z)|\, dz \\
&= \sum_{j=0}^{\infty} \int_{2^{j+2}B \setminus 2^{j+1}B} \frac{|x-y|\rho^*(x,|x-z|)}{|x-z|^{n+1}} |b(z)-b_{2B}||f(z)|\, dz.
\end{aligned}
$$

Since ρ^* satisfies the doubling condition (see Remark 2.2.1), we have

$$
\begin{aligned}
&\int_{2^{j+2}B \setminus 2^{j+1}B} \frac{|x-y|\rho^*(x,|x-z|)}{|x-z|^{n+1}} |b(z)-b_{2B}||f(z)|\, dz \\
&\lesssim \frac{r\rho^*(2^{j+2}B)}{(2^{j+2}r)^{n+1}} \int_{2^{j+2}B \setminus 2^{j+1}B} |b(z)-b_{2B}||f(z)|\, dz \\
&\lesssim \frac{\rho^*(2^{j+2}B)}{2^{j+2}} \left(⨍_{2^{j+2}B} |b(z)-b_{2B}|^{\eta'}\, dz \right)^{1/\eta'} \left(⨍_{2^{j+2}B} |f(z)|^{\eta}\, dz \right)^{1/\eta}.
\end{aligned}
$$

By Lemma 2.3.7 and Remark 2.3.1 we have

$$
\begin{aligned}
|F_3(y)| &\lesssim \|b\|_{\mathcal{L}^{(1,\psi)}} \sum_{j=0}^{\infty} \frac{j+1}{2^{j+2}} \rho^*(2^{j+2}B)\psi(2^{j+2}B) \left(⨍_{2^{j+2}B} |f(z)|^{\eta}\, dz \right)^{1/\eta} \\
&\lesssim \|b\|_{\mathcal{L}^{(1,\psi)}} \left(M_{(\rho^*\psi)^\eta}(|f|^\eta)(x) \right)^{1/\eta}.
\end{aligned}
$$

Therefore,

$$
⨍_B |F_3(y)|\, dy \lesssim \|b\|_{\mathcal{L}^{(1,\psi)}} \left(M_{(\rho^*\psi)^\eta}(|f|^\eta)(x) \right)^{1/\eta}.
$$

Then we have (2.4.6) and the conclusion. □

2.5 Estimate by the sharp maximal operator

In this section we prove the following proposition and its corollary.

Proposition 2.5.1. *Let $1 \le p < \infty$ and $\varphi : \mathbb{R}^n \times (0,\infty) \to (0,\infty)$. Then, for $f \in L^1_{\mathrm{loc}}(\mathbb{R}^n)$,*

$$
\|f\|_{\mathcal{L}^{(p,\varphi)}} \le C \|M^\sharp f\|_{L^{(p,\varphi)}}, \tag{2.5.1}
$$

where C is a positive constant independent of f.

Corollary 2.5.2. *Let $1 \le p < \infty$ and $\varphi : \mathbb{R}^n \times (0,\infty) \to (0,\infty)$. Assume that $\varphi \in \mathcal{G}^{\mathrm{dec}}$ and that φ satisfies (2.1.4). For $f \in L^1_{\mathrm{loc}}(\mathbb{R}^n)$, if $\lim_{r\to\infty} f_{B(0,r)} = 0$, then*

$$\|f\|_{L^{(p,\varphi)}} \le C \|M^\sharp f\|_{L^{(p,\varphi)}}, \tag{2.5.2}$$

where C is a positive constant independent of f.

The condition $\lim_{r\to\infty} f_{B(0,r)} = 0$ was considered by Fujii [20] in 1989. We first prove Corollary 2.5.2 by using Proposition 2.5.1.

Proof of Corollary 2.5.2. By Lemma 2.3.5 we have that, for every $f \in \mathcal{L}^{(p,\varphi)}(\mathbb{R}^n)$, $f_{B(0,r)}$ converges as $r \to \infty$ and $\|f - \lim_{r\to\infty} f_{B(0,r)}\|_{L^{(p,\varphi)}} \lesssim \|f\|_{\mathcal{L}^{(p,\varphi)}}$. Since $\lim_{r\to\infty} f_{B(0,r)} = 0$ by the assumption, using Proposition 2.5.1, we have the conclusion. □

To prove Proposition 2.5.1 we define local versions of the dyadic maximal operator and the dyadic sharp maximal operator. For any cube $Q \subset \mathbb{R}^n$ centered at $a \in \mathbb{R}^n$ and with sidelength $2r > 0$, we denote by $\mathcal{Q}^{\mathrm{dy}}(Q)$ the set of all dyadic cubes with respect to Q, that is,

$$\mathcal{Q}^{\mathrm{dy}}(Q) = \left\{ Q_{j,k} = a + \prod_{i=1}^{n} [2^{-j}k_i r, 2^{-j}(k_i+1)r) : j \in \mathbb{Z},\ k = (k_1,\dots,k_n) \in \mathbb{Z}^n \right\}.$$

For any cube $Q \subset \mathbb{R}^n$, let

$$M_Q^{\mathrm{dy}} f(x) = \sup_{R \in \mathcal{Q}^{\mathrm{dy}}(Q),\, x \in R \subset Q} \fint_R |f(y)|\,dy,$$
$$M_Q^{\sharp,\mathrm{dy}} f(x) = \sup_{R \in \mathcal{Q}^{\mathrm{dy}}(Q),\, x \in R \subset Q} \fint_R |f(y) - f_R|\,dy.$$

Lemma 2.5.3 (Tsutsui [67], Komori-Furuya [33])**.** *Let Q be a cube and $f \in L^1(Q)$. Then, for any $0 < \gamma \le 1$ and $\lambda > |f|_Q$,*

$$\begin{aligned} &|\{x \in Q : M_Q^{\mathrm{dy}} f(x) > 2\lambda, M_Q^{\sharp,\mathrm{dy}} f(x) \le \gamma\lambda\}| \\ &\qquad \le 2^n \gamma |\{x \in Q : M_Q^{\mathrm{dy}} f(x) > \lambda\}|. \end{aligned} \tag{2.5.3}$$

Using Lemma 2.5.3, we have the following lemma, which is a special case of [52, Lemma 4.4]. We give a proof for readers' convenience.

Lemma 2.5.4. *There exists a positive constant C, for any cube Q and any function $f \in L^1(Q)$,*

$$\|f - f_Q\|_{L^p(Q)} \le C\|M_Q^{\sharp,\mathrm{dy}} f\|_{L^p(Q)}. \tag{2.5.4}$$

Proof. By the good λ inequality (2.5.3) and the standard argument we have the following boundedness: There exists a positive constant C, for any cube Q and any function $f \in L^1(Q)$,

$$\|M_Q^{\mathrm{dy}} f\|_{L^p(Q)} \le C\left(\|M_Q^{\sharp,\mathrm{dy}} f\|_{L^p(Q)} + |Q|^{1/p}|f|_Q\right). \tag{2.5.5}$$

Actually, for any $L > 2|f|_Q$,

$$\begin{aligned}
&\int_0^L p\lambda^{p-1}|\{x \in Q : M_Q^{\mathrm{dy}} f(x) > \lambda\}|\, d\lambda \\
&= \int_0^{2|f|_Q} p\lambda^{p-1}|\{x \in Q : M_Q^{\mathrm{dy}} f(x) > \lambda\}|\, d\lambda \\
&\qquad + \int_{2|f|_Q}^L p\lambda^{p-1}|\{x \in Q : M_Q^{\mathrm{dy}} f(x) > \lambda\}|\, d\lambda \\
&\le (2|f|_Q)^p|Q| + 2^p \int_{|f|_Q}^{L/2} p\lambda^{p-1}|\{x \in Q : M_Q^{\mathrm{dy}} f(x) > 2\lambda\}|\, d\lambda.
\end{aligned}$$

By the good λ inequality (2.5.3) we have

$$\begin{aligned}
&2^p \int_{|f|_Q}^{L/2} p\lambda^{p-1}|\{x \in Q : M_Q^{\mathrm{dy}} f(x) > 2\lambda\}|\, d\lambda \\
&\le 2^{n+p}\gamma \int_{|f|_Q}^{L/2} p\lambda^{p-1}|\{x \in Q : M_Q^{\mathrm{dy}} f(x) > \lambda\}|\, d\lambda \\
&\qquad + 2^p \int_{|f|_Q}^{L/2} p\lambda^{p-1}|\{x \in Q : M_Q^{\sharp,\mathrm{dy}} f(x) > \gamma\lambda\}|\, d\lambda \\
&\le 2^{n+p}\gamma \int_0^L p\lambda^{p-1}|\{x \in Q : M_Q^{\mathrm{dy}} f(x) > \lambda\}|\, d\lambda \\
&\qquad + 2^p\gamma^{-p} \int_0^\infty p\lambda^{p-1}|\{x \in Q : M_Q^{\sharp,\mathrm{dy}} f(x) > \lambda\}|\, d\lambda.
\end{aligned}$$

Then, for small $\gamma > 0$,

$$\begin{aligned}
&(1 - 2^{n+p}\gamma) \int_0^L p\lambda^{p-1}|\{x \in Q : M_Q^{\mathrm{dy}} f(x) > \lambda\}|\, d\lambda \\
&\le (2|f|_Q)^p|Q| + 2^p\gamma^{-p} \int_0^\infty p\lambda^{p-1}|\{x \in Q : M_Q^{\sharp,\mathrm{dy}} f(x) > \lambda\}|\, d\lambda.
\end{aligned}$$

Letting $L \to \infty$, we have (2.5.5).

Now, substitute $f - f_Q$ for f in (2.5.5). Then

$$\begin{aligned}\|f - f_Q\|_{L^p(Q)} &\le \|M_Q^{\rm dy}(f - f_Q)\|_{L^p(Q)} \\ &\lesssim \|M_Q^{\sharp,{\rm dy}} f\|_{L^p(Q)} + |Q|^{1/p} ⨍_Q |f - f_Q| \\ &\le \|M_Q^{\sharp,{\rm dy}} f\|_{L^p(Q)} + |Q|^{1/p} \inf_{x\in Q} M_Q^{\sharp,{\rm dy}} f(x).\end{aligned}$$

Since

$$|Q|^{1/p} \inf_{x\in Q} M_Q^{\sharp,{\rm dy}} f(x) = \left(\int_Q \left[\inf_{x\in Q} M_Q^{\sharp,{\rm dy}} f(x) \right]^p dy \right)^{1/p} \le \|M_Q^{\sharp,{\rm dy}} f\|_{L^p(Q)},$$

we have the conclusion. □

Proof of Proposition 2.5.1. For any ball $B = B(x, r)$, take the cube Q centered at x and with sidelength $2r$. Then $B \subset Q$. By Lemma 2.5.4 we have

$$\begin{aligned}\left(\frac{1}{\varphi(B)} ⨍_B |f - f_B|^p \right)^{1/p} &\le 2 \left(\frac{1}{\varphi(B)} \frac{|Q|}{|B|} ⨍_Q |f - f_Q|^p \right)^{1/p} \\ &\lesssim \left(\frac{1}{\varphi(B)} ⨍_Q (M_Q^{\sharp,{\rm dy}} f)^p \right)^{1/p} \\ &\lesssim \|M^\sharp f\|_{L^{(p,\varphi)}(\mathbb{R}^n)}.\end{aligned}$$

This shows the conclusion. □

2.6 Proofs of the theorems

We first note that, for $0 < \theta < \infty$, we have

$$\| |g|^\theta \|_{L^{(p,\varphi)}} = \left(\|g\|_{L^{(p\theta,\varphi)}} \right)^\theta, \tag{2.6.1}$$

Proof of Theorem 2.1.1 (i). By the assumption we have that T is bounded on $L^{(p,\varphi)}(\mathbb{R}^n)$, see [38, Theorem 2]. Let $1 < \eta < p$. Then, from (2.1.7) it follows that

$$\psi(x, r)^\eta \varphi(x, r)^{\eta/p} \le C_0{}^\eta \varphi(x, r)^{\eta/q}.$$

By Theorem 2.2.3 with this condition we have the boundedness of M_{ψ^η} from $L^{(p/\eta,\varphi)}(\mathbb{R}^n)$ to $L^{(q/\eta,\varphi)}(\mathbb{R}^n)$. Using this boundedness and (2.6.1), we have

$$\begin{aligned}\|(M_{\psi^\eta}(|Tf|^\eta))^{1/\eta}\|_{L^{(q,\varphi)}} &= \left(\|M_{\psi^\eta}(|Tf|^\eta)\|_{L^{(q/\eta,\varphi)}}\right)^{1/\eta} \lesssim \left(\||Tf|^\eta\|_{L^{(p/\eta,\varphi)}}\right)^{1/\eta} \\ &= \|Tf\|_{L^{(p,\varphi)}} \lesssim \|f\|_{L^{(p,\varphi)}},\end{aligned}$$

and

$$\begin{aligned}\|(M_{\psi^\eta}(|f|^\eta))^{1/\eta}\|_{L^{(q,\varphi)}} &= \left(\|M_{\psi^\eta}(|f|^\eta)\|_{L^{(q/\eta,\varphi)}}\right)^{1/\eta} \lesssim \left(\||f|^\eta\|_{L^{(p/\eta,\varphi)}}\right)^{1/\eta} \\ &= \|f\|_{L^{(p,\varphi)}}.\end{aligned}$$

Then, using Proposition 2.4.1, we have

$$\|M^\sharp([b,T]f)\|_{L^{(q,\varphi)}} \lesssim \|b\|_{\mathcal{L}^{(1,\psi)}}\|f\|_{L^{(p,\varphi)}}. \tag{2.6.2}$$

Therefore, if we show that, for $B_r = B(0,r)$,

$$⨍_{B_r} [b,T]f \to 0 \quad \text{as} \quad r\to\infty, \tag{2.6.3}$$

then we have

$$\|[b,T]f\|_{L^{(q,\varphi)}} \lesssim \|b\|_{\mathcal{L}^{(1,\psi)}}\|f\|_{L^{(p,\varphi)}}, \tag{2.6.4}$$

by Corollary 2.5.2.

In the following we show (2.6.3).

Case 1: First we show (2.6.3) for all $f \in L^{(p,\varphi)}(\mathbb{R}^n)$ with compact support. Let $\operatorname{supp} f \subset B_s = B(0,s)$ with $s \geq 1$. Then $f \in L^p(\mathbb{R}^n)$ and $b \in L^{p_0}_{\mathrm{loc}}(\mathbb{R}^n)$ for all $p_0 \in [1,\infty)$. Since T is bounded on Lebesgue spaces, we see that $(bTf)\chi_{B_{2s}}$ and $T(bf)\chi_{B_{2s}}$ are in $L^1(\mathbb{R}^n)$ and that

$$⨍_{B_r} (bTf)\chi_{B_{2s}} \to 0, \quad ⨍_{B_r} T(bf)\chi_{B_{2s}} \to 0 \quad \text{as } r\to\infty.$$

If $x \notin B_{2s}$ and $y \in B(0,s)$, then $|x|/2 \leq |x-y| \leq (3/2)|x|$. By (2.1.1) and (2.1.3) we have

$$|Tf(x)| \lesssim \frac{1}{|x|^n}\|f\|_{L^1}, \quad |T(bf)(x)| \lesssim \frac{1}{|x|^n}\|bf\|_{L^1}, \quad x \notin B_{2s}, \tag{2.6.5}$$

which yields

$$b_{B_{2s}} \fint_{B_r} (Tf)(1-\chi_{B_{2s}}) \to 0, \quad \fint_{B_r} (T(bf))(1-\chi_{B_{2s}}) \to 0 \quad \text{as } r \to \infty.$$

Next, we show

$$\fint_{B_r} (b - b_{B_{2s}})(Tf)(1-\chi_{B_{2s}}) \to 0 \quad \text{as } r \to \infty. \tag{2.6.6}$$

Then we have (2.6.3) for all $f \in L^{(p,\varphi)}(\mathbb{R}^n)$ with compact support.

To show (2.6.6), take $\epsilon \in (0,1)$ such that $1+1/q-1/p > \epsilon$, and let $\nu = 1/(1-\epsilon)$. Then

$$\left| \fint_{B_r} (b - b_{B_{2s}})(Tf)(1-\chi_{B_{2s}}) \right| \le \left(\fint_{B_r} |b - b_{B_{2s}}|^{\nu'} \right)^{1/\nu'} \left(\fint_{B_r} |(Tf)(1-\chi_{B_{2s}})|^{\nu} \right)^{1/\nu}.$$

From Lemma 2.3.7, Remark 2.3.1 and (2.1.7) it follows that, for $r > 4s \ge 4$,

$$\begin{aligned} \left(\fint_{B_r} |b - b_{B_{2s}}|^{\nu'} \right)^{1/\nu'} &\lesssim \int_{2s}^{r} \frac{\psi(0,t)}{t}\, dt\, \|b\|_{\mathcal{L}^{(1,\psi)}} \\ &\lesssim \psi(0,r) \log r\, \|b\|_{\mathcal{L}^{(1,\psi)}} \lesssim \varphi(0,r)^{1/q-1/p} \log r\, \|b\|_{\mathcal{L}^{(1,\psi)}}. \end{aligned} \tag{2.6.7}$$

From (2.6.5) it follows that

$$\left(\int_{B_r \setminus B_{2s}} |Tf(x)|^{\nu}\, dx \right)^{1/\nu} \lesssim \left(\int_{B_r \setminus B_{2s}} \left(\frac{1}{|x|^n} \|f\|_{L^1} \right)^{\nu} dx \right)^{1/\nu} \lesssim \|f\|_{L^1}. \tag{2.6.8}$$

By (2.6.7) and (2.6.8) we have

$$\begin{aligned} &\left| \fint_{B_r} (b - b_{B_{2s}})(Tf)(1-\chi_{B_{2s}}) \right| \\ &\lesssim \varphi(0,r)^{1/q-1/p} \log r\, \|b\|_{\mathcal{L}^{(1,\psi)}} \frac{1}{r^{n/\nu}} \|f\|_{L^1} \\ &= \frac{\log r}{r^{n(1+1/q-1/p-\epsilon)}} \left(\frac{1}{r^n \varphi(0,r)} \right)^{1/p-1/q} \|b\|_{\mathcal{L}^{(1,\psi)}} \|f\|_{L^1} \\ &\to 0 \quad \text{as } r \to \infty, \end{aligned}$$

since $r^n\varphi(0,r)$ is almost increasing. Therefore, we have (2.6.3) and (2.6.4) for all $f \in L^{(p,\varphi)}(\mathbb{R}^n)$ with compact support.

Case 2: For general $f \in L^{(p,\varphi)}(\mathbb{R}^n)$, using Case 1, we have

$$\|[b,T](f\chi_{B_{2r}})\|_{L^{(q,\varphi)}} \lesssim \|b\|_{\mathcal{L}^{(1,\psi)}}\|f\chi_{B_{2r}}\|_{L^{(p,\varphi)}} \le \|b\|_{\mathcal{L}^{(1,\psi)}}\|f\|_{L^{(p,\varphi)}}.$$

Then

$$\left|\fint_{B_r}[b,T](f\chi_{B_{2r}})\right| \le \varphi(0,r)^{1/q}\|[b,T](f\chi_{B_{2r}})\|_{L^{(q,\varphi)}} \lesssim \varphi(0,r)^{1/q}\|b\|_{\mathcal{L}^{(1,\psi)}}\|f\|_{L^{(p,\varphi)}}.$$

Combining this and Lemma 2.3.10, we have

$$\left|\fint_{B_r}[b,T]f\right| \lesssim \varphi(0,r)^{1/q}\|b\|_{\mathcal{L}^{(1,\psi)}}\|f\|_{L^{(p,\varphi)}},$$

which implies (2.6.3). Therefore, we have (2.6.4) for all $f \in L^{(p,\varphi)}(\mathbb{R}^n)$. The proof is complete. □

Proof of Theorem 2.1.1 (ii). We use the method by Janson [28]. Since $1/K(z)$ is infinitely differentiable in an open set, we may choose $z_0 \ne 0$ and $\delta > 0$ such that $1/K(z)$ can be expressed in the neighborhood $|z-z_0| < 2\delta$ as an absolutely convergent Fourier series, $1/K(z) = \sum a_j e^{iv_j\cdot z}$. (The exact form of the vectors v_j is irrelevant.)

Set $z_1 = z_0/\delta$. If $|z - z_1| < 2$, we have the expansion

$$\frac{1}{K(z)} = \frac{\delta^{-n}}{K(\delta z)} = \delta^{-n}\sum a_j e^{iv_j\cdot\delta z}.$$

Choose now any ball $B = B(x_0, r)$. Set $y_0 = x_0 - rz_1$ and $B' = B(y_0, r)$. Then, if $x \in B$ and $y \in B'$,

$$\left|\frac{x-y}{r} - z_1\right| \le \left|\frac{x-x_0}{r}\right| + \left|\frac{y-y_0}{r}\right| \le 2.$$

Denote $\operatorname{sgn}(f(x) - f_{B'})$ by $s(x)$. Then

$$\begin{aligned}
&\int_B |b(x) - b_{B'}|\,dx = \int_B (b(x)-b_{B'})s(x)\,dx = \frac{1}{|B'|}\int_B\int_{B'}(b(x)-b(y))s(x)\,dy\,dx\\
&= \frac{1}{|B'|}\int_{\mathbb{R}^n}\int_{\mathbb{R}^n}(b(x)-b(y))\frac{r^nK(x-y)}{K(\frac{x-y}{r})}s(x)\chi_B(x)\chi_{B'}(y)\,dy\,dx\\
&= \frac{r^n\delta^{-n}}{|B'|}\int_{\mathbb{R}^n}\int_{\mathbb{R}^n}(b(x)-b(y))K(x-y)\sum a_j e^{iv_j\cdot\delta\frac{x-y}{r}}s(x)\chi_B(x)\chi_{B'}(y)\,dy\,dx.
\end{aligned}$$

Here, we set $C = \delta^{-n}|B(0,1)|^{-1}$ and

$$g_j(y) = e^{-iv_j\cdot\delta\frac{y}{r}}\chi_{B'}(y), \quad h_j(x) = e^{iv_j\cdot\delta\frac{x}{r}}s(x)\chi_B(x).$$

Then

$$\begin{aligned}\int_B |b(x) - b_{B'}|\,dx &= C\sum a_j \int_{\mathbb{R}^n}\int_{\mathbb{R}^n}(b(x)-b(y))K(x-y)g_j(y)h_j(x)\,dy\,dx \\ &= C\sum a_j \int_{\mathbb{R}^n}([b,T]g_j)(x)h_j(x)\,dx \\ &\le C\sum |a_j| \int_{\mathbb{R}^n}|([b,T]g_j)(x)||h_j(x)|\,dx \\ &= C\sum |a_j| \int_B |([b,T]g_j)(x)|\,dx \\ &\le C\sum |a_j||B|\varphi(B)^{1/q}\|[b,T]g_j\|_{L^{(q,\varphi)}} \\ &\le C\|[b,T]\|_{L^{(p,\varphi)}\to L^{(q,\varphi)}}|B|\varphi(B)^{1/q}\sum |a_j|\|g_j\|_{L^{(p,\varphi)}}.\end{aligned}$$

Since φ is in $\mathcal{G}^{\mathrm{dec}}$ and satisfies (1.2.7), by [39, Lemma 4.2] we can conclude that $\|g_j\|_{L^{(p,\varphi)}} = \|\chi_{B'}\|_{L^{(p,\varphi)}} \sim \frac{1}{\varphi(B')^{1/p}}$. We also see that $\varphi(B') \sim \varphi(B)$, since $|x_0 - y_0| = r|z_1|$. Then

$$\int_B |b(x) - b_{B'}|\,dx \lesssim \|[b,T]\|_{L^{(p,\varphi)}\to L^{(q,\varphi)}}|B|\varphi(B)^{1/q-1/p}.$$

By (2.1.8) we have

$$\frac{1}{\psi(B)}\fint_B |b(x) - b_B|\,dx \le \frac{2}{\psi(B)}\fint_B |b(x) - b_{B'}|\,dx \lesssim \|[b,T]\|_{L^{(p,\varphi)}\to L^{(q,\varphi)}}.$$

That is, $\|b\|_{\mathcal{L}^{(1,\psi)}} \lesssim \|[b,T]\|_{L^{(p,\varphi)}\to L^{(q,\varphi)}}$ and we have the conclusion. □

Proof of Theorem 2.1.2 (i). By Theorem 2.2.2 with the assumption (2.1.12) we have the boundedness of I_ρ from $L^{(p,\varphi)}(\mathbb{R}^n)$ to $L^{(\tilde{p},\varphi)}(\mathbb{R}^n)$. Let $1 < \eta < p$ and $\rho^*(x,r) = \int_0^r \frac{\rho(x,t)}{t}\,dt$. Then, from (2.1.12) and (2.1.13) it follows that

$$\begin{aligned}(\rho^*(x,r)\psi(x,r))^\eta\varphi(x,r)^{\eta/p} &\le C_0{}^\eta\varphi(x,r)^{\eta/q}, \\ \psi(x,r)^\eta\varphi(x,r)^{\eta/\tilde{p}} &\le C_1{}^\eta\varphi(x,r)^{\eta/q}.\end{aligned}$$

By Theorem 2.2.3 with these conditions we have the boundedness of $M_{(\rho^*\psi)^\eta}$ from $L^{(p/\eta,\varphi)}(\mathbb{R}^n)$ to $L^{(q/\eta,\varphi)}(\mathbb{R}^n)$ and of M_{ψ^η} from $L^{(\tilde{p}/\eta,\varphi)}(\mathbb{R}^n)$ to $L^{(q/\eta,\varphi)}(\mathbb{R}^n)$. Using these boundedness and (2.6.1), we have

$$\|(M_{\psi^\eta}(|I_\rho f|^\eta))^{1/\eta}\|_{L^{(q,\varphi)}} = \left(\|M_{\psi^\eta}(|I_\rho f|^\eta)\|_{L^{(q/\eta,\varphi)}}\right)^{1/\eta} \lesssim \left(\||I_\rho f|^\eta\|_{L^{(\tilde{p}/\eta,\varphi)}}\right)^{1/\eta}$$
$$= \|I_\rho f\|_{L^{(\tilde{p},\varphi)}} \lesssim \|f\|_{L^{(p,\varphi)}},$$

and

$$\|(M_{(\rho^*\psi)^\eta}(|f|^\eta))^{1/\eta}\|_{L^{(q,\varphi)}} = \left(\|M_{(\rho^*\psi)^\eta}(|f|^\eta)\|_{L^{(q/\eta,\varphi)}}\right)^{1/\eta} \lesssim \left(\||f|^\eta\|_{L^{(p/\eta,\varphi)}}\right)^{1/\eta}$$
$$= \|f\|_{L^{(p,\varphi)}}.$$

Then, using Proposition 2.4.2, we have

$$\|M^\sharp([b, I_\rho]f)\|_{L^{(q,\varphi)}} \lesssim \|b\|_{\mathcal{L}^{(1,\psi)}}\|f\|_{L^{(p,\varphi)}}. \tag{2.6.9}$$

Therefore, if we show that, for $B_r = B(0,r)$,

$$⨍_{B_r} [b, I_\rho]f \to 0 \quad \text{as} \quad r \to \infty, \tag{2.6.10}$$

then we have

$$\|[b, I_\rho]f\|_{L^{(q,\varphi)}} \lesssim \|b\|_{\mathcal{L}^{(1,\psi)}}\|f\|_{L^{(p,\varphi)}}, \tag{2.6.11}$$

by Corollary 2.5.2.

In the following we show (2.6.10).

Case 1: First we show (2.6.10) for all $f \in L^{(p,\varphi)}(\mathbb{R}^n)$ with compact support. Let $\operatorname{supp} f \subset B_s = B(0,s)$ with $s \ge 1$. Then $f \in L^p(\mathbb{R}^n)$ and $b \in L^{p_0}_{\mathrm{loc}}(\mathbb{R}^n)$ for all $p_0 \in [1,\infty)$. Since $\frac{\rho(x,|y|)}{|y|^n}$ is locally integrable with respect to y, we see that $(bI_\rho f)\chi_{B_{2s}}$ and $I_\rho(bf)\chi_{B_{2s}}$ are in $L^1(\mathbb{R}^n)$ and that

$$⨍_{B_r} (bI_\rho f)\chi_{B_{2s}} \to 0, \quad ⨍_{B_r} I_\rho(bf)\chi_{B_{2s}} \to 0 \quad \text{as } r \to \infty.$$

If $x \notin B_{2s}$ and $y \in B(0,s)$, then $|y| < |x-y|$ and $|x|/2 \le |x-y| \le (3/2)|x|$. Since ρ satisfies (1.2.7),

$$\rho(x, |x-y|) \sim \rho(y, |x-y|) \sim \rho(0, |x-y|) \le \sup_{|x|/2 \le t \le (3/2)|x|} \rho(0,t). \tag{2.6.12}$$

Then we have

$$\frac{\rho(x,|x-y|)}{|x-y|^n} \lesssim \frac{\sup_{|x|/2\le t\le(3/2)|x|}\rho(0,t)}{|x|^n} \sim \sup_{|x|/2 \le t \le (3/2)|x|} \frac{\rho(0,t)}{t^n},$$

and

$$|I_\rho f(x)| \lesssim \sup_{|x|/2\le t\le (3/2)|x|} \frac{\rho(0,t)}{t^n}\|f\|_{L^1}, \quad |I_\rho(bf)(x)| \lesssim \sup_{|x|/2\le t\le (3/2)|x|} \frac{\rho(0,t)}{t^n}\|bf\|_{L^1}.$$

From (2.1.10) it follows that $\frac{\rho(0,t)}{t^n} \to 0$ as $t \to \infty$, which yields

$$b_{B_{2s}} \fint_{B_r} (I_\rho f)(1-\chi_{B_{2s}}) \to 0, \quad \fint_{B_r} (I_\rho(bf))(1-\chi_{B_{2s}}) \to 0 \quad \text{as } r \to \infty.$$

Next, we show

$$\fint_{B_r} (b - b_{B_{2s}})(I_\rho f)(1-\chi_{B_{2s}}) \to 0 \quad \text{as } r \to \infty. \tag{2.6.13}$$

Then we have (2.6.10) for all $f \in L^{(p,\varphi)}(\mathbb{R}^n)$ with compact support.

To show (2.6.13), take $\epsilon \in (0,1)$ such that $1+1/q-1/p > \epsilon$, and let $\nu = 1/(1-\epsilon)$. Then

$$\left|\fint_{B_r} (b-b_{B_{2s}})(I_\rho f)(1-\chi_{B_{2s}})\right| \le \left(\fint_{B_r} |b-b_{B_{2s}}|^{\nu'}\right)^{1/\nu'} \left(\fint_{B_r} |(I_\rho f)(1-\chi_{B_{2s}})|^{\nu}\right)^{1/\nu}.$$

From Lemma 2.3.7, Remark 2.3.1 and (2.1.13) it follows that

$$\begin{aligned}\left(\fint_{B_r} |b-b_{B_{2s}}|^{\nu'}\right)^{1/\nu'} &\lesssim \int_{2s}^{r} \frac{\psi(0,t)}{t}\,dt\,\|b\|_{\mathcal{L}^{(1,\psi)}} \\ &\lesssim \psi(0,r)\log r\,\|b\|_{\mathcal{L}^{(1,\psi)}} \lesssim \varphi(0,r)^{1/q-1/\tilde{p}}\log r\,\|b\|_{\mathcal{L}^{(1,\psi)}}.\end{aligned} \tag{2.6.14}$$

For $j = 0,1,2,\dots$, from (2.6.12) and (1.2.4) it follows that

$$\begin{aligned}&\left(\int_{2^{j+2}B_s\setminus 2^{j+1}B_s} |I_\rho f(x)|^\nu\,dx\right)^{1/\nu} \\ &\lesssim \left(\int_{2^{j+2}B_s\setminus 2^{j+1}B_s} \left(\frac{\sup_{|x|/2\le t\le(3/2)|x|}\rho(0,t)}{|x|^n}\|f\|_{L^1}\right)^\nu dx\right)^{1/\nu} \\ &\lesssim (2^j s)^{(-n\nu+n)/\nu} \sup_{2^j s\le t\le 3\cdot 2^{j+1}s} \rho(0,t)\,\|f\|_{L^1} \lesssim \int_{2^jK_1 s}^{3\cdot 2^j K_2 s} \frac{\rho(0,t)}{t}\,dt\,\|f\|_{L^1},\end{aligned}$$

since $s \ge 1$. Take the integer j_0 such that $r \le 2^{j_0+2}s < 2r$. Then, by (2.1.12),

$$\begin{aligned}\left(\fint_{B_r} |(I_\rho f)(1-\chi_{B_{2s}})|^\nu\right)^{1/\nu} &\le \frac{1}{r^{n/\nu}} \sum_{j=0}^{j_0} \left(\int_{2^{j+2}B_s\setminus 2^{j+1}B_s} |I_\rho f|^\nu\right)^{1/\nu} \\ &\lesssim \frac{1}{r^{n/\nu}} \int_0^{3K_2 r/2} \frac{\rho(0,t)}{t}\,dt\,\|f\|_{L^1} \lesssim \frac{1}{r^{n/\nu}}\varphi(0,r)^{1/\tilde{p}-1/p}\,\|f\|_{L^1}.\end{aligned} \tag{2.6.15}$$

By (2.6.14) and (2.6.15) we have

$$\begin{aligned}
&\left|\fint_{B_r}(b-b_{B_{2s}})(I_\rho f)(1-\chi_{B_{2s}})\right| \\
&\lesssim \varphi(0,r)^{1/q-1/\tilde{p}}\log r\frac{1}{r^{n/\nu}}\varphi(0,r)^{1/\tilde{p}-1/p}\,\|b\|_{\mathcal{L}^{(1,\psi)}}\|f\|_{L^1} \\
&= \frac{\log r}{r^{n(1+1/q-1/p-\epsilon)}}\left(\frac{1}{r^n\varphi(0,r)}\right)^{1/p-1/q}\|b\|_{\mathcal{L}^{(1,\psi)}}\|f\|_{L^1} \\
&\to 0 \quad \text{as } r\to\infty,
\end{aligned}$$

since $r^n\varphi(0,r)$ is almost increasing. Therefore, we have (2.6.10) and (2.6.11) for all $f\in L^{(p,\varphi)}(\mathbb{R}^n)$ with compact support.

Case 2: For general $f\in L^{(p,\varphi)}(\mathbb{R}^n)$, using Case 1, we have

$$\|[b,I_\rho](f\chi_{B_{2r}})\|_{L^{(q,\varphi)}}\lesssim\|b\|_{\mathcal{L}^{(1,\psi)}}\|f\chi_{B_{2r}}\|_{L^{(p,\varphi)}}\le\|b\|_{\mathcal{L}^{(1,\psi)}}\|f\|_{L^{(p,\varphi)}}.$$

Then

$$\fint_{B_r}[b,I_\rho](f\chi_{B_{2r}})\le\varphi(0,r)^{1/q}\|[b,I_\rho](f\chi_{B_{2r}})\|_{L^{(q,\varphi)}}\lesssim\varphi(0,r)^{1/q}\|b\|_{\mathcal{L}^{(1,\psi)}}\|f\|_{L^{(p,\varphi)}}.$$

Combining this and Lemma 2.3.13, we have

$$\fint_{B_r}[b,I_\rho]f\lesssim\varphi(0,r)^{1/q}\|b\|_{\mathcal{L}^{(1,\psi)}}\|f\|_{L^{(p,\varphi)}},$$

which implies (2.6.10). Therefore, we have (2.6.11) for all $f\in L^{(p,\varphi)}(\mathbb{R}^n)$. The proof is complete. □

Proof of Theorem 2.1.2 (ii). In a similar way to the proof of Theorem 2.1.1 (ii), we can conclude that $\|b\|_{\mathcal{L}^{(1,\psi)}}\lesssim\|[b,I_\alpha]\|_{L^{(p,\varphi)}\to L^{(q,\varphi)}}$, by calculating $|z|^{n-\alpha}$ instead of $1/K(z)$. □

Proof of Theorem 2.1.3. By Lemma 2.3.5 we have that, for every $b\in\mathcal{L}^{(p_0,\psi)}(\mathbb{R}^n)$, $b_{B(0,r)}$ converges as $r\to\infty$ and $\|b-\lim_{r\to\infty}b_{B(0,r)}\|_{L^{(p_0,\psi)}}\sim\|b\|_{\mathcal{L}^{(p_0,\psi)}}$. Let $b_0=b-\lim_{r\to\infty}b_{B(0,r)}$. Then $\|b_0\|_{L^{(p_0,\psi)}}\sim\|b\|_{\mathcal{L}^{(p_0,\psi)}}$ and $[b,T]f=b_0Tf-T(b_0f)$. Since φ and ψ satisfy (2.1.4), by Lemma 2.3.4 we have

$$\begin{aligned}
\int_r^\infty\frac{\theta(x,t)}{t}\,dt &= \int_r^\infty\frac{\psi(x,t)^{q/p_0}\varphi(x,t)^{q/p}}{t^{q/p_0}t^{q/p}}\,dt \\
&\le\left(\int_r^\infty\frac{\psi(x,t)}{t}\,dt\right)^{q/p_0}\left(\int_r^\infty\frac{\varphi(x,t)}{t}\,dt\right)^{q/p} \\
&\lesssim\psi(x,r)^{q/p_0}\varphi(x,r)^{q/p}=\theta(x,r).
\end{aligned}$$

Hence T is bounded on $L^{(p,\varphi)}(\mathbb{R}^n)$ and on $L^{(q,\theta)}(\mathbb{R}^n)$. By these boundedness and Lemma 2.3.6 we have

$$\begin{aligned}\|[b,T]f\|_{L^{(q,\theta)}} &\le \|b_0 Tf\|_{L^{(q,\theta)}} + \|T(b_0 f)\|_{L^{(q,\theta)}} \\ &\lesssim \|b_0\|_{L^{(p_0,\psi)}} \|Tf\|_{L^{(p,\varphi)}} + \|b_0 f\|_{L^{(q,\theta)}} \\ &\lesssim \|b_0\|_{L^{(p_0,\psi)}} \|f\|_{L^{(p,\varphi)}} \sim \|b\|_{\mathcal{L}^{(p_0,\psi)}} \|f\|_{L^{(p,\varphi)}}.\end{aligned}$$

This is the conclusion. □

Proof of Theorem 2.1.4. We use the same method as the proof of Theorem 2.1.3. For $b \in \mathcal{L}^{(p_0,\varphi)}(\mathbb{R}^n)$, let $b_0 = b - \lim_{r\to\infty} b_{B(0,r)}$. Then $\|b_0\|_{L^{(p_0,\varphi)}} \sim \|b\|_{\mathcal{L}^{(p_0,\varphi)}}$ and $[b,I_\rho]f = b_0 I_\rho f - I_\rho(b_0 f)$. Let $1/p + 1/p_0 = 1/q_0$. Then $\varphi^{1/p}\varphi^{1/p_0} = \varphi^{1/q_0}$ and $\varphi^{1/\tilde{p}}\varphi^{1/p_0} = \varphi^{1/q}$. Since φ^{1/p_0} is almost decreasing, from (2.1.12) it follows that

$$\begin{aligned}&\int_0^r \frac{\rho(x,t)}{t}\,dt\,\varphi(x,r)^{1/q_0} + \int_r^\infty \frac{\rho(x,t)\varphi(x,t)^{1/q_0}}{t}\,dt \\ &\quad\lesssim \left(\int_0^r \frac{\rho(x,t)}{t}\,dt\,\varphi(x,r)^{1/p} + \int_r^\infty \frac{\rho(x,t)\varphi(x,t)^{1/p}}{t}\,dt\right)\varphi(x,r)^{1/p_0} \lesssim \varphi(x,t)^{1/q}.\end{aligned}$$

Hence I_ρ is bounded from $L^{(p,\varphi)}(\mathbb{R}^n)$ to $L^{(\tilde{p},\varphi)}(\mathbb{R}^n)$ and from $L^{(q_0,\varphi)}(\mathbb{R}^n)$ to $L^{(q,\varphi)}(\mathbb{R}^n)$. By these boundedness and Lemma 2.3.6 we have

$$\begin{aligned}\|[b,I_\rho]f\|_{L^{(q,\varphi)}} &\le \|b_0 I_\rho f\|_{L^{(q,\varphi)}} + \|I_\rho(b_0 f)\|_{L^{(q,\varphi)}} \\ &\lesssim \|b_0\|_{L^{(p_0,\varphi)}} \|I_\rho f\|_{L^{(\tilde{p},\varphi)}} + \|b_0 f\|_{L^{(q_0,\varphi)}} \\ &\lesssim \|b_0\|_{L^{(p_0,\varphi)}} \|f\|_{L^{(p,\varphi)}} \sim \|b\|_{\mathcal{L}^{(p_0,\varphi)}} \|f\|_{L^{(p,\varphi)}}.\end{aligned}$$

This is the conclusion. □

Chapter 3

Compactness – Sufficiency

3.1 Theorems

First, we state our main results in this chapter. We denote by $\overline{C^\infty_{\rm comp}(\mathbb{R}^n)}^{\mathcal{L}^{(1,\psi)}(\mathbb{R}^n)}$ the closure of $C^\infty_{\rm comp}(\mathbb{R}^n)$ with respect to $\mathcal{L}^{(1,\psi)}(\mathbb{R}^n)$. If $\psi \equiv 1$, then $\mathcal{L}^{(1,\psi)}(\mathbb{R}^n) = \mathrm{BMO}$ and $\overline{C^\infty_{\rm comp}(\mathbb{R}^n)}^{\mathrm{BMO}(\mathbb{R}^n)} = \mathrm{CMO}(\mathbb{R}^n)$.

For the compactness of the commutators $[b, T]$ and $[b, I_\rho]$, we consider the following condition on ψ: There exists a positive constant C such that, for all $x \in \mathbb{R}^n$ and $r \in (0, \infty)$,

$$\int_r^\infty \frac{\psi(x,t)}{t^2}\,dt \le C\frac{\psi(x,r)}{r}. \tag{3.1.1}$$

Then our main results are the following:

Theorem 3.1.1. *Let $1 < p \le q < \infty$ and $\varphi, \psi : \mathbb{R}^n \times (0,\infty) \to (0,\infty)$. Assume the same condition as Theorem 2.1.1. Assume also that, for all $f \in C^\infty_{\rm comp}(\mathbb{R}^n)$,*

$$Tf(x) = \lim_{\epsilon\to+0}\int_{|x-y|\ge\epsilon} K(x,y)f(y)\,dy, \quad a.e.\ x \in \mathbb{R}^n, \tag{3.1.2}$$

and that φ and ψ satisfy (1.2.7) and (3.1.1), respectively. If $b \in \overline{C^\infty_{\rm comp}(\mathbb{R}^n)}^{\mathcal{L}^{(1,\psi)}(\mathbb{R}^n)}$, then the commutator $[b, T]$ is compact from $L^{(p,\varphi)}(\mathbb{R}^n)$ to $L^{(q,\varphi)}(\mathbb{R}^n)$.

Observe that the Hilbert transform ($n = 1$, $K(x,y) = (x-y)/|x-y|^2$) and the Riesz transforms ($n \ge 2$, $K(x,y) = (x_j - y_j)/|x-y|^{n+1}$, $j = 1,\dots,n$) are Calderón-Zygmund operators satisfying (3.1.2).

Remark 3.1.1. It is known by [69] that, if T is a Calderón-Zygmund operator of type $\omega \in \Omega$, then the truncated maximal operator T_* of T is bounded from $L^p(\mathbb{R}^n)$ to itself and $L^1(\mathbb{R}^n)$ to $\mathrm{w}L^1(\mathbb{R}^n)$ (weak-L^1 space), where

$$T_*f(x) = \sup_{\epsilon>0} \left| \int_{|x-y|>\epsilon} K(x,y)f(y)\,dy \right|.$$

Consequently, (3.1.2) holds for all $f \in L^p(\mathbb{R}^n)$, $1 \le p < \infty$. Moreover, by Remark 2.3.2, we see that the equality

$$[b,T]f(x) = \lim_{\epsilon\to+0} \int_{|x-y|>\epsilon} (b(x)-b(y))K(x,y)f(y)\,dy \quad \text{a.e. } x \in \mathbb{R}^n \tag{3.1.3}$$

holds for all $f \in L^{(p,\varphi)}(\mathbb{R}^n)$ under the assumption of Theorem 3.1.1.

Remark 3.1.2. It is known that a Calderón-Zygmund operator is equal to a Calderón-Zygmund singular integral operator plus a bounded function times the identity operator, see Grafakos [22, p. 221]. A Calderón-Zygmund operator satisfying (3.1.2) is one of Calderón-Zygmund singular integral operators.

Theorem 3.1.2. *Let $1 < p < q < \infty$ and $\rho, \varphi, \psi : \mathbb{R}^n \times (0,\infty) \to (0,\infty)$. Assume the same condition as Theorem 2.1.2. Assume also that φ and ψ satisfy* (1.2.7) *and* (3.1.1)*, respectively. If $b \in \overline{C^\infty_{\mathrm{comp}}(\mathbb{R}^n)}^{\mathcal{L}^{(1,\psi)}(\mathbb{R}^n)}$, then the commutator $[b, I_\rho]$ is compact from $L^{(p,\varphi)}(\mathbb{R}^n)$ to $L^{(q,\varphi)}(\mathbb{R}^n)$.*

Remark 3.1.3. By Remark 2.3.3, we see that the equality

$$[b,I_\rho]f(x) = \lim_{\epsilon\to+0} \int_{|x-y|>\epsilon} (b(x)-b(y))\frac{\rho(x,|x-y|)}{|x-y|^n}f(y)\,dy \quad \text{a.e. } x \in \mathbb{R}^n \tag{3.1.4}$$

holds for all $f \in L^{(p,\varphi)}(\mathbb{R}^n)$ under the assumption of Theorem 3.1.2.

3.2 Musielak-Orlicz spaces

To prove the main results we recall Young functions and Musielak-Orlicz spaces. In this section we show the inclusion relation between generalized Morrey spaces with variable growth condition and Musielak-Orlicz spaces.

Let $\bar{\Phi}$ be the set of all functions $\Phi : [0,\infty] \to [0,\infty]$ such that

$$\lim_{t\to+0} \Phi(t) = \Phi(0) = 0 \quad \text{and} \quad \lim_{t\to\infty} \Phi(t) = \Phi(\infty) = \infty,$$

where the second statement means one of the following properties:

(i) $\Phi(t) \in [0,\infty)$ for all $t \in [0,\infty)$ and $\lim\limits_{t\to\infty} \Phi(t) = \infty$.

(ii) There exists $b \in (0,\infty)$ such that $\Phi(t) = \infty$ for all $t \in (b,\infty]$.

In what follows, if a function $\Phi : [0,\infty) \to [0,\infty)$ satisfies $\lim\limits_{t\to+0} \Phi(t) = \Phi(0) = 0$ and $\lim\limits_{t\to\infty} \Phi(t) = \infty$, then we always regards that $\Phi(\infty) = \infty$ and that $\Phi \in \bar{\mathit{\Phi}}$. Let

$$a(\Phi) = \sup\{t \geq 0 : \Phi(t) = 0\}, \quad b(\Phi) = \inf\{t \geq 0 : \Phi(t) = \infty\}.$$

Definition 3.2.1 (Young function)**.** A function $\Phi \in \bar{\mathit{\Phi}}$ is called a Young function (or sometimes also called an Orlicz function) if Φ is increasing on $[0,\infty]$ and convex on $[0, b(\Phi))$. Moreover, if $b(\Phi) < \infty$, then

$$\lim_{t\to b(\Phi)-0} \Phi(t) = \Phi(b(\Phi)) \ (\leq \infty).$$

Let $\mathit{\Phi}_Y$ be the set of all Young functions.

If $\Phi \in \mathit{\Phi}_Y$ satisfies $a(\Phi) = 0$ and $b(\Phi) = \infty$, then Φ is continuous on $[0,\infty)$ and bijective from $[0,\infty]$ to itself.

Next we recall the generalized inverse in the sense of O'Neil [56]. For a Young function Φ, let

$$\Phi^{-1}(u) = \begin{cases} \inf\{t \geq 0 : \Phi(t) > u\}, & u \in [0,\infty), \\ \infty, & u = \infty. \end{cases}$$

If Φ is bijective from $[0,\infty]$ to itself, then Φ^{-1} is the usual inverse function of Φ. We have the following property of the Young function Φ and its inverse ([56, Property 1.3]):

$$\Phi(\Phi^{-1}(t)) \leq t \leq \Phi^{-1}(\Phi(t)) \text{ for all } t \in [0,\infty]. \tag{3.2.1}$$

For a Young function Φ, its complementary function is defined by

$$\widetilde{\Phi}(t) = \begin{cases} \sup\{tu - \Phi(u) : u \in [0,\infty)\}, & t \in [0,\infty), \\ \infty, & t = \infty. \end{cases}$$

Then $\widetilde{\Phi}$ is also a Young function, and $(\Phi, \widetilde{\Phi})$ is called a complementary pair. For example, $\Phi(t) = t$, then

$$\widetilde{\Phi}(t) = \begin{cases} 0, & t \in [0,1], \\ \infty, & t \in (0,\infty]. \end{cases}$$

Definition 3.2.2. Let $\mathit{\Phi}_Y^v$ be the set of all $\Phi : \mathbb{R}^n \times [0,\infty] \to [0,\infty]$ such that $\Phi(x,\cdot)$ is a Young function for every $x \in \mathbb{R}^n$, and that $\Phi(\cdot,t)$ is measurable on $\mathbb{R}^n$ for every $t \in [0,\infty]$.

For $\Phi \in \mathit{\Phi}_Y^v$ and $x \in \mathbb{R}^n$, let

$$\Phi^{-1}(x,u) = \begin{cases} \inf\{t \ge 0 : \Phi(x,t) > u\}, & u \in [0,\infty), \\ \infty, & u = \infty. \end{cases}$$

We also define the complementary function $\widetilde{\Phi} : \mathbb{R}^n \times [0,\infty] \to [0,\infty]$ by

$$\widetilde{\Phi}(x,t) = \begin{cases} \sup\{tu - \Phi(x,u) : u \in [0,\infty)\}, & t \in [0,\infty), \\ \infty, & t = \infty. \end{cases}$$

Definition 3.2.3 (Musielak-Orlicz space)**.** For a function $\Phi \in \mathit{\Phi}_Y^v$, let

$$L^\Phi(\mathbb{R}^n) = \left\{ f : \int_{\mathbb{R}^n} \Phi(x, \varepsilon|f(x)|)\, dx < \infty \ \text{ for some } \ \varepsilon > 0 \right\},$$

$$\|f\|_{L^\Phi} = \inf\left\{ \lambda > 0 : \int_{\mathbb{R}^n} \Phi\left(x, \frac{|f(x)|}{\lambda}\right) dx \le 1 \right\}.$$

Then $\|\cdot\|_{L^\Phi}$ is a norm, which is called the Luxemburg-Nakano norm, and thereby $L^\Phi(\mathbb{R}^n)$ is a Banach space.

Let $(\Phi, \widetilde{\Phi})$ be a complementary pair of functions in $\mathit{\Phi}_Y^v$. Then it is known that

$$t \le \Phi^{-1}(x,t)\widetilde{\Phi}^{-1}(x,t) \le 2t, \quad t \in [0,\infty]. \tag{3.2.2}$$

It is also known that

$$\int_{\mathbb{R}^n} |f(x)g(x)|\, dx \le 2\|f\|_{L^\Phi}\|g\|_{L^{\tilde{\Phi}}}. \tag{3.2.3}$$

We first prove the following proposition:

Proposition 3.2.1. *Let $\varphi : \mathbb{R}^n \times (0,\infty) \to (0,\infty)$. Assume that φ is in $\mathcal{G}^{\rm dec}$ and satisfies* (1.2.7) *and* (1.2.9)*. Then there exists a Young function $\Phi_\varphi : \mathbb{R}^n \times [0,\infty] \to [0,\infty]$ and a positive constant C such that, for all balls $B = B(x,r)$,*

$$C^{-1}\varphi(B) \le \Phi_\varphi^{-1}(x, 1/|B|) \le C\varphi(B). \tag{3.2.4}$$

Moreover, there exist positive constants C' and C'' such that, for all balls B,

$$\|\chi_B\|_{L^{\Phi_\varphi}} \le C'\frac{1}{\varphi(B)}, \quad \|\chi_B\|_{L^{\tilde{\Phi}_\varphi}} \le C''|B|\varphi(B), \tag{3.2.5}$$

where $\widetilde{\Phi}_\varphi$ is the complementary function of Φ_φ.

To prove the above proposition we need the notion of pseudo-concavity. We say that a function $h : (0,\infty) \to (0,\infty)$ is pseudo-concave if there exist a concave function h_0 and a positive constant C such that, for all $u \in (0,\infty)$.

$$h(u) \le h_0(u) \le Ch(u).$$

Then the following characterization is known by Peetre [57]:

Lemma 3.2.2 ([57])**.** *Let $h : (0,\infty) \to (0,\infty)$. Then h is pseudo-concave if and only if there exists a positive constant C such that, for all $u, v \in (0,\infty)$,*

$$h(v) \le C \max(1, v/u) h(u). \tag{3.2.6}$$

Remark 3.2.1. If (3.2.6) holds for some constant C, then h_0 defined by

$$h_0(u) = \sup\left\{ \sum_i \alpha_i h(u_i) : \alpha_i \ge 0, \sum_i \alpha_i = 1, u = \sum_i \alpha_i u_i \text{ (finite sum)} \right\} \tag{3.2.7}$$

is concave and the relation

$$h(u) \le h_0(u) \le 2Ch(u)$$

holds for all $u \in (0,\infty)$, see [57]. Note that, if h is continuous, then

$$h_0(u) = \sup\left\{ \sum_i \alpha_i h(u_i) : \alpha_i, u_i/u \in \mathbb{Q}, \alpha_i \ge 0, \sum_i \alpha_i = 1, u = \sum_i \alpha_i u_i \text{ (finite sum)} \right\}.$$

Proof of Proposition 3.2.1. First note that we always assume that $\varphi(x,t)$ is measurable with respect to x and t. By Remark 1.2.2 we may assume that $\varphi(x,t)$ is continuous with respect to t for each x. Let $h_x(u) = h(x,u) = \varphi(x, u^{-1/n})$. First we show that h_x is pseudo-concave. Let $u, v \in (0,\infty)$. If $u > v$, then, by the almost decreasingness of φ, we have

$$h_x(v) = \varphi(x, v^{-1/n}) \lesssim \varphi(x, u^{-1/n}) = h_x(u).$$

If $u < v$, then, by the almost increasingness of $r \mapsto \varphi(x,r)r^n$, we have

$$h_x(v) = v\varphi(x, v^{-1/n})v^{-1} \lesssim v\varphi(x, u^{-1/n})u^{-1} = \frac{v}{u} h_x(u).$$

In the above two inequalities the implicit constants are independent of x. That is, there exists a positive constant C such that, for all $x \in \mathbb{R}^n$ and $u, v \in (0, \infty)$,

$$h_x(v) \le C \max(1, v/u) h_x(u).$$

Then by Lemma 3.2.2 and Remark 3.2.1 there exists a function $h_0(x,u)$ which is measurable with respect to x and concave with respect to u such that, for all x and u,

$$\varphi(x, u^{-1/n}) \le h_0(x,u) \le 2C\varphi(x, u^{-1/n}).$$

Moreover, by (1.2.9) and the concavity we see that

$$\lim_{u\to+0} h_0(x,u) = 0, \quad \lim_{u\to\infty} h_0(x,u) = \infty$$

and that $h_0(x,\cdot)$ is strictly increasing and bijective from $(0,\infty)$ to itself. Let

$$\Phi_\varphi(x,t) = \begin{cases} 0, & t = 0, \\ h_0^{-1}(x,t), & t \in (0,\infty), \\ \infty, & t = \infty, \end{cases}$$

where h_0^{-1} is the inverse function with respect to t for each x. Then $\Phi_\varphi \in \mathit{\Phi}_Y^v$ and satisfies

$$\varphi(x, t^{-1/n}) \le \Phi_\varphi^{-1}(x,t) \le 2C\varphi(x, t^{-1/n}), \quad t \in (0,\infty).$$

This shows (3.2.4). In this case $\Phi_\varphi(x,\cdot)$ is bijective from $[0,\infty]$ to itself for every $x \in \mathbb{R}^n$.

Next we show (3.2.5). Let $B = B(x,r)$. Since φ satisfies (1.2.7), we have that, for $y \in B$, $\varphi(x,r) \sim \varphi(y,r) \lesssim \Phi_\varphi^{-1}(y, 1/|B|)$, that is, $\varphi(B)/C' \le \Phi_\varphi^{-1}(y,1/|B|)$ for some positive constant C'. Then

$$\int_{\mathbb{R}^n} \Phi_\varphi\left(y, \frac{\chi_B(y)}{C'/\varphi(B)}\right) dy \le \int_B \Phi_\varphi(y, \Phi_\varphi^{-1}(y, 1/|B|))\, dy = 1.$$

This shows that $\|\chi_B\|_{L^{\Phi_\varphi}} \le C'/\varphi(B)$. Similarly, from (3.2.2) it follows that

$$\frac{1}{\widetilde{\Phi}_\varphi^{-1}(y,1/|B|)} \le |B|\Phi_\varphi^{-1}(y,1/|B|) \lesssim |B|\varphi(y,r) \lesssim |B|\varphi(x,r),$$

that is, $1/(C''|B|\varphi(B)) \le \widetilde{\Phi}_\varphi^{-1}(y,1/|B|)$ for some positive constant C''. Then

$$\int_{\mathbb{R}^n} \widetilde{\Phi}_\varphi\left(y, \frac{\chi_B(y)}{C''|B|\varphi(B)}\right) dy \le \int_B \widetilde{\Phi}_\varphi(y, \widetilde{\Phi}_\varphi^{-1}(y,1/|B|))\, dy \le 1,$$

where we use (3.2.1) at the last inequality. This shows that $\|\chi_B\|_{L^{\tilde{\Phi}_\varphi}} \le C''|B|\varphi(B)$. The proof is complete. □

Now we show the following inclusion relation:

Proposition 3.2.3. *Let $1 \le q < \infty$ and $\varphi : \mathbb{R}^n \times (0,\infty) \to (0,\infty)$. Assume that φ is in $\mathcal{G}^{\mathrm{dec}}$ and satisfies (1.2.7) and (1.2.9). Then there exists a Young function $\Phi_{q,\varphi}$ such that*

$$L^{\Phi_{q,\varphi}}(\mathbb{R}^n) \subset L^{(q,\varphi)}(\mathbb{R}^n) \quad \textit{and} \quad \|f\|_{L^{(q,\varphi)}} \le C\|f\|_{L^{\Phi_{q,\varphi}}},$$

where C is a positive constant independent of $f \in L^{\Phi_{q,\varphi}}(\mathbb{R}^n)$.

Proof. For φ, take a Young function Φ_φ as in Proposition 3.2.1, and set $\Phi_{q,\varphi}(x,t) = \Phi_\varphi(x,t^q)$. Then $\Phi_{q,\varphi}$ is also a Young function. By generalized Hölder's inequality (3.2.3) and Proposition 3.2.1 we have that, for all balls B,

$$\frac{1}{\varphi(B)} \fint_B |f|^q \le \frac{2}{\varphi(B)|B|} \||f|^q\|_{L^{\Phi_\varphi}} \|\chi_B\|_{L^{\tilde{\Phi}_\varphi}} \lesssim \||f|^q\|_{L^{\Phi_\varphi}} = \|f\|^q_{L^{\Phi_{q,\varphi}}},$$

where $\widetilde{\Phi}_\varphi$ is the complementary function of Φ_φ. This shows the conclusion. □

Remark 3.2.2. By Proposition 3.2.1 we see that, for all balls B,

$$\|\chi_B\|_{L^{\Phi_{q,\varphi}}} \lesssim \frac{1}{\varphi(B)^{1/q}}.$$

3.3 Compactness criterion on generalized Morrey spaces

We consider the integral operator

$$T_0 f(x) = \int_{\mathbb{R}^n} K_0(x,y) f(y)\, dy, \quad x \in \mathbb{R}^n, \tag{3.3.1}$$

for a kernel function $K_0 : \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{C}$. In this section we prove the following proposition:

Proposition 3.3.1. *Let $1 < p \le q < \infty$ and $\varphi : \mathbb{R}^n \times (0,\infty) \to (0,\infty)$. Assume that φ is in $\mathcal{G}^{\mathrm{dec}}$ and satisfies (1.2.7) and (1.2.9). If $K_0 \in L^\infty_{\mathrm{comp}}(\mathbb{R}^n \times \mathbb{R}^n)$. then T_0 defined by (3.3.1) is a compact operator from $L^{(p,\varphi)}(\mathbb{R}^n)$ to $L^{(q,\varphi)}(\mathbb{R}^n)$.*

To prove Proposition 3.3.1 we use Proposition 3.2.3 and the following lemma whose proof method is known, see for example [30] or [61]. We give the proof for readers' convenience.

Lemma 3.3.2. *Let $\Phi \in \mathit{\Phi}_Y^v$ and $p, p' \in (1,\infty)$ with $1/p + 1/p'$. If*

$$\|K_0\|_{L^\Phi(\mathbb{R}^n;L^{p'}(\mathbb{R}^n))} := \left\| \left(\int_{\mathbb{R}^n} |K_0(\cdot,y)|^{p'}\,dy \right)^{1/p'} \right\|_{L^\Phi} < \infty,$$

then T_0 is compact from $L^p(\mathbb{R}^n)$ to $L^\Phi(\mathbb{R}^n)$ and

$$\|T_0\|_{L^p \to L^\Phi} \le \|K_0\|_{L^\Phi(\mathbb{R}^n;L^{p'}(\mathbb{R}^n))}, \tag{3.3.2}$$

where $\|\cdot\|_{L^p\to L^\Phi}$ is the operator norm from $L^p(\mathbb{R}^n)$ to $L^\Phi(\mathbb{R}^n)$.

Proof. By Hölder's inequality we have

$$|T_0 f(x)| \le \int_{\mathbb{R}^n} |K_0(x,y)||f(y)|\,dy \le \left(\int_{\mathbb{R}^n} |K_0(x,y)|^{p'}\,dy \right)^{1/p'} \|f\|_{L^p}.$$

Then

$$\|T_0 f\|_{L^\Phi} \le \left\| \left(\int_{\mathbb{R}^n} |K_0(\cdot,y)|^{p'}\,dy \right)^{1/p'} \right\|_{L^\Phi} \|f\|_{L^p}.$$

This shows (3.3.2). Next we show the compactness. For any $\epsilon > 0$, there exist a finite number of bounded measurable sets $E_1, E_2, \dots, E_k, F_1, F_2, \dots, F_k$ and $z_1, z_2, \dots, z_k \in \mathbb{C}$ such that

$$\|K_0 - K_{0,\epsilon}\|_{L^\Phi(\mathbb{R}^n;L^{p'}(\mathbb{R}^n))} < \epsilon, \quad K_{0,\epsilon}(x,y) = \sum_{j=1}^k z_j \chi_{E_j}(x)\chi_{F_j}(y).$$

This shows that T_0 can be approximated by a finite rank operator $T_{0,\epsilon}$ whose kernel is $K_{0,\epsilon}$. Therefore, T_0 is compact. □

Proof of Propositon 3.3.1. For q and φ, take a Young function $\Phi_{q,\varphi}$ as in Proposition 3.2.3. Then we see that $\|K_0\|_{L^{\Phi_{q,\varphi}}(\mathbb{R}^n;L^{p'}(\mathbb{R}^n))} < \infty$ by Remark 3.2.2. Let B_0 be a ball in $\mathbb{R}^n$ such that $\operatorname{supp} K_0 \subset B_0 \times B_0$. Then $T_0 : L^{(p,\varphi)}(\mathbb{R}^n) \to L^{(q,\varphi)}(\mathbb{R}^n)$ can be factorized as

$$T_0 : L^{(p,\varphi)}(\mathbb{R}^n) \xrightarrow{T_1} L^p(\mathbb{R}^n) \xrightarrow{T_2} L^{\Phi_{q,\varphi}}(\mathbb{R}^n) \xrightarrow{T_3} L^{(q,\varphi)}(\mathbb{R}^n),$$

where

$$T_1 : f \mapsto \chi_{B_0} f, \quad T_2 : f \mapsto Tf, \quad T_3 : f \mapsto \chi_{B_0} f,$$

since

$$T_0 f(x) = \chi_{B_0}(x) \int_{\mathbb{R}^n} K(x,y) \chi_{B_0}(y) f(y)\, dy, \quad x \in \mathbb{R}^n.$$

The operator T_1 is clearly bounded and T_2 is compact by Lemma 3.3.2. The operator T_3 is also bounded by Proposition 3.2.3. Thus $T_0 = T_3 T_2 T_1$ is compact. □

3.4 Lemmas

We first recall the definition of generalized fractional maximal operators and a theorem. For a function $\rho : \mathbb{R}^n \times (0,\infty) \to (0,\infty)$, let

$$M_\rho f(x) = \sup_{B \ni x} \rho(B) \fint_B |f(y)|\, dy,$$

where the supremum is taken over all balls B containing x. If $\rho(B) = |B|^{\alpha/n}$, then M_ρ is the usual fractional maximal operator M_α If $\rho \equiv 1$, then M_ρ is the Hardy-Littlewood maximal operator M, that is,

$$Mf(x) = \sup_{B \ni x} \fint_B |f(y)|\, dy.$$

Then the following boundedness of M_ρ is proven in Chapter 2.

Theorem 3.4.1 (Theorem 2.2.3)**.** *Let $1 < p < q < \infty$ and $\rho, \varphi : \mathbb{R}^n \times (0,\infty) \to (0,\infty)$. Assume that φ is in $\mathcal{G}^{\rm dec}$ and satisfies* (1.2.9)*. Assume also that there exists a positive constant C_0 such that, for all $x \in \mathbb{R}^n$ and $r \in (0,\infty)$,*

$$\rho(x,r)\varphi(x,r)^{1/p} \le C_0 \varphi(x,r)^{1/q}.$$

Then M_ρ is bounded from $L^{(p,\varphi)}(\mathbb{R}^n)$ to $L^{(q,\varphi)}(\mathbb{R}^n)$.

We also use the following lemmas:

Lemma 3.4.2 ([38, Lemma 2], [45, Lemma 7.1])**.** *Let φ satisfy the doubling condition* (1.2.5) *and* (2.1.4)*, that is,*

$$\int_r^\infty \frac{\varphi(x,t)}{t}\, dt \le C\, \varphi(x,r).$$

Then there exists positive constants ϵ and C_ϵ such that, for all $x \in \mathbb{R}^n$ and $r \in (0,\infty)$,

$$\int_r^\infty \frac{\varphi(x,t)t^\epsilon}{t}\,dt \le C_\epsilon \varphi(x,r) r^\epsilon.$$

Moreover, for all $p \in (0,\infty)$, there exists a positive constant C_p such that, for all $x \in \mathbb{R}^n$ and $r > 0$,

$$\int_r^\infty \frac{\varphi(x,t)^{1/p}}{t}\,dt \le C_p\,\varphi(x,r)^{1/p}.$$

Remark 3.4.1. If φ is in $\mathcal{G}^{\mathrm{dec}}$ and satisfies (2.1.4), then φ satisfies (1.2.9). Actually, φ satisfies the doubling condition and the following inequalities hold:

$$\varphi(x,r) \lesssim \int_r^{2r} \frac{\varphi(x,t)}{t}\,dt \le \int_r^\infty \frac{\varphi(x,t)}{t}\,dt \lesssim \varphi(x,r).$$

Then we see that $\lim_{r\to+0} \varphi(x,r) = \infty$ and that $\lim_{r\to\infty} \varphi(x,r) = 0$.

Lemma 3.4.3. *If ψ satisfies* (3.1.1)*, then there exist constants $\theta \in (0,1)$ and $C \in [1,\infty)$ such that, for all $b \in C^\infty_{\mathrm{comp}}(\mathbb{R}^n)$ and all $x, y \in \mathbb{R}^n$ with $|x-y| < 1$,*

$$|b(x) - b(y)| \le C\|\nabla b\|_{L^\infty} |x-y|^\theta \psi(x, |x-y|).$$

Proof. By the assumption (3.1.1) and Lemma 3.4.2 we see that there exists a constant $\theta \in (0,1)$ such that

$$\int_r^\infty \frac{\psi(x,t)}{t^{2-\theta}}\,dt \le C_\theta \frac{\psi(x,r)}{r^{1-\theta}}.$$

On the other hand, by the almost increasingness of ψ,

$$\frac{\psi(x,r)}{r^{1-\theta}} \sim \psi(x,r)\int_r^{2r} \frac{1}{t^{2-\theta}}\,dt \lesssim \int_r^{2r} \frac{\psi(x,t)}{t^{2-\theta}}\,dt \le \int_r^\infty \frac{\psi(x,t)}{t^{2-\theta}}\,dt.$$

This shows that

$$\int_r^\infty \frac{\psi(x,t)}{t^{2-\theta}}\,dt \sim \frac{\psi(x,r)}{r^{1-\theta}}$$

and that $r \mapsto \frac{\psi(x,r)}{r^{1-\theta}}$ is almost decreasing, that is,

$$r^{1-\theta} \lesssim \psi(x,r) \quad \text{for} \quad r \in (0,1]. \tag{3.4.1}$$

Then, for $|x-y| < 1$,

$$|b(x) - b(y)| \le \|\nabla b\|_{L^\infty}|x-y| \lesssim \|\nabla b\|_{L^\infty}|x-y|^\theta \psi(x,|x-y|).$$ □

3.5 Proofs of the theorems

Now we prove Theorem 3.1.1. For $0 < \epsilon < R < \infty$, let

$$T_\epsilon f(x) = \int_{|x-y|\geq\epsilon} K(x,y)f(y)\,dy, \quad T_{\epsilon,R}f(x) = \int_{\epsilon\leq|x-y|<R} K(x,y)f(y)\,dy.$$

From Remark 3.1.1 and Remark 2.3.2, it follows that

$$[b,T]f(x) = \lim_{\epsilon\to+0}[b,T_\epsilon]f(x),\ [b,T_\epsilon]f(x) = \lim_{R\to\infty}[b,T_{\epsilon,R}]f(x) \quad \text{a.e. } x \in \mathbb{R}^n,$$

for all $f \in L^{(p,\varphi)}(\mathbb{R}^n)$. Since $T_{\epsilon,R}$ is compact by Proposition 3.3.1 and Remark 3.4.1, it is enough to show the following proposition to prove Theorem 3.1.1.

Proposition 3.5.1. *Under the assumption in Theorem 3.1.1, we have*

(i) $\lim_{\epsilon\to+0} \|[b,T_\epsilon] - [b,T]\|_{L^{(p,\varphi)}\to L^{(q,\varphi)}} = 0,$

(ii) $\lim_{R\to\infty} \|[b,T_{\epsilon,R}] - [b,T_\epsilon]\|_{L^{(p,\varphi)}\to L^{(q,\varphi)}} = 0,$

where $\|\cdot\|_{L^{(p,\varphi)}\to L^{(q,\varphi)}}$ *is the operator norm from* $L^{(p,\varphi)}(\mathbb{R}^n)$ *to* $L^{(q,\varphi)}(\mathbb{R}^n)$.

We first state a lemma.

Lemma 3.5.2. *Let* $\theta \in (0,1]$. *Assume that* ψ *satisfies the doubling condition* (1.2.5). *Then there exists a positive constant* C *such that, for all* $x \in \mathbb{R}^n$ *and* $\epsilon \in (0,1]$,

$$\int_{B(x,\epsilon)} \frac{\psi(x,|x-y|)|f(y)|}{|x-y|^{n-\theta}}\,dy \leq C\epsilon^\theta M_\psi f(x).$$

Proof. Since ψ satisfies the doubling condition, we have

$$\begin{aligned}
&\int_{B(x,\epsilon)} \frac{\psi(x,|x-y|)|f(y)|}{|x-y|^{n-\theta}}\,dy \\
&= \sum_{j=0}^{\infty} \int_{B(x,2^{-j}\epsilon)\setminus B(x,2^{-j-1}\epsilon)} \frac{\psi(x,|x-y|)|f(y)|}{|x-y|^{n-\theta}}\,dy \\
&\sim \sum_{j=0}^{\infty} \frac{\psi(x,2^{-j}\epsilon)}{(2^{-j-1}\epsilon)^{n-\theta}} \int_{B(x,2^{-j}\epsilon)\setminus B(x,2^{-j-1}\epsilon)} |f(y)|\,dy \\
&\lesssim \sum_{j=0}^{\infty} (2^{-j-1}\epsilon)^\theta M_\psi f(x) \sim \epsilon^\theta M_\psi f(x).
\end{aligned}$$

□

Proof of Proposition 3.5.1. (i) Let $f \in L^{(p,\varphi)}(\mathbb{R}^n)$ and $\epsilon \in (0,1]$. Then, from (3.1.3) it follows that

$$[b,T]f(x) - [b,T_\epsilon]f(x) = \lim_{\eta\to 0}\int_{\eta\le|x-y|<\epsilon} \frac{(b(x)-b(y))}{|x-y|^n} f(y)\,dy, \quad \text{a.e. } x.$$

By Lemmas 3.4.3 and 3.5.2 we have

$$\int_{B(x,\epsilon)} \frac{|b(x)-b(y)|}{|x-y|^n}|f(y)|\,dy \lesssim \int_{B(x,\epsilon)} \frac{\psi(x,|x-y|)}{|x-y|^{n-\theta}}|f(y)|\,dy \lesssim \epsilon^\theta M_\psi f(x),$$

for some $\theta \in (0,1)$. Hence, by Theorem 2.2.3 with the assumption (2.1.7) we have

$$\|[b,T]f - [b,T_\epsilon]f\|_{L^{(q,\varphi)}} \lesssim \epsilon^\theta \|M_\psi f\|_{L^{(q,\varphi)}} \lesssim \epsilon^\theta \|f\|_{L^{(p,\varphi)}}.$$

This shows (i).

(ii) Let $\operatorname{supp} b \subset B_0 = B(0,R_0)$. Then

$$\begin{aligned}
&|[b,T_\epsilon]f(x) - [b,T_{\epsilon,R}]f(x)| \\
&\le \int_{|x-y|>R} \frac{|b(x)-b(y)|}{|x-y|^n}|f(y)|\,dy \\
&\lesssim \int_{|x-y|>R} (\chi_{B_0}(x)+\chi_{B_0}(y)) \left(\int_{|x-y|}^\infty \frac{1}{t^{n+1}}\,dt\right)|f(y)|\,dy \\
&= \int_0^\infty \int_{\mathbb{R}^n} \chi_{\{R<|x-y|<t\}}(y,t)(\chi_{B_0}(x)+\chi_{B_0}(y))\frac{1}{t^{n+1}}|f(y)|\,dy\,dt \\
&\le \int_R^\infty \left(\int_{B(x,t)} (\chi_{B_0}(x)+\chi_{B_0}(y))|f(y)|\,dy\right)\frac{1}{t^{n+1}}\,dt.
\end{aligned}$$

Let

$$\begin{aligned}
E_1(x) &= \int_R^\infty \left(\int_{B(x,t)} \chi_{B_0}(x)|f(y)|\,dy\right)\frac{1}{t^{n+1}}\,dt, \\
E_2(x) &= \int_R^\infty \left(\int_{B(x,t)} \chi_{B_0}(y)|f(y)|\,dy\right)\frac{1}{t^{n+1}}\,dt.
\end{aligned}$$

Then

$$|[b,T_\epsilon]f(x) - [b,T_{\epsilon,R}]f(x)| \lesssim E_1(x)+E_2(x).$$

By the inequality

$$\int_{B(x,t)} |f(y)|\,dy \le |B(x,t)|\left(\fint_{B(x,t)} |f(y)|^p\,dy\right)^{1/p} \lesssim \varphi(x,t)^{1/p} t^n \|f\|_{L^{(p,\varphi)}},$$

Lemma 3.4.2 and (1.2.7) of φ we have that, for large R,

$$\begin{aligned}E_1(x) &\lesssim \chi_{B_0}(x) \int_R^\infty \frac{\varphi(x,t)^{1/p}}{t} \, dt \, \|f\|_{L^{(p,\varphi)}} \\ &\lesssim \chi_{B_0}(x)\varphi(x,R)^{1/p}\|f\|_{L^{(p,\varphi)}} \lesssim \chi_{B_0}(x)\varphi(0,R)^{1/p}\|f\|_{L^{(p,\varphi)}}.\end{aligned}$$

Then

$$\|E_1\|_{L^{(q,\varphi)}} \lesssim \|\chi_{B_0}\|_{L^{(q,\varphi)}}\varphi(0,R)^{1/p}\|f\|_{L^{(p,\varphi)}}.$$

Next we estimate $\|E_2\|_{L^{(q,\varphi)}}$. If $y \in B_0 \cap B(x,t)$ and t is large, then

$$\|\chi_{B(y,t)}\|_{L^{\Phi_{q,\varphi}}} = \|\chi_{B(y,t)}\|_{L^{\Phi_\varphi}}^{1/q} \lesssim \frac{1}{\varphi(y,t)^{1/q}} \lesssim \frac{1}{\varphi(0,t)^{1/q}},$$

where Φ_φ and $\Phi_{q,\varphi}$ are as in Propositions 3.2.1 and 3.2.3, respectively. Hence

$$\begin{aligned}\|E_2\|_{L^{(q,\varphi)}} &\lesssim \int_R^\infty \left(\int_{\mathbb{R}^n} \|\chi_{B(\cdot,t)}(y)\|_{L^{(q,\varphi)}} \chi_{B_0}(y)|f(y)| \, dy \right) \frac{1}{t^{n+1}} \, dt \\ &\lesssim \int_R^\infty \sup_{y\in B_0} \|\chi_{B(y,t)}\|_{L^{(q,\varphi)}} \left(\int_{\mathbb{R}^n} \chi_{B_0}(y)|f(y)| \, dy \right) \frac{1}{t^{n+1}} \, dt \\ &\lesssim \int_R^\infty \sup_{y\in B_0} \|\chi_{B(y,t)}\|_{L^{\Phi_{q,\varphi}}} \frac{1}{t^{n+1}} \, dt \left(\int_{B_0} |f(y)| \, dy \right) \\ &\lesssim \int_R^\infty \frac{1}{\varphi(0,t)^{1/q} t^{n+1}} \, dt \, \varphi(B_0)^{1/p}|B_0|\|f\|_{L^{(p,\varphi)}}.\end{aligned}$$

By the almost increasingness of $r \mapsto \varphi(0,r)r^n$ we have

$$\int_R^\infty \frac{1}{\varphi(0,t)^{1/q}t^{n+1}} \, dt \lesssim \frac{1}{(\varphi(0,R)R^n)^{1/q}} \int_R^\infty \frac{1}{t^{n-n/q+1}} \, dt \lesssim \frac{1}{(\varphi(0,R)R^n)^{1/q}R^{n-n/q}}.$$

Therefore,

$$\begin{aligned}&\|[b,T_\epsilon]f(x) - [b,T_{\epsilon,R}]f\|_{L^{(q,\varphi)}} \\ &\lesssim \left(\|\chi_{B_0}\|_{L^{(q,\varphi)}}\varphi(0,R)^{1/q} + \frac{\varphi(B_0)^{1/p}|B_0|}{(\varphi(0,R)R^n)^{1/q}R^{n-n/q}} \right) \|f\|_{L^{(p,\varphi)}}.\end{aligned}$$

Since $\varphi(0,R) \to 0$ and $\dfrac{1}{(\varphi(0,R)R^n)^{1/q}R^{n-n/q}} \to 0$ as $R \to \infty$, we have (ii). $\square$

Next we prove Theorem 3.1.2. For $0 < \epsilon < R < \infty$, let

$$\begin{aligned}I_{\rho,\epsilon}f(x) &= \int_{|x-y|\ge\epsilon} \frac{\rho(x,|x-y|)}{|x-y|^n} f(y) \, dy, \\ I_{\rho,\epsilon,R}f(x) &= \int_{\epsilon\le|x-y|<R} \frac{\rho(x,|x-y|)}{|x-y|^n} f(y) \, dy.\end{aligned}$$

From Remark 2.3.3, it follows that

$$[b, I_\rho]f(x) = \lim_{\epsilon\to+0}[b, I_{\rho,\epsilon}]f(x),\ [b, I_{\rho,\epsilon}]f(x) = \lim_{R\to\infty}[b, I_{\rho,\epsilon,R}]f(x) \quad \text{a.e. } x \in \mathbb{R}^n,$$

for all $f \in L^{(p,\varphi)}(\mathbb{R}^n)$. Since $I_{\rho,\epsilon,R}$ is compact by Proposition 3.3.1 and Remark 3.4.1, it is enough to show the following proposition to prove Theorem 3.1.2.

Proposition 3.5.3. *Under the assumption in Theorem 3.1.2, we have*

(i) $\lim_{\epsilon\to+0} \|[b, I_{\rho,\epsilon}] - [b, I_\rho]\|_{L^{(p,\varphi)}\to L^{(q,\varphi)}} = 0,$

(ii) $\lim_{R\to\infty} \|[b, I_{\rho,\epsilon,R}] - [b, I_{\rho,\epsilon}]\|_{L^{(p,\varphi)}\to L^{(q,\varphi)}} = 0,$

where $\|\cdot\|_{L^{(p,\varphi)}\to L^{(q,\varphi)}}$ *is the operator norm from* $L^{(p,\varphi)}(\mathbb{R}^n)$ *to* $L^{(q,\varphi)}(\mathbb{R}^n)$.

We need a lemma to prove the above proposition.

Lemma 3.5.4. *Let* $\theta \in (0, 1]$. *Assume that* ψ *and* ρ^* *satisfy the doubling condition* (1.2.5). *Then there exists a positive constant* C *such that, for all* $x \in \mathbb{R}^n$ *and* $\epsilon \in (0, 1]$,

$$\int_{B(x,\epsilon)} \frac{\psi(x, |x-y|)\rho^*(x, |x-y|)|f(y)|}{|x-y|^{n-\theta}}\, dy \le C\epsilon^\theta M_{\psi\rho^*}f(x).$$

Proof. Since ψ and ρ^* satisfies the doubling condition, we have

$$\begin{aligned}
&\int_{B(x,\epsilon)} \frac{\psi(x, |x-y|)\rho^*(x, |x-y|)|f(y)|}{|x-y|^{n-\theta}}\, dy \\
&= \sum_{j=0}^{\infty} \int_{B(x,2^{-j}\epsilon)\setminus B(x,2^{-j-1}\epsilon)} \frac{\psi(x, |x-y|)\rho^*(x, |x-y|)|f(y)|}{|x-y|^{n-\theta}}\, dy \\
&\sim \sum_{j=0}^{\infty} \frac{\psi(x, 2^{-j}\epsilon)\rho^*(x, 2^{-j}\epsilon)}{(2^{-j-1}\epsilon)^{n-\theta}} \int_{B(x,2^{-j}\epsilon)\setminus B(x,2^{-j-1}\epsilon)} |f(y)|\, dy \\
&\lesssim \sum_{j=0}^{\infty} (2^{-j-1}\epsilon)^\theta M_{\psi\rho^*}f(x) \sim \epsilon^\theta M_{\psi\rho^*}f(x).
\end{aligned}$$

□

Proof of Proposition 3.5.3. (i) Let $f \in L^{(p,\varphi)}(\mathbb{R}^n)$ and $\epsilon \in (0, 1]$. Then, from (3.1.4) it follows that

$$[b, I_\rho]f(x) - [b, I_{\rho,\epsilon}]f(x) = \lim_{\eta\to 0} \int_{\eta\le|x-y|<\epsilon} (b(x) - b(y)) \frac{\rho(x, |x-y|)}{|x-y|^n} f(y)\, dy, \quad \text{a.e. } x.$$

By Remark 2.2.1 and Lemmas 3.4.3 and 3.5.4 we have

$$\begin{aligned}
&\int_{B(x,\epsilon)} |b(x)-b(y)| \frac{\rho(x,|x-y|)}{|x-y|^n} |f(y)|\,dy \\
&\lesssim \int_{B(x,\epsilon)} \frac{\psi(x,|x-y|)\rho^*(x,|x-y|)}{|x-y|^{n-\theta}} |f(y)|\,dy \\
&\lesssim \epsilon^\theta M_{\psi\rho^*} f(x),
\end{aligned}$$

for some $\theta \in (0,1)$. Hence, by Theorem 2.2.3 with the assumptions (2.1.12) and (2.1.13) we have

$$\|[b,I_\rho]f - [b,I_{\rho,\epsilon}]f\|_{L^{(q,\varphi)}} \lesssim \epsilon^\theta \|M_{\psi\rho^*}f\|_{L^{(q,\varphi)}} \lesssim \epsilon^\theta \|f\|_{L^{(p,\varphi)}}.$$

This shows (i).

(ii) Let $\operatorname{supp} b \subset B_0 = B(0,R_0)$. Using the relation

$$\frac{\rho(x,r)}{r^n} \le \frac{C}{r^n} \int_{K_1 r}^{K_2 r} \frac{\rho(x,t)}{t}\,dt \sim \int_{K_1 r}^{K_2 r} \frac{\rho(x,t)}{t^{n+1}}\,dt \le \int_{K_1 r}^{\infty} \frac{\rho(x,t)}{t^{n+1}}\,dt,$$

we have

$$\begin{aligned}
&|[b,I_{\rho,\epsilon}]f(x) - [b,I_{\rho,\epsilon,R}]f(x)| \\
&\le \int_{|x-y|>R} |b(x)-b(y)| \frac{\rho(x,|x-y|)}{|x-y|^n} |f(y)|\,dy \\
&\lesssim \int_{|x-y|>R} (\chi_{B_0}(x) + \chi_{B_0}(y)) \left(\int_{K_1|x-y|}^{\infty} \frac{\rho(x,t)}{t^{n+1}}\,dt \right) |f(y)|\,dy \\
&= \int_0^\infty \int_{\mathbb{R}^n} \chi_{\{R<|x-y|<t/K_1\}}(y,t) (\chi_{B_0}(x) + \chi_{B_0}(y)) \frac{\rho(x,t)}{t^{n+1}} |f(y)|\,dy\,dt \\
&\le \int_{K_1 R}^{\infty} \left(\int_{B(x,t/K_1)} (\chi_{B_0}(x) + \chi_{B_0}(y)) |f(y)|\,dy \right) \frac{\rho(x,t)}{t^{n+1}}\,dt.
\end{aligned}$$

Let

$$\begin{aligned}
E_1(x) &= \int_{K_1 R}^{\infty} \left(\int_{B(x,t/K_1)} \chi_{B_0}(x) |f(y)|\,dy \right) \frac{\rho(x,t)}{t^{n+1}}\,dt, \\
E_2(x) &= \int_{K_1 R}^{\infty} \left(\int_{B(x,t/K_1)} \chi_{B_0}(y) |f(y)|\,dy \right) \frac{\rho(x,t)}{t^{n+1}}\,dt.
\end{aligned}$$

Then

$$|[b,I_{\rho,\epsilon}]f(x) - [b,I_{\rho,\epsilon,R}]f(x)| \lesssim E_1(x) + E_2(x).$$

By the inequality

$$\int_{B(x,t/K_1)} |f(y)|\,dy \le |B(x,t/K_1)| \left(\fint_{B(x,t/K_1)} |f(y)|^p\,dy \right)^{1/p} \lesssim \varphi(x,t)^{1/p} t^n \|f\|_{L^{(p,\varphi)}}$$

and the assumptions (2.1.12) and (1.2.7) on φ we have that, for large R,

$$\begin{aligned} E_1(x) &\lesssim \chi_{B_0}(x) \int_{K_1 R}^{\infty} \frac{\rho(x,t)\varphi(x,t)^{1/p}}{t}\,dt\, \|f\|_{L^{(p,\varphi)}} \\ &\lesssim \chi_{B_0}(x)\varphi(x,R)^{1/q}\|f\|_{L^{(p,\varphi)}} \lesssim \chi_{B_0}(x)\varphi(0,R)^{1/q}\|f\|_{L^{(p,\varphi)}}. \end{aligned}$$

Then

$$\|E_1\|_{L^{(q,\varphi)}} \lesssim \|\chi_{B_0}\|_{L^{(q,\varphi)}} \varphi(0,R)^{1/q} \|f\|_{L^{(p,\varphi)}}.$$

Next we estimate $\|E_2\|_{L^{(q,\varphi)}}$. If $y \in B_0 \cap B(x,t/K_1)$ and $R < t$ is large, then $\rho(x,t) \sim \rho(y,t) \sim \rho(0,t)$ and

$$\|\chi_{B(y,t/K_1)}\|_{L^{\Phi_{q,\varphi}}} = \|\chi_{B(y,t/K_1)}\|_{L^{\Phi_\varphi}}^{1/q} \lesssim \frac{1}{\varphi(y,t/K_1)^{1/q}} \lesssim \frac{1}{\varphi(0,t)^{1/q}},$$

where Φ_φ and $\Phi_{q,\varphi}$ are as in Propositions 3.2.1 and 3.2.3, respectively. From (1.2.4), (2.1.12) and (2.1.13) it follows that

$$\rho(0,r)\varphi(0,r)^{1/p} \lesssim \int_{K_1 r}^{K_2 r} \frac{\rho(0,t)\varphi(0,t)^{1/p}}{t}\,dt \lesssim \varphi(0,r)^{1/\tilde{p}} \lesssim \frac{\varphi(0,r)^{1/q}}{\psi(0,r)}, \quad r > 0,$$

which implies

$$\|\chi_{B(y,t/K_1)}\|_{L^{\Phi_{q,\varphi}}} \lesssim \frac{1}{\varphi(0,t)^{1/q}} \lesssim \frac{1}{\rho(0,t)\varphi(0,t)^{1/p}\psi(0,t)}.$$

Hence

$$\begin{aligned} \|E_2\|_{L^{(q,\varphi)}} &\lesssim \int_{K_1 R}^{\infty} \left(\int_{\mathbb{R}^n} \|\chi_{B(\cdot,t/K_1)}(y)\|_{L^{(q,\varphi)}} \chi_{B_0}(y)|f(y)|\,dy \right) \frac{\rho(0,t)}{t^{n+1}}\,dt \\ &\lesssim \int_{K_1 R}^{\infty} \sup_{y\in B_0} \|\chi_{B(y,t/K_1)}\|_{L^{(q,\varphi)}} \left(\int_{\mathbb{R}^n} \chi_{B_0}(y)|f(y)|\,dy \right) \frac{\rho(0,t)}{t^{n+1}}\,dt \\ &\lesssim \int_{K_1 R}^{\infty} \sup_{y\in B_0} \|\chi_{B(y,t/K_1)}\|_{L^{\Phi_{q,\varphi}}} \frac{\rho(0,t)}{t^{n+1}}\,dt \left(\int_{B_0} |f(y)|\,dy \right) \\ &\lesssim \int_{K_1 R}^{\infty} \frac{1}{\varphi(0,t)^{1/p}\psi(0,t)t^{n+1}}\,dt\, \varphi(B_0)^{1/p}|B_0|\|f\|_{L^{(p,\varphi)}}. \end{aligned}$$

Since $r \mapsto \varphi(0,r)r^n$ and ψ are almost increasing,

$$\begin{aligned}
&\int_{K_1R}^{\infty} \frac{1}{\varphi(0,t)^{1/p}\psi(0,t)t^{n+1}}\,dt \\
&\lesssim \frac{1}{(\varphi(0,R)R^n)^{1/p}\psi(0,R)} \int_{K_1R}^{\infty} \frac{1}{t^{n-n/p+1}}\,dt \\
&\lesssim \frac{1}{(\varphi(0,R)R^n)^{1/p}\psi(0,R)R^{n-n/p}}.
\end{aligned}$$

Therefore,

$$\begin{aligned}
&\|[b,I_{\rho,\epsilon}]f(x) - [b,I_{\rho,\epsilon,R}]f\|_{L^{(q,\varphi)}} \\
&\lesssim \left(\|\chi_{B_0}\|_{L^{(q,\varphi)}}\varphi(0,R)^{1/q} + \frac{\varphi(B_0)^{1/p}|B_0|}{(\varphi(0,R)R^n)^{1/p}\psi(0,R)R^{n-n/p}} \right) \|f\|_{L^{(p,\varphi)}}.
\end{aligned}$$

Since $\varphi(0,R) \to 0$ and $\dfrac{1}{(\varphi(0,R)R^n)^{1/p}\psi(0,R)R^{n-n/p}} \to 0$ as $R \to \infty$, we have (ii). The proof is complete. □

Chapter 4

A generalizetion of the characterization of CMO

4.1 Theorem and Corollaries

For a function $f \in L^1_{\mathrm{loc}}(\mathbb{R}^n)$ and a ball $B \subset \mathbb{R}^n$, we denote by $\mathrm{MO}(f, B)$ the mean oscillation of f on B, that is,

$$\mathrm{MO}(f,B) = \fint_B |f(y) - f_B|\,dy. \tag{4.1.1}$$

Then our main results in this chapter are the following:

Theorem 4.1.1. *Let ϕ be in $\mathcal{G}^{\mathrm{inc}}$ and satisfy* (1.2.7)*. Assume that*

$$\lim_{r\to+0} \inf_{x\in\mathbb{R}^n} \frac{\phi(x,r)}{r} = \infty, \quad \lim_{r\to\infty} \inf_{x\in\mathbb{R}^n} r^n \phi(x,r) = \infty. \tag{4.1.2}$$

Let $f \in \mathcal{L}_{1,\phi}(\mathbb{R}^n)$. Then $f \in \overline{C^\infty_{\mathrm{comp}}(\mathbb{R}^n)}^{\mathcal{L}_{1,\phi}(\mathbb{R}^n)}$ if and only if f satisfies the following three conditions:

(i) $\displaystyle \lim_{r\to+0} \sup_{x\in\mathbb{R}^n} \frac{\mathrm{MO}(f,B(x,r))}{\phi(x,r)} = 0.$

(ii) $\displaystyle \lim_{r\to\infty} \sup_{x\in\mathbb{R}^n} \frac{\mathrm{MO}(f,B(x,r))}{\phi(x,r)} = 0.$

(iii) $\displaystyle \lim_{|x|\to\infty} \frac{\mathrm{MO}(f,B(x,r))}{\phi(x,r)} = 0$ *for each $r > 0$.*

Remark 4.1.1. We do not need (4.1.2) to prove that, if f satisfies (i)–(iii), then $f \in \overline{C^\infty_{\mathrm{comp}}(\mathbb{R}^n)}^{\mathcal{L}_{1,\phi}(\mathbb{R}^n)}$. We do not need (1.2.7) to prove that, if $f \in \overline{C^\infty_{\mathrm{comp}}(\mathbb{R}^n)}^{\mathcal{L}_{1,\phi}(\mathbb{R}^n)}$, then f satisfies (i)–(iii).

If $\phi \equiv 1$, then the theorem above is the same as Theorem 1.1.3. If $\phi(x,r) \equiv r^\alpha$, then we have the following corollary.

Corollary 4.1.2 ([55]). *Let $f \in \mathrm{Lip}_\alpha(\mathbb{R}^n)$, $0 < \alpha < 1$. Then $f \in \overline{C^\infty_{\mathrm{comp}}(\mathbb{R}^n)}^{\mathrm{Lip}_\alpha(\mathbb{R}^n)}$ if and only if f satisfies the following three conditions:*

(i) $\displaystyle \lim_{r\to+0} \sup_{x\in\mathbb{R}^n} \frac{\mathrm{MO}(f,B(x,r))}{r^\alpha} = 0.$

(ii) $\displaystyle \lim_{r\to\infty} \sup_{x\in\mathbb{R}^n} \frac{\mathrm{MO}(f,B(x,r))}{r^\alpha} = 0.$

(iii) $\displaystyle \lim_{|x|\to\infty} \mathrm{MO}(f,B(x,r)) = 0$ *for each $r > 0$.*

As another corollary, we consider the Lipschitz (Hölder) space with variable exponent. For $\alpha(\cdot) : \mathbb{R}^n \to [0,\infty)$ and $\alpha_* \in [0,\infty)$, let $\mathrm{Lip}^{\alpha_*}_{\alpha(\cdot)}(\mathbb{R}^n)$ be the set of all functions f such that the following functional is finite:

$$\|f\|_{\mathrm{Lip}^{\alpha_*}_{\alpha(\cdot)}} = \max\left\{ \sup_{0<|x-y|<1} \frac{2|f(x)-f(y)|}{|x-y|^{\alpha(x)}+|x-y|^{\alpha(y)}}, \sup_{|x-y|\ge 1} \frac{|f(x)-f(y)|}{|x-y|^{\alpha_*}} \right\},$$

see [46, Definition 2.1 and Remark 2.2]. For these $\alpha(\cdot)$ and α_*, let

$$\phi(x,r) = \begin{cases} r^{\alpha(x)}, & 0 < r < 1, \\ r^{\alpha_*}, & 1 \le r < \infty. \end{cases} \tag{4.1.3}$$

If

$$0 \le \inf_{x\in\mathbb{R}^n} \alpha(x) \le \sup_{x\in\mathbb{R}^n} \alpha(x) < 1, \quad 0 \le \alpha_* < 1, \tag{4.1.4}$$

then ϕ is in $\mathcal{G}^{\mathrm{inc}}$ and satisfies (4.1.2). If $\alpha(\cdot)$ is log-Hölder continuous also, that is, there exists a positive constant C such that, for all $x, y \in \mathbb{R}^n$,

$$|\alpha(x) - \alpha(y)| \le \frac{C}{\log(e/|x-y|)} \quad \text{if} \quad 0 < |x-y| < 1,$$

then ϕ satisfies (1.2.7), see [46, Proposition 3.3]. Moreover, if $\inf_{x\in\mathbb{R}^n} \alpha(x) > 0$ and $\alpha_* > 0$, then $\mathcal{L}_{1,\phi}(\mathbb{R}^n) = \mathrm{Lip}^{\alpha_*}_{\alpha(\cdot)}(\mathbb{R}^n)$ with equivalent norms, see [46, Corollary 3.5]. Hence we have the following corollary.

Corollary 4.1.3. *Let $\phi : \mathbb{R}^n \times (0,\infty) \to (0,\infty)$ be defined by* (4.1.3)*. Assume that $\alpha(\cdot)$ and α_* satisfy* (4.1.4) *and that $\alpha(\cdot)$ is log-Hölder continuous. Let $f \in \mathcal{L}_{1,\phi}(\mathbb{R}^n)$. Then $f \in \overline{C^\infty_{\mathrm{comp}}(\mathbb{R}^n)}^{\mathcal{L}_{1,\phi}(\mathbb{R}^n)}$ if and only if f satisfies the following three conditions:*

(i) $\displaystyle\lim_{r\to+0}\sup_{x\in\mathbb{R}^n}\frac{\mathrm{MO}(f,B(x,r))}{r^{\alpha(x)}} = 0.$

(ii) $\displaystyle\lim_{r\to\infty}\sup_{x\in\mathbb{R}^n}\frac{\mathrm{MO}(f,B(x,r))}{r^{\alpha_*}} = 0.$

(iii) $\displaystyle\lim_{|x|\to\infty}\mathrm{MO}(f,B(x,r)) = 0$ *for each $r > 0$.*

Moreover, if $\inf_{x\in\mathbb{R}^n}\alpha(x) > 0$ and $\alpha_* > 0$, then $f \in \overline{C^\infty_{\mathrm{comp}}(\mathbb{R}^n)}^{\mathrm{Lip}^{\alpha_*}_{\alpha(\cdot)}(\mathbb{R}^n)}$ if and only if f satisfies the above three conditions.

4.2 Lemmas and a proposition

In this section we show three lemmas and one proposition to prove Theorem 4.1.1.

First, let η be a function on $\mathbb{R}^n$ such that

$$\operatorname{supp}\eta \subset \overline{B(0,1)}, \quad 0 \le \eta \le 2 \quad \text{and} \quad \int_{B(0,1)} \eta(y)\,dy = |B(0,1)|, \tag{4.2.1}$$

and let $\bar\eta_r(x) = |B(0,r)|^{-1}\eta(x/r)$. Then, for $f \in L^1_{\mathrm{loc}}(\mathbb{R}^n)$,

$$\bar\eta_r * f(x) = \fint_{B(x,r)} \eta((x-y)/r) f(y)\,dy. \tag{4.2.2}$$

If $\eta = \chi_{B(0,1)}$, then $\bar\eta_r * f(x) = f_{B(x,r)}$. If $\eta \in C^\infty_{\mathrm{comp}}(\mathbb{R}^n)$, then (4.2.2) is a mollifier. We can choose $\eta \in C^\infty_{\mathrm{comp}}(\mathbb{R}^n)$ which satisfies (4.2.1) and

$$\|\nabla\eta\|_{L^\infty} \le c_n \tag{4.2.3}$$

for some positive constant c_n dependent only on n.

For two balls B_1 and B_2, if $B_1 \subset B_2$, then

$$|f_{B_1} - f_{B_2}| \le \frac{|B_2|}{|B_1|}\mathrm{MO}(f,B_2), \tag{4.2.4}$$

and

$$\mathrm{MO}(f,B_1) \le 2\frac{|B_2|}{|B_1|}\mathrm{MO}(f,B_2). \tag{4.2.5}$$

The first lemma is an extension of (4.2.4).

Lemma 4.2.1. *If $B_1 = B(x,r) \subset B_2$, then*

$$|\bar{\eta}_r * f(x) - f_{B_2}| \le 2\frac{|B_2|}{|B_1|}\mathrm{MO}(f, B_2). \tag{4.2.6}$$

Proof. From (4.2.1) and (4.2.2) it follows that

$$\begin{aligned}
|\bar{\eta}_r * f(x) - f_{B_2}| &= \left| \fint_{B_1} \eta((x-y)/r) f(y)\, dy - f_{B_2} \right| \\
&= \left| \fint_{B_1} \eta((x-y)/r)\big(f(y) - f_{B_2}\big)\, dy \right| \\
&\le 2 \fint_{B_1} \big|f(y) - f_{B_2}\big|\, dy \\
&\le 2\frac{|B_2|}{|B_1|} \fint_{B_2} |f(y) - f_{B_2}|\, dy = 2\frac{|B_2|}{|B_1|}\mathrm{MO}(f, B_2).
\end{aligned}$$

□

Lemma 4.2.2. *For any ball $B(x,r)$,*

$$\fint_{B(x,r)} |f(y) - \bar{\eta}_r * f(y)|\, dy \le 2^{n+2}\mathrm{MO}(f, B(x,2r)). \tag{4.2.7}$$

Proof. Let $B = B(x,r)$. From Lemma 4.2.1 it follows that

$$\begin{aligned}
\fint_B |f(y) - \bar{\eta}_r * f(y)|\, dy &\le \fint_B \big(|f(y) - f_{2B}| + |f_{2B} - \bar{\eta}_r * f(y)|\big)\, dy \\
&\le \fint_B |f(y) - f_{2B}|\, dy + 2^{n+1}\mathrm{MO}(f, 2B) \\
&\le 2^{n+2}\mathrm{MO}(f, 2B).
\end{aligned}$$

□

Lemma 4.2.3. *Let η be in $C^\infty_{\mathrm{comp}}(\mathbb{R}^n)$ and satisfy* (4.2.1). *If $y, z \in B(x,r)$, then*

$$|\bar{\eta}_r * f(y) - \bar{\eta}_r * f(z)| \le 2^n \|\nabla \eta\|_{L^\infty} \frac{|y-z|}{r}\mathrm{MO}(f, B(x,2r)). \tag{4.2.8}$$

Proof. Letting $\tilde{f}(x) = f(x) - f_{B(x,2r)}$, we have

$$\begin{aligned}
&|\bar{\eta}_r * f(y) - \bar{\eta}_r * f(z)| = |\bar{\eta}_r * \tilde{f}(y) - \bar{\eta}_r * \tilde{f}(z)| \\
&= \left| \frac{1}{|B(x,r)|} \int_{B(x,2r)} \big(\eta((y-w)/r) - \eta((z-w)/r)\big)\tilde{f}(w)\, dw \right| \\
&\le 2^n \fint_{B(x,2r)} \big|\big(\eta((y-w)/r) - \eta((z-w)/r)\big)\tilde{f}(w)\big|\, dw \\
&\le 2^n \frac{|y-z|}{r} \|\nabla \eta\|_{L^\infty} \fint_{B(x,2r)} |\tilde{f}(w)|\, dw,
\end{aligned}$$

which shows the conclusion. □

Proposition 4.2.4. *Let η be in $C^\infty_{\mathrm{comp}}(\mathbb{R}^n)$ and satisfy* (4.2.1) *and* (4.2.3)*. Let ϕ be in $\mathcal{G}^{\mathrm{inc}}$ and satisfy* (1.2.7)*. Then there exists a positive constant C, dependent only on n and ϕ, such that, for all $r > 0$,*

$$\|f - \bar{\eta}_r * f\|_{\mathcal{L}_{1,\phi}} \le C \sup_{x \in \mathbb{R}^n,\ 0<t\le 2r} \frac{\mathrm{MO}(f, B(x,t))}{\phi(x,t)}. \tag{4.2.9}$$

Before we prove Proposition 4.2.4 we state its corollary, which is a variant of Theorem 4.1.1.

Corollary 4.2.5. *Let η be in $C^\infty_{\mathrm{comp}}(\mathbb{R}^n)$ and satisfy* (4.2.1) *and* (4.2.3)*. Let ϕ be in $\mathcal{G}^{\mathrm{inc}}$ and satisfy* (1.2.7)*. Then there exists a positive constant C, dependent only on n and ϕ, such that, for all $f \in \mathcal{L}_{1,\phi}(\mathbb{R}^n)$ and $r > 0$,*

$$\|\bar{\eta}_r * f\|_{\mathcal{L}_{1,\phi}} \le C\|f\|_{\mathcal{L}_{1,\phi}}. \tag{4.2.10}$$

Moreover, if f satisfies (i) *in Theorem 4.1.1, then $\bar{\eta}_r * f \to f$ in $\mathcal{L}_{1,\phi}(\mathbb{R}^n)$ as $r \to +0$.*

Proof of Proposition 4.2.4. We show that

$$\frac{\mathrm{MO}(f - \bar{\eta}_r * f, B(x,t))}{\phi(x,t)}$$

is dominated by the right hand side of (4.2.9) for each ball $B(x,t)$.

Case 1. $0 < t \le r$: From Lemma 4.2.3 it follows that

$$\begin{aligned}
&\frac{1}{\phi(x,t)} ⨍_{B(x,t)} |\bar{\eta}_r * f(y) - (\bar{\eta}_r * f)_{B(x,t)}|\, dy \\
&\le \frac{1}{\phi(x,t)} ⨍_{B(x,t)} ⨍_{B(x,t)} |\bar{\eta}_r * f(y) - \bar{\eta}_r * f(z)|\, dz\, dy \\
&\le \frac{2^n \|\nabla \eta\|_{L^\infty}}{\phi(x,t)} \left(⨍_{B(x,t)} ⨍_{B(x,t)} \frac{|y-z|}{r}\, dz\, dy \right) \mathrm{MO}(f, B(x,2r)) \\
&\le 2^n c_n \frac{2t}{r\phi(x,t)} \mathrm{MO}(f, B(x,2r)) \\
&\le C_{n,\phi} \frac{\mathrm{MO}(f, B(x,2r))}{\phi(x,2r)}.
\end{aligned}$$

In the above we used the almost decreasingness of $r \mapsto \phi(x,r)/r$ for the last inequality. Hence

$$\begin{aligned}
&\frac{\mathrm{MO}(f-\bar{\eta}_r * f, B(x,t))}{\phi(x,t)} \\
&= \frac{1}{\phi(x,t)} \fint_{B(x,t)} |f(y) - \bar{\eta}_r * f(y) - (f-\bar{\eta}_r * f)_{B(x,t)}| \\
&\le \frac{1}{\phi(x,t)} \fint_{B(x,t)} |f(y) - f_{B(x,t)}|\, dy \\
&\quad + \frac{1}{\phi(x,t)} \fint_{B(x,t)} |\bar{\eta}_r * f(y) - (\bar{\eta}_r * f)_{B(x,t)}|\, dy \\
&\le \frac{\mathrm{MO}(f, B(x,t))}{\phi(x,t)} + C_{n,\phi} \frac{\mathrm{MO}(f, B(x,2r))}{\phi(x,2r)}.
\end{aligned}$$

Case 2. $t > r$: Take balls $\{B(x_j, r)\}_j$ such that

$$B(x,t) \subset \bigcup_j B(x_j, r) \subset B(x, 2t), \quad \sum_j |B(x_j, r)| \le C_n |B(x,t)|,$$

where C_n is a positive constant depending only on n. Then, using Lemma 4.2.2, we have

$$\begin{aligned}
&\mathrm{MO}(f-\bar{\eta}_r * f, B(x,t)) \\
&\le \frac{2}{|B(x,t)|} \int_{B(x,t)} |f(y) - \bar{\eta}_r * f(y)|\, dy \\
&\le \frac{2}{|B(x,t)|} \sum_j \int_{B(x_j,r)} |f(y) - \bar{\eta}_r * f(y)|\, dy \\
&\le \frac{2}{|B(x,t)|} \sum_j |B(x_j,r)| 2^{n+2} \mathrm{MO}(f, B(x_j, 2r)) \\
&\le 2^{n+3} C_n \sup_j \mathrm{MO}(f, B(x_j, 2r)).
\end{aligned}$$

By the almost increasingness of ϕ, (1.2.7) and the doubling condition of ϕ we have

$$\phi(x_j, 2r) \lesssim \phi(x_j, 2t) \sim \phi(x, 2t) \lesssim \phi(x,t).$$

Therefore,

$$\frac{\mathrm{MO}(f-\bar{\eta}_r * f, B(x,t))}{\phi(x,t)} \le C'_{n,\phi} \sup_j \frac{\mathrm{MO}(f, B(x_j, 2r))}{\phi(x_j, 2r)}.$$

The proof is complete. □

4.3 Proof of the theorem

Proof of Theorem 4.1.1. **Part 1.** Let $f \in C^\infty_{\mathrm{comp}}(\mathbb{R}^n)$. Then, from the inequality

$$\fint_{B(x,r)} |f(y) - f_B|\, dy \le 2r\|\nabla f\|_{L^\infty}$$

and (4.1.2) it follows that

$$\lim_{r\to+0} \sup_{x\in\mathbb{R}^n} \frac{\mathrm{MO}(f, B(x,r))}{\phi(x,r)} \le \lim_{r\to+0} \sup_{x\in\mathbb{R}^n} \frac{2r}{\phi(x,r)}\|\nabla f\|_{L^\infty} = 0.$$

On the other hand, from the inequality

$$\fint_{B(x,r)} |f(y) - f_B|\, dy \le \frac{2|\operatorname{supp} f|\|f\|_{L^\infty}}{|B(x,r)|}$$

and (4.1.2) it follows that

$$\lim_{r\to\infty} \sup_{x\in\mathbb{R}^n} \frac{\mathrm{MO}(f, B(x,r))}{\phi(x,r)} \le \lim_{r\to\infty} \sup_{x\in\mathbb{R}^n} \frac{2|\operatorname{supp} f|\|f\|_{L^\infty}}{\phi(x,r)|B(x,r)|} = 0.$$

For each $r > 0$, take $x \in \mathbb{R}^n$ such that $\operatorname{supp} f \cap B(x,r) = \emptyset$. Then

$$\frac{\mathrm{MO}(f, B(x,r))}{\phi(x,r)} = 0.$$

That is, f satisfies (i), (ii) and (iii).

Let $f \in \overline{C^\infty_{\mathrm{comp}}(\mathbb{R}^n)}^{\mathcal{L}_{1,\phi}(\mathbb{R}^n)}$. Then, for any $\epsilon > 0$, there exists $g \in C^\infty_{\mathrm{comp}}(\mathbb{R}^n)$ such that, $\displaystyle\sup_{x\in\mathbb{R}^n,\ r>0} \frac{\mathrm{MO}(f-g, B(x,r))}{\phi(x,r)} < \epsilon$. Therefore, f satisfies (i), (ii) and (iii).

Part 2. Let f satisfy (i), (ii) and (iii). For any $\epsilon > 0$, from (i) and (ii) there exist integers i_ϵ and k_ϵ ($i_\epsilon < k_\epsilon$) such that

$$\sup\left\{\frac{\mathrm{MO}(f, B(x,r))}{\phi(x,r)} : x \in \mathbb{R}^n,\ 0 < r \le 2^{i_\epsilon}\right\} < \epsilon$$

and

$$\sup\left\{\frac{\mathrm{MO}(f, B(x,r))}{\phi(x,r)} : x \in \mathbb{R}^n,\ r \ge 2^{k_\epsilon}\right\} < \epsilon.$$

From (iii) it follows that

$$\lim_{|x|\to\infty} \max\left\{\frac{\mathrm{MO}(f, B(x,2^\ell))}{\phi(x,2^\ell)} : \ell = i_\epsilon, i_\epsilon + 1, \dots, k_\epsilon\right\} = 0.$$

By (4.2.5) and the doubling condition of ϕ we have

$$\sup_{2^{\ell-1}\le r\le 2^\ell}\frac{\mathrm{MO}(f,B(x,r))}{\phi(x,r)}\le C\frac{\mathrm{MO}(f,B(x,2^\ell))}{\phi(x,2^\ell)},\quad \ell=i_\epsilon,i_\epsilon+1,\dots,k_\epsilon,$$

where the positive constant C is dependent only on n and ϕ. Consequently,

$$\lim_{|x|\to\infty}\sup_{2^{i_\epsilon}\le r\le 2^{k_\epsilon}}\frac{\mathrm{MO}(f,B(x,r))}{\phi(x,r)}=0.$$

Then there exists an integer j_ϵ such that $j_\epsilon>k_\epsilon(>i_\epsilon)$ and

$$\sup\left\{\frac{\mathrm{MO}(f,B(x,r))}{\phi(x,r)}:B(x,r)\cap B(0,2^{j_\epsilon})=\emptyset\right\}<\epsilon.$$

Using i_ϵ, k_ϵ and j_ϵ, we set

$$\begin{aligned}\mathcal{B}_1&=\left\{B(x,r):x\in\mathbb{R}^n,\ 0<r\le 2^{i_\epsilon}\right\},\\ \mathcal{B}_2&=\left\{B(x,r):x\in\mathbb{R}^n,\ r\ge 2^{k_\epsilon}\right\},\\ \mathcal{B}_3&=\left\{B(x,r):B(x,r)\cap B(0,2^{j_\epsilon})=\emptyset\right\}.\end{aligned}$$

Then $\mathrm{MO}(f,B)/\phi(B)<\epsilon$ if $B\in\mathcal{B}_1\cup\mathcal{B}_2\cup\mathcal{B}_3$.

We define a C^∞-function f_1 as follows: Let η be in $C^\infty_{\mathrm{comp}}(\mathbb{R}^n)$ and satisfy (4.2.1) and (4.2.3), and let

$$f_1=\bar\eta_{r_1}*f,\quad r_1=2^{i_\epsilon-1}.$$

Then, from Proposition 4.2.4 it follows that

$$\|f-f_1\|_{\mathcal{L}_{1,\phi}}\le C_{n,\phi}\sup_{B\in\mathcal{B}_1}\frac{\mathrm{MO}(f,B)}{\phi(B)}\le C_{n,\phi}\,\epsilon, \tag{4.3.1}$$

where the positive constant $C_{n,\phi}$ is dependent only on n and ϕ, and independent of r_1. This also shows that

$$\begin{aligned}\frac{\mathrm{MO}(f_1,B)}{\phi(B)}&\le\|f-f_1\|_{\mathcal{L}_{1,\phi}}+\frac{\mathrm{MO}(f,B)}{\phi(B)}\\ &\le(C_{n,\phi}+1)\epsilon\quad\text{for}\quad B\in\mathcal{B}_1\cup\mathcal{B}_2\cup\mathcal{B}_3.\end{aligned} \tag{4.3.2}$$

Next we define a C^∞-function f_2 as follows: Let $h\in C^\infty_{\mathrm{comp}}(\mathbb{R}^n)$ satisfy

$$\chi_{B(0,1)}\le h\le\chi_{B(0,2)},\quad \|\nabla h\|_{L^\infty}\le 2,$$

and let

$$f_2 = (f_1 - (f_1)_{B(0,4r_2)})h_{r_2} + (f_1)_{B(0,4r_2)}, \quad h_{r_2}(x) = h(x/r_2), \quad r_2 = 2^{j_\epsilon+1}.$$

Then $f_2 - (f_1)_{B(0,4r_2)} \in C^\infty_{\mathrm{comp}}(\mathbb{R}^n)$, that is,

$$\min_{g \in C^\infty_{\mathrm{comp}}(\mathbb{R}^n)} \|f_2 - g\|_{\mathcal{L}_{1,\phi}} = 0. \tag{4.3.3}$$

In the following, using (4.3.2), we will show that there exists a positive constant $\widetilde{C}_{n,\phi}$, dependent only on n and ϕ, such that

$$\|f_1 - f_2\|_{\mathcal{L}_{1,\phi}} \le \widetilde{C}_{n,\phi}\, \epsilon. \tag{4.3.4}$$

Once we show (4.3.4), combining this with (4.3.1) and (4.3.3), we obtain that $f \in \overline{C^\infty_{\mathrm{comp}}(\mathbb{R}^n)}^{\mathcal{L}_{1,\phi}(\mathbb{R}^n)}$.

Now, take a ball $B = B(z,r)$ arbitrarily.

Case 1. $r \ge r_2/2$: In this case $B \in \mathcal{B}_2$.

Case 1-1. If $B \cap B(0,2r_2) = \emptyset$, then $f_2 = (f_1)_{B(0,4r_2)}$ on B, that is, $\mathrm{MO}(f_2,B) = 0$. Hence, by (4.3.2) we have

$$\frac{\mathrm{MO}(f_1 - f_2, B)}{\phi(B)} = \frac{\mathrm{MO}(f_1, B)}{\phi(B)} \le (C_{n,\phi} + 1)\epsilon.$$

Case 1-2. If $B \cap B(0,2r_2) \ne \emptyset$, then, using the almost increasingness, the nearness condition (1.2.7) and the doubling condition (1.2.5) of ϕ, we have

$$\phi(0,4r_2) \lesssim \phi(0,8r) \sim \phi(z,8r) \sim \phi(B), \quad |B(0,4r_2)| \le 8^n |B|,$$

and then

$$\begin{aligned}
\frac{\mathrm{MO}(f_2,B)}{\phi(B)} &= \frac{\mathrm{MO}((f_1 - (f_1)_{B(0,4r_2)})h_{r_2}, B)}{\phi(B)} \\
&\le \frac{2}{\phi(B)} \fint_B |(f_1(y) - (f_1)_{B(0,4r_2)})h_{r_2}|\, dy \\
&\le \frac{2}{\phi(B)|B|} \int_{B(0,4r_2)} |f_1(y) - (f_1)_{B(0,4r_2)}|\, dy \\
&\lesssim \frac{\mathrm{MO}(f_1, B(0,4r_2))}{\phi(B(0,4r_2))}.
\end{aligned}$$

Since both B and $B(0,4r_2)$ are in $\mathcal{B}_2$, from (4.3.2) it follows that

$$\frac{\mathrm{MO}(f_1-f_2,B)}{\phi(B)} \le \frac{\mathrm{MO}(f_1,B)}{\phi(B)} + \frac{\mathrm{MO}(f_2,B)}{\phi(B)} \le C'_{n,\phi}\,\epsilon,$$

where $C'_{n,\phi}$ is dependent only on n and ϕ.

Case 2. $r < r_2/2$:

Case 2-1. If $B \subset B(0,r_2)$, then $\mathrm{MO}(f_1-f_2,B)=0$, since

$$f_1 - f_2 = \big(f_1 - (f_1)_{B(0,4r_2)}\big)\big(1-h_{r_2}\big) = 0 \quad \text{on} \quad B(0,r_2).$$

Case 2-2. If $B \cap B(0,2r_2) = \emptyset$, then $B \in \mathcal{B}_3$ and $f_2 = (f_1)_{B(0,4r_2)}$ on B. Hence

$$\frac{\mathrm{MO}(f_1-f_2,B)}{\phi(B)} = \frac{\mathrm{MO}(f_1,B)}{\phi(B)} \le (C_{n,\phi}+1)\epsilon.$$

Case 2-3. If $B \cap (B(0,2r_2)\setminus B(0,r_2)) \neq \emptyset$, then $B \subset B(0,4r_2)\setminus B(0,r_2/2)$, since $r < r_2/2$, and hence $B \in \mathcal{B}_3$. Choose a sequence of balls $\{B_\ell\}_{\ell=0}^{m+1}$ such that

$$\begin{cases} B(0,4r_2) = B_0 \supset B_1 \supset \cdots \supset B_m \supset B_{m+1} = B, & \\ |B_\ell| = 2^n |B_{\ell+1}|, & \ell = 0,\dots,m-1, \\ |B_m| \le 2^n |B_{m+1}|, & \\ B_\ell \in \mathcal{B}_2, & \ell = 0,1,2,3, \\ B_\ell \in \mathcal{B}_3, & \ell = 4,\dots,m+1. \end{cases}$$

Note that the balls above are not concentric. Then, using (4.2.4) and (4.3.2), we have

$$\begin{aligned} |(f_1)_{B(0,4r_2)} - (f_1)_B| &\le \sum_{\ell=0}^{m} |(f_1)_{B_\ell} - (f_1)_{B_{\ell+1}}| \\ &\le 2^n \sum_{\ell=0}^{m} \phi(B_\ell) \max\left\{\frac{\mathrm{MO}(f_1,B_\ell)}{\phi(B_\ell)} : \ell = 0,1,\dots,m\right\} \\ &\le 2^n (C_{n,\phi}+1) \sum_{\ell=0}^{m} \phi(B_\ell)\,\epsilon. \end{aligned}$$

Since ϕ is in $\mathcal{G}^{\mathrm{inc}}$ and satisfies the nearness condition (1.2.7), the inequalities

$$\phi(B_\ell)/(2^{2-\ell} r_2) \le C_\phi\, \phi(B)/r, \quad \ell = 0,1,\dots,m,$$

hold for some positive constant C_ϕ dependent only on ϕ. Then

$$\sum_{\ell=0}^{m} \phi(B_\ell) \le \sum_{\ell=0}^{m} C_\phi \frac{(2^{2-\ell} r_2)\phi(B)}{r} \le 2^3 C_\phi \frac{r_2 \phi(B)}{r}.$$

Hence,

$$|(f_1)_{B(0,4r_2)} - (f_1)_B| \le C''_{n,\phi} \frac{r_2\phi(B)}{r}\, \epsilon, \tag{4.3.5}$$

where $C''_{n,\phi} = 2^{n+3}(C_{n,\phi}+1)C_\phi$. Next, let

$$C_{f_1} = \big((f_1)_B - (f_1)_{B(0,4r_2)}\big)\big(1 - (h_{r_2})_B\big).$$

Then

$$\begin{aligned}
&\big(f_1(y) - f_2(y)\big) - C_{f_1} \\
&= \big(f_1(y) - (f_1)_{B(0,4r_2)}\big)\big(1 - h_{r_2}(y)\big) - \big((f_1)_B - (f_1)_{B(0,4r_2)}\big)\big(1 - (h_{r_2})_B\big) \\
&= \Big(\big(f_1(y) - (f_1)_B\big)\big(1 - h_{r_2}(y)\big)\Big) + \Big(\big(h_{r_2}(y) - (h_{r_2})_B\big)\big((f_1)_{B(0,4r_2)} - (f_1)_B\big)\Big),
\end{aligned}$$

and then, for $y \in B = B(z,r)$,

$$\begin{aligned}
&\Big|\big(f_1(y) - f_2(y)\big) - C_{f_1}\Big| \\
&\le \big|f_1(y) - (f_1)_B\big| + 2r\|\nabla h_{r_2}\|_{L^\infty}\big|(f_1)_{B(0,4r_2)} - (f_1)_B\big| \\
&\le \big|f_1(y) - (f_1)_B\big| + 2r\frac{2}{r_2} \times C''_{n,\phi}\frac{r_2\phi(B)}{r}\,\epsilon,
\end{aligned}$$

where we used (4.3.5) in the last inequality. Hence,

$$\frac{1}{\phi(B)} \fint_B \big|\big(f_1(y) - f_2(y)\big) - C_f\big|\,dy \le \frac{\mathrm{MO}(f_1, B)}{\phi(B)} + 2^2 C''_{n,\phi}\,\epsilon \le C'''_{n,\phi}\,\epsilon,$$

where $C'''_{n,\phi}$ is dependent only on n and ϕ, which shows

$$\frac{\mathrm{MO}(f_1 - f_2, B)}{\phi(B)} \le 2C'''_{n,\phi}\,\epsilon.$$

The proof is complete. □

Chapter 5

Compactness – Necessity

5.1 Theorems

In this section, as an application of Theorem 4.1.1, we give a characterization of compact commutators $[b,T]$ and $[b,I_\alpha]$ with $b \in \mathcal{L}_{1,\phi}(\mathbb{R}^n)$ on generalized Morrey spaces $L^{(p,\varphi)}(\mathbb{R}^n)$ with variable growth condition.

In Theorems 3.1.1, 3.1.2, we state sufficient conditions for the compactness of the commutators $[b,T]$ and $[b,I_\rho]$ on $L^{(p,\varphi)}(\mathbb{R}^n)$. In this section, to characterize the compactness, we give necessary conditions. To prove the results we apply Theorem 4.1.1 in the final section.

Theorem 5.1.1. *Let $1 < p \le q < \infty$ and $\varphi, \psi : \mathbb{R}^n \times (0,\infty) \to (0,\infty)$. Let T be a Calderón-Zygmund operator of convolution type with kernel $K : \mathbb{R}^n \setminus \{0\} \to \mathbb{R}$. Assume the same condition on φ, ψ and T as Theorem 2.1.1 (ii). Assume also that there exists a positive constant μ_0 such that*

(5.1.1)
$$\limsup_{r\to+0} \sup_{x\in\mathbb{R}^n} \varphi(x,r)^{1/p}\psi(x,r)r^{n/q} \le \mu_0 \inf_{x\in\mathbb{R}^n,\ r\in(0,1]} \varphi(x,r)^{1/p}\psi(x,r)r^{n/q},$$

(5.1.2)
$$\sup_{x\in\mathbb{R}^n,\ r\in[1,\infty)} \varphi(x,r)^{1/p}\psi(x,r)r^{n/q} \le \mu_0 \liminf_{r\to\infty} \inf_{x\in\mathbb{R}^n} \varphi(x,r)^{1/p}\psi(x,r)r^{n/q},$$

(5.1.3)
$$\limsup_{|x|\to\infty} \varphi(x,r)^{1/p}\psi(x,r) \le \mu_0 \liminf_{|x|\to\infty} \varphi(x,r)^{1/p}\psi(x,r) \text{ for every } r > 0.$$

Let b be a real valued function in $L^1_{\rm loc}(\mathbb{R}^n)$. If $[b,T]$ is well defined on $L^{(p,\varphi)}(\mathbb{R}^n)$ and compact from $L^{(p,\varphi)}(\mathbb{R}^n)$ to $L^{(q,\varphi)}(\mathbb{R}^n)$, then b is in $\overline{C^\infty_{\rm comp}(\mathbb{R}^n)}^{\mathcal{L}_{1,\psi}(\mathbb{R}^n)}$.

We note that the Riesz transforms fall under the scope of Theorem 5.1.1.

Remark 5.1.1. If φ and ψ satisfy

$$\begin{cases} \lim\limits_{r\to+0} \sup\limits_{x\in\mathbb{R}^n} \varphi(x,r)^{1/p}\psi(x,r)r^{n/q} = 0, \\ \lim\limits_{r\to\infty} \inf\limits_{x\in\mathbb{R}^n} \varphi(x,r)^{1/p}\psi(x,r)r^{n/q} = \infty, \\ \lim\limits_{|x|\to\infty} \varphi(x,r)^{1/p}\psi(x,r) \text{ exists for every } r > 0, \end{cases} \tag{5.1.4}$$

or

$$\mu_0^{-1} \le \varphi(x,r)^{1/p}\psi(x,r)r^{n/q} \le \mu_0 \text{ for all } x \in \mathbb{R}^n, r \in (0,\infty), \tag{5.1.5}$$

then the conditions (5.1.1), (5.1.2) and (5.1.3) hold.

Example 5.1.1. Let $1 < p \le q < \infty$ and $\beta(\cdot), \lambda(\cdot) : \mathbb{R}^n \to (-\infty,\infty)$. Assume that

$$\begin{aligned} 0 \le \inf_{x\in\mathbb{R}^n} \beta(x) \le \sup_{x\in\mathbb{R}^n} \beta(x) \le 1, \quad 0 \le \beta_* \le 1, \\ -n \le \inf_{x\in\mathbb{R}^n} \lambda(x) \le \sup_{x\in\mathbb{R}^n} \lambda(x) < 0, \quad -n \le \lambda_* < 0. \end{aligned}$$

Let

$$\psi(x,r) = \begin{cases} r^{\beta(x)}, \\ r^{\beta_*}, \end{cases} \quad \varphi(x,r) = \begin{cases} r^{\lambda(x)}, & 0 < r < 1, \\ r^{\lambda_*}, & 1 \le r < \infty. \end{cases}$$

Assume that $\lambda(\cdot)$ is log-Hölder continuous, that $\lim\limits_{|x|\to\infty} \beta(x)$ and $\lim\limits_{|x|\to\infty} \lambda(x)$ exist and that

$$\begin{aligned} \inf_{x\in\mathbb{R}^n} (\beta(x) + \lambda(x)/p) > -n/q, \quad \beta_* + \lambda_*/p > -n/q, \\ \beta(x) + \lambda(x)/p \le \lambda(x)/q, \quad \beta_* + \lambda_*/p \ge \lambda_*/q. \end{aligned}$$

Then φ satisfies (1.2.7) and φ and ψ satisfy (2.1.8) and (5.1.4). Let $b \in L^1_{\rm loc}(\mathbb{R}^n)$. If a Calderón-Zygmund operator T satisfies the assumption in Theorem 5.1.1, and if $[b,T]$ is compact from $L^{(p,\varphi)}(\mathbb{R}^n)$ to $L^{(q,\varphi)}(\mathbb{R}^n)$, then b is in $\overline{C^\infty_{\rm comp}(\mathbb{R}^n)}^{\mathcal{L}_{1,\psi}(\mathbb{R}^n)}$.

We also take the cases

$$\psi(x,r) = \begin{cases} r^{\beta(x)}(1/\log(e/r))^{\beta_1(x)}, & 0 < r < 1, \\ r^{\beta_*}(\log(er))^{\beta_{**}}, & 1 \le r < \infty, \end{cases}$$

etc.

Theorem 5.1.2. *Let $1 < p < q < \infty$, $0 < \alpha < n$ and $\varphi, \psi : \mathbb{R}^n \times (0,\infty) \to (0,\infty)$. Assume the same condition on φ, ψ and α as Theorem 2.1.2 (ii). Assume also that there exists a positive constant μ_0 such that*

(5.1.6)
$$\limsup_{r\to+0} \sup_{x\in\mathbb{R}^n} \varphi(x,r)^{1/p}\psi(x,r)r^{\alpha+n/q} \le \mu_0 \inf_{x\in\mathbb{R}^n,\ r\in(0,1]} \varphi(x,r)^{1/p}\psi(x,r)r^{\alpha+n/q},$$

(5.1.7)
$$\sup_{x\in\mathbb{R}^n,\ r\in[1,\infty)} \varphi(x,r)^{1/p}\psi(x,r)r^{\alpha+n/q} \le \mu_0 \liminf_{r\to\infty} \inf_{x\in\mathbb{R}^n} \varphi(x,r)^{1/p}\psi(x,r)r^{\alpha+n/q},$$

(5.1.8)
$$\limsup_{|x|\to\infty} \varphi(x,r)^{1/p}\psi(x,r) \le \mu_0 \liminf_{|x|\to\infty} \varphi(x,r)^{1/p}\psi(x,r) \text{ for every } r > 0.$$

Let b be a real valued function in $L^1_{\mathrm{loc}}(\mathbb{R}^n)$. If $[b, I_\alpha]$ is well defined on $L^{(p,\varphi)}(\mathbb{R}^n)$ and compact from $L^{(p,\varphi)}(\mathbb{R}^n)$ to $L^{(q,\varphi)}(\mathbb{R}^n)$, then b is in $\overline{C^\infty_{\mathrm{comp}}(\mathbb{R}^n)}^{\mathcal{L}_{1,\psi}(\mathbb{R}^n)}$.

We can take similar examples to Example 5.1.1 for the compactness of $[b, I_\alpha]$.

We will prove Theorems 5.1.1 and 5.1.2 in the following sections by using Theorem 4.1.1.

5.2 Lemmas

In this section we show several lemmas to prove Theorems 5.1.1 and 5.1.2 in Section 5.3.

Lemma 5.2.1 ([37, Corollary 2.4])**.** *There exists a positive constant c_n dependent only on n such that, for all $x \in \mathbb{R}^n$ and $r, s \in (0,\infty)$,*

$$|f_{B(x,r)} - f_{B(x,s)}| \le c_n \int_r^{2s} \frac{\mathrm{MO}(f, B(x,t))}{t}\, dt, \quad \text{if } r < s.$$

The next lemma is well known as the John-Nirenberg inequality.

Lemma 5.2.2 ([29])**.** *For all cubes Q_0 and all $t > 0$,*

$$|\{x \in Q_0 : |f(x) - f_{Q_0}| > t\}| \le e|Q_0| \exp\left(-At/\sup\{\mathrm{MO}(f,Q) : Q \subset Q_0\}\right),$$

with $A = (2^n e)^{-1}$.

For the constants e and A in the above lemma, see [22, Theorem 3.1.6].

Corollary 5.2.3. *Assume that $\psi \in \mathcal{G}^{\rm inc}$. Let $\nu > 1$ and $f \in \mathcal{L}_{1,\psi}(\mathbb{R}^n)$ with $\|f\|_{\mathcal{L}_{1,\psi}} = 1$. Then, for all balls B_0 and all $t > 0$,*

$$|\{x \in \nu B_0 : |f(x) - f_{B_0}| > t + A_0 \nu \psi(B_0)\}| \le A_1 \nu^n |B_0| \exp\left(-A_2 t/(\nu \psi(B_0))\right),$$

where the constants A_0, A_1 and A_2 are dependent only on n and ψ.

Proof. We denote by v_n the volume of the unit ball. Let Q_0 be the smallest cube containing νB_0. Then

$$\nu B_0 \subset Q_0 \subset \sqrt{n}\nu B_0, \quad \frac{|Q_0|}{|B_0|} = \frac{(2\nu)^n}{v_n}.$$

By this relation, Lemma 5.2.1 and $\|f\|_{\mathcal{L}_{1,\psi}} = 1$ we have

$$\begin{aligned} |f_{B_0} - f_{Q_0}| &\le |f_{B_0} - f_{\sqrt{n}\nu B_0}| + |f_{\sqrt{n}\nu B_0} - f_{Q_0}| \\ &\le c_n \int_1^{2\sqrt{n}\nu} \frac{\mathrm{MO}(f, tB_0)}{t}\, dt + \frac{|\sqrt{n}\nu B_0|}{|Q_0|} \mathrm{MO}(f, \sqrt{n}\nu B_0) \\ &\le c_n \int_1^{2\sqrt{n}\nu} \frac{\psi(tB_0)}{t}\, dt + (\sqrt{n}/2)^n v_n \psi(\sqrt{n}\nu B_0) \\ &\le A_0 \nu \psi(B_0), \end{aligned}$$

where the constant A_0 is dependent only on n and ψ. Since

$$\begin{aligned} &|f(x) - f_{B_0}| > t + A_0 \nu \psi(B_0) \\ &\qquad \Rightarrow \quad |f(x) - f_{B_0}| > t + |f_{B_0} - f_{Q_0}| \quad \Rightarrow \quad |f(x) - f_{Q_0}| > t, \end{aligned}$$

we have

$$\begin{aligned} &|\{x \in \nu B_0 : |f(x) - f_{B_0}| > t + A_0 \nu \psi(B_0)\}| \\ &\le |\{x \in \nu B_0 : |f(x) - f_{Q_0}| > t\}| \\ &\le |\{x \in Q_0 : |f(x) - f_{Q_0}| > t\}| \\ &\le e|Q_0| \exp\left(-At/\sup\{\mathrm{MO}(f, Q) : Q \subset Q_0\}\right) \\ &= \frac{e(2\nu)^n}{v_n} |B_0| \exp\left(-At/\sup\{\mathrm{MO}(f, Q) : Q \subset Q_0\}\right) \quad \text{with } A = (2^n e)^{-1}. \end{aligned}$$

In the above the third inequality follows from the John-Nirenberg inequality. For any cube $Q \subset Q_0$, take the smallest ball B containing Q. Then

$$Q \subset B \subset \sqrt{n}\nu B_0, \quad \frac{|B|}{|Q|} = (\sqrt{n}/2)^n v_n.$$

Hence

$$\mathrm{MO}(f,Q) \le \frac{2|B|}{|Q|}\mathrm{MO}(f,B) = 2(\sqrt{n}/2)^n v_n \mathrm{MO}(f,B).$$

That is,

$$\begin{aligned}\sup\{\mathrm{MO}(f,Q) : Q \subset Q_0\} &\le 2(\sqrt{n}/2)^n v_n \sup\left\{\mathrm{MO}(f,B) : B \subset \sqrt{n}\nu B_0\right\} \\ &\le 2(\sqrt{n}/2)^n v_n \sup\left\{\psi(B) : B \subset \sqrt{n}\nu B_0\right\} \\ &\le A_2'\nu\psi(B_0),\end{aligned}$$

where the constant A_2' is dependent only on n and ψ. Letting $A_1 = e2^n/v_n$ and $A_2 = A/A_2'$, we have the conclusion. □

In the following lemma we used the idea in [10].

Lemma 5.2.4. *Let b be a real valued function in $L^1_{\mathrm{loc}}(\mathbb{R}^n)$. For any ball B, let*

(5.2.1)
$$f^B(z) = \varphi(B)^{1/p}\Big(\mathrm{sgn}(b(z)-b_B) - c_0\Big)\chi_B(z), \text{ where } c_0 = ⨍_B \mathrm{sgn}(b(z)-b_B)\,dz.$$

Then

(5.2.2) $$\mathrm{supp}\, f^B \subset B, \quad \int_{\mathbb{R}^n} f^B(z)\,dz = 0,$$

(5.2.3) $$f^B(z)(b(z)-b_B) \ge 0,$$

(5.2.4) $$\int_{\mathbb{R}^n} f^B(z)(b(z)-b_B)\,dz = \varphi(B)^{1/p}|B|\mathrm{MO}(b,B),$$

(5.2.5) $$\|f^B\|_{L^{(p,\varphi)}} \le C,$$

where C is a constant dependent only on n and φ.

Proof. The first assertion (5.2.2) is clear. Since $\int_B (b(z)-b_B)\,dz = 0$, it is easy to check $|c_0| < 1$. Then we have

$$f^B(z)(b(z)-b_B) = \varphi(B)^{1/p}\Big(|b(z)-b_B| - c_0(b(z)-b_B)\Big)\chi_B(z) \ge 0$$

and

$$\begin{aligned}\int_{\mathbb{R}^n} f^B(z)(b(z)-b_B)\,dz &= \varphi(B)^{1/p}\int_B \Big(|b(z)-b_B| - c_0(b(z)-b_B)\Big)\,dz \\ &= \varphi(B)^{1/p}\int_B |b(z)-b_B|\,dz \\ &= \varphi(B)^{1/p}|B|\mathrm{MO}(b,B).\end{aligned}$$

Finally, let $B = B(x,r)$. We show that, for any $B' = B(x',r')$,

$$\frac{1}{\varphi(B')} \fint_{B'} |f^B(z)|^p \, dz \le C.$$

If $B \cap B' \ne \emptyset$ and $r' \le r$, then $\varphi(x,r) \sim \varphi(x,2r) \sim \varphi(x',2r) \lesssim \varphi(x',r')$ by (1.2.5), (1.2.7) and the almost decreasingness of φ. Hence

$$\frac{1}{\varphi(B')} \fint_{B'} |f^B(z)|^p \, dz \le \frac{\varphi(B)}{\varphi(B')} \le C.$$

If $B \cap B' \ne \emptyset$ and $r' > r$, then $\varphi(x,r)r^n \lesssim \varphi(x,2r')(2r')^n \sim \varphi(x',2r')(2r')^n \sim 2^n\varphi(x',r')(r')^n$ by the almost increasingness of $t \mapsto \varphi(x,t)t^n$, (1.2.7) and (1.2.5). Hence

$$\frac{1}{\varphi(B')} \fint_{B'} |f^B(z)|^p \, dz \le \frac{\varphi(B)|B|}{\varphi(B')|B'|} \le C. \qquad \square$$

Lemma 5.2.5. *Let $p,q \in (1,\infty)$. Let T be a convolution type singular integral operator such that*

$$Tf(x) = p.v. \int_{\mathbb{R}^n} K(x-y) f(y) \, dy \tag{5.2.6}$$

with homogeneous kernel $K : \mathbb{R}^n \setminus \{0\} \to \mathbb{R}$ satisfying $K(x) = |x|^{-n} K(x/|x|)$, $\int_{S^{n-1}} K = 0$, $K \in C^\infty(S^{n-1})$ and $K \not\equiv 0$, Assume that $\varphi \in \mathcal{G}^{\mathrm{dec}}$ and $\psi \in \mathcal{G}^{\mathrm{inc}}$. Assume also that ψ satisfies (3.1.1). *Let b be a real valued function and $\|b\|_{\mathcal{L}_{1,\psi}} = 1$. For any ball B, define f^B by* (5.2.1). *Then, for any constants $\epsilon_0, \mu_0 \in (0,\infty)$, there exist constants $\nu_1, \nu_2 \in [2,\infty)$ $(\nu_1 < \nu_2)$, $\nu_3 \in (0,\infty)$ and $\nu_4 \in (0,1)$ such that, for all balls B satisfying $\mathrm{MO}(b,B)/\psi(B) \ge \epsilon_0$, the following three inequalities hold:*

$$\left(\frac{1}{|B|} \int_{\nu_2 B \setminus \nu_1 B} \left| [b,T] f^B(y) \right|^q dy \right)^{1/q} \ge \nu_3 \varphi(B)^{1/p} \psi(B), \tag{5.2.7}$$

$$\left(\frac{1}{|B|} \int_{\mathbb{R}^n \setminus \nu_2 B} \left| [b,T] f^B(y) \right|^q dy \right)^{1/q} \le \frac{\nu_3}{4\mu_0} \varphi(B)^{1/p} \psi(B), \tag{5.2.8}$$

and, for any measurable set $E \subset \nu_2 B \setminus \nu_1 B$ satisfying $|E|/|B| \le \nu_4$,

$$\left(\frac{1}{|B|} \int_{E} \left| [b,T] f^B(y) \right|^q dy \right)^{1/q} \le \frac{\nu_3}{4} \varphi(B)^{1/p} \psi(B). \tag{5.2.9}$$

The Riesz transforms fall under the scope of Lemma 5.2.5

Proof. **Step 1.** Since $K \in C^\infty(S^{n-1})$ and $K \not\equiv 0$, by normalization we may assume that $|K(y') - K(z')| \le |y' - z'|$ for all $y', z' \in S^{n-1}$ and that

$$\sigma(\{x' \in S^{n-1} : K(x') \ge 2\epsilon_1\}) > 0.$$

for some constant $\epsilon_1 \in (0,1)$, where σ is the area measure on S^{n-1}. Let

$$\Lambda = \{x' \in S^{n-1} : K(x') \ge 2\epsilon_1\}.$$

Then

$$y' \in \Lambda,\ z' \in S^{n-1} \text{ and } |y' - z'| \le \epsilon_1 \quad \Rightarrow \quad K(z') \ge \epsilon_1, \tag{5.2.10}$$

since $K(y') \ge 2\epsilon_1$ and $|K(y') - K(z')| \le |y' - z'| \le \epsilon_1$. Set $\ell = 2/\epsilon_1 > 2$.
Step 2. Let $B = B(x,r)$ satisfy $\mathrm{MO}(b,B)/\psi(B) \ge \epsilon_0$. We show that

$$|T((b - b_B)f^B)(y)| \ge \frac{\varphi(B)^{1/p}\psi(B)|B|}{(2|y-x|)^n}\epsilon_1\epsilon_0 \quad \text{for } y \notin \ell B \text{ and } \frac{y-x}{|y-x|} \in \Lambda, \tag{5.2.11}$$

$$|T((b - b_B)f^B)(y)| \le 2^n C_K \frac{\varphi(B)^{1/p}\psi(B)|B|}{|y-x|^n} \quad \text{for } y \notin \ell B, \tag{5.2.12}$$

$$|(b(y) - b_B)T(f^B)(y)| \le C_K \frac{r|b(y) - b_B|\varphi(B)^{1/p}|B|}{|y-x|^{n+1}} \quad \text{for } y \notin \ell B, \tag{5.2.13}$$

where the constant C_K is dependent only on the kernel K.

Now, for $y \notin \ell B$ and $z \in B$, we have

$$\left|\frac{y-x}{|y-x|} - \frac{y-z}{|y-z|}\right| \le \left|\frac{y-x}{|y-x|} - \frac{y-z}{|y-x|}\right| + \left|\frac{y-z}{|y-x|} - \frac{y-z}{|y-z|}\right| \le \frac{2|z-x|}{|y-x|} \le \frac{2}{\ell} = \epsilon_1.$$

In this case, if $\frac{y-x}{|y-x|} \in \Lambda$ also, then $K(\frac{y-z}{|y-z|}) \ge \epsilon_1$ by (5.2.10), and then

$$K(y-z) \ge \frac{\epsilon_1}{|y-z|^n} \ge \frac{\epsilon_1}{(2|y-x|)^n}.$$

Hence, from (5.2.3) and (5.2.4) it follows that, for $y \notin \ell B$ and $\frac{y-x}{|y-x|} \in \Lambda$,

$$|T((b - b_B)f^B)(y)| = \int_B K(y-z)(b(z) - b_B)f^B(z)\,dz \ge \frac{\varphi(B)^{1/p}|B|\mathrm{MO}(b,B)}{(2|y-x|)^n}\epsilon_1,$$

which shows (5.2.11), since $\mathrm{MO}(b,B) \ge \psi(B)\epsilon_0$. On the other hand, for $y \notin \ell B$ and $z \in B$, we have

$$|K(y-z)| \le \frac{C_K}{|y-z|^n} \le \frac{2^n C_K}{|y-x|^n}.$$

Then, from (5.2.3) and (5.2.4) it follows that, for $y \notin \ell B$,

$$|T((b-b_B)f^B)(y)| \le 2^n C_K \frac{\varphi(B)^{1/p}|B|\mathrm{MO}(b,B)}{|y-x|^n},$$

which shows (5.2.12), since $\|b\|_{\mathcal{L}_{1,\psi}} = 1$. Finally, from (5.2.2) and (5.2.5) it follows that, for $y \notin \ell B$,

$$\begin{aligned}
|(b(y)-b_B)T(f^B)(y)| &= \left|(b(y)-b_B)\int_B \big(K(y-z)f^B(z) - K(y-x)f^B(z)\big)\,dz\right| \\
&\le |b(y)-b_B| \int_B \frac{C_K|z-x|}{|y-x|^{n+1}}|f^B(z)|\,dz \\
&\le C_K \frac{r|b(y)-b_B|\varphi(B)^{1/p}|B|}{|y-x|^{n+1}},
\end{aligned}$$

which is (5.2.13).

Step 3. Let $\kappa = n - n/q > 0$. From the condition (3.1.1) it follows that $t \mapsto \psi(x,t)/t^{1-\theta}$ is almost decreasing for some constant $\theta \in (0,1)$, see [38, Lemma 2] or [45, Lemma 7.1]. In this step, using (5.2.13), we show

$$\text{(5.2.14)} \quad \left(\int_{\mathbb{R}^n \setminus 2^{j_0}B} |(b(y)-b_B)T(f^B)(y)|^q\,dy\right)^{1/q} \\ \le C_1 (2^{j_0})^{-\kappa-\theta} \varphi(B)^{1/p}|B|^{1/q}\psi(B),$$

where the constant C_1 is independent of B and $j_0 \in \mathbb{Z}$ satisfying $j_0 \ge \log_2 \ell$

By Lemma 5.2.1 and $\|b\|_{\mathcal{L}_{1,\psi}} = 1$ we have

$$\begin{aligned}
\left(\fint_{2^{j+1}B} |b(y)-b_B|^q\,dy\right)^{1/q} &\le \left(\fint_{2^{j+1}B} |b(y)-b_{2^{j+1}B}|^q\,dy\right)^{1/q} + |b_{2^{j+1}B} - b_B| \\
&\le c_n \int_r^{2^{j+2}r} \frac{\psi(x,t)}{t}\,dt, \quad j = 1, 2, \dots .
\end{aligned}$$

Then, for $j_0 \geq \log_2 \ell$, by (5.2.13),

$$\begin{aligned}
&\left(\int_{\mathbb{R}^n\setminus 2^{j_0}B} |(b(y)-b_B)T(f^B)(y)|^q\,dy\right)^{1/q} \\
&\leq C_K r\varphi(B)^{1/p}|B|\sum_{j=j_0}^{\infty}\left(\int_{2^{j+1}B\setminus 2^jB}\frac{|b(y)-b_B|^q}{|y-x|^{q(n+1)}}\,dy\right)^{1/q} \\
&\lesssim r\varphi(B)^{1/p}|B|\sum_{j=j_0}^{\infty}\frac{|2^{j+1}B|^{1/q}}{(2^jr)^{n+1}}\int_r^{2^{j+2}r}\frac{\psi(x,t)}{t}\,dt \\
&\lesssim r\varphi(B)^{1/p}|B|\int_{2^{j_0}r}^{\infty}s^{-n+n/q-2}\left(\int_r^s\frac{\psi(x,t)}{t}\,dt\right)ds.
\end{aligned}$$

Recall that $\kappa = n - n/q > 0$, and let

$$I_1 = \int_{2^{j_0}r}^{\infty} s^{-\kappa-2}\left(\int_r^{2^{j_0}r}\frac{\psi(x,t)}{t}\,dt\right)ds, \quad I_2 = \int_{2^{j_0}r}^{\infty} s^{-\kappa-2}\left(\int_{2^{j_0}r}^{s}\frac{\psi(x,t)}{t}\,dt\right)ds.$$

Then

$$\left(\int_{\mathbb{R}^n\setminus 2^{j_0}B} |(b(y)-b_B)T(f^B)(y)|^q\,dy\right)^{1/q} \lesssim r\varphi(B)^{1/p}|B|(I_1+I_2). \tag{5.2.15}$$

Using the almost decreasingness of $t \mapsto \psi(x,t)/t^{1-\theta}$, we have

$$\begin{aligned}
I_1 &= \frac{(2^{j_0}r)^{-\kappa-1}}{\kappa+1}\int_r^{2^{j_0}r}\frac{\psi(x,t)}{t}\,dt \lesssim (2^{j_0}r)^{-\kappa-1}\frac{\psi(x,r)}{r^{1-\theta}}\int_r^{2^{j_0}r}t^{-\theta}\,dt \\
&\lesssim (2^{j_0}r)^{-\kappa-1}\frac{\psi(x,r)}{r^{1-\theta}}(2^{j_0}r)^{1-\theta} \sim (2^{j_0})^{-\kappa-\theta}\frac{\psi(B)}{r}|B|^{-1+1/q}.
\end{aligned}$$

and

$$\begin{aligned}
I_2 &= \int_{2^{j_0}r}^{\infty}\frac{\psi(x,t)}{t}\left(\int_t^{\infty}s^{-\kappa-2}\,ds\right)dt = \int_{2^{j_0}r}^{\infty}\frac{\psi(x,t)}{t}\frac{t^{-\kappa-1}}{\kappa+1}\,dt \\
&\lesssim \frac{\psi(x,2^{j_0}r)}{(2^{j_0}r)^{1-\theta}}\int_{2^{j_0}r}^{\infty}t^{-\kappa-\theta-1}\,dt \lesssim \frac{\psi(x,r)}{r^{1-\theta}}(2^{j_0}r)^{-\kappa-\theta} \sim (2^{j_0})^{-\kappa-\theta}\frac{\psi(B)}{r}|B|^{-1+1/q}.
\end{aligned}$$

Hence, combining (5.2.15) with the estimates of I_1 and I_2, we have (5.2.14).

Step 4. Recall that $\kappa = n - n/q > 0$. We show (5.2.7) and (5.2.8). From (5.2.11) and (5.2.14) it follows that, for $j_1 > j_0$,

$$
\begin{aligned}
&\left(\int_{2^{j_1}B\setminus 2^{j_0}B} |[b,T]f^B(y)|^q\,dy\right)^{1/q}\\
&\geq \left(\int_{2^{j_1}B\setminus 2^{j_0}B} |T((b(y)-b_B)f^B)(y)|^q\,dy\right)^{1/q}\\
&\quad - \left(\int_{\mathbb{R}^n\setminus 2^{j_0}B} |(b(y)-b_B)T(f^B)(y)|^q\,dy\right)^{1/q}\\
&\geq \varphi(B)^{1/p}\psi(B)|B|\left(\int_{(2^{j_1}B\setminus 2^{j_0}B)\cap\left\{y:\frac{y-x}{|y-x|}\in\Lambda\right\}} \frac{1}{(2|y-x|)^{nq}}\,dy\right)^{1/q}\epsilon_1\epsilon_0\\
&\quad - C_1(2^{j_0})^{-\kappa-\theta}\varphi(B)^{1/p}|B|^{1/q}\psi(B)\\
&\geq \varphi(B)^{1/p}|B|^{1/q}\psi(B)\left(C_2\big((2^{j_0})^{-\kappa q}-(2^{j_1})^{-\kappa q}\big)^{1/q}\epsilon_1\epsilon_0 - C_1(2^{j_0})^{-\kappa-\theta}\right),
\end{aligned}
$$

where the constant C_2 is independent of B, j_0 and j_1. From (5.2.12) and (5.2.14) it follows that

$$
\begin{aligned}
&\left(\int_{\mathbb{R}^n\setminus 2^{j_1}B} |[b,T]f^B(y)|^q\,dy\right)^{1/q}\\
&\leq 2^nC\varphi(B)^{1/p}\psi(B)|B|\left(\int_{\mathbb{R}^n\setminus 2^{j_1}B} \frac{1}{|y-x|^{nq}}\,dy\right)^{1/q}\\
&\quad + C_1(2^{j_1})^{-\kappa-\theta}\varphi(B)^{1/p}|B|^{1/q}\psi(B)\\
&\leq \varphi(B)^{1/p}|B|^{1/q}\psi(B)\left(C_3(2^{j_1})^{-\kappa}+C_1(2^{j_1})^{-\kappa-\theta}\right),
\end{aligned}
$$

where the constant C_3 is independent of B, j_0 and j_1. Therefore, we can choose $\nu_1 = 2^{j_0}$, $\nu_2 = 2^{j_1}$ and $\nu_3 > 0$ such that (5.2.7) and (5.2.8) hold.

Step 5. We show (5.2.9). Let $E \subset \nu_2 B\setminus \nu_1 B$. From (5.2.12) and (5.2.13) it follows that

$$
\begin{aligned}
&\left(\int_E |[b,T]f^B(y)|^q\,dy\right)^{1/q} \qquad (5.2.16)\\
&\leq 2^nC_K\varphi(B)^{1/p}\psi(B)|B|\left(\int_E \frac{1}{|y-x|^{nq}}\,dy\right)^{1/q}\\
&\quad + C_Kr\varphi(B)^{1/p}|B|\left(\int_E \frac{|b(y)-b_B|^q}{|y-x|^{(n+1)q}}\,dy\right)^{1/q}\\
&\leq C_{K,n}(\nu_1)^{-n}\varphi(B)^{1/p}\psi(B)|E|^{1/q}
\end{aligned}
$$

$$+ C_{K,n}(\nu_1)^{-n-1}\varphi(B)^{1/p}\left(\int_E |b(y)-b_B|^q\,dy\right)^{1/q}.$$

Let $\tilde{b} = b - b_B$, and let

$$\lambda(\omega) = |\{x \in E : |\tilde{b}(x)| > \omega\}| \quad \text{and} \quad \tilde{b}^*(t) = \inf\{\omega > 0 : \lambda(\omega) \le t\}.$$

Since $E \subset \nu_2 B$, by Corollary 5.2.3 we have

$$\lambda(\omega + A_0\nu_2\psi(B)) \le A_1{\nu_2}^n|B|\exp\left(-A_2\omega/(\nu_2\psi(B))\right).$$

Hence

$$\lambda(\omega) \le A_1{\nu_2}^n|B|\exp\left(-A_2(\omega - A_0\nu_2\psi(B))/(\nu_2\psi(B))\right).$$

Since

$$t = A_1{\nu_2}^n|B|\exp\left(-A_2(\omega - A_0\nu_2\psi(B))/(\nu_2\varphi(B))\right)$$
$$\Leftrightarrow \quad \omega = \nu_2\psi(B)\left(A_0 + \frac{1}{A_2}\log\frac{A_1{\nu_2}^n|B|}{t}\right),$$

we see that

$$\tilde{b}^*(t) \le \nu_2\psi(B)\left(A_0 + \frac{1}{A_2}\log\frac{A_1{\nu_2}^n|B|}{t}\right) \le A_3\nu_2\psi(B)\left(1 + \log\frac{A_1{\nu_2}^n|B|}{t}\right),$$

with $A_3 = \max(1, A_0)/\min(1, A_2)$. Then

(5.2.17)
$$\begin{aligned}\int_E |b(x)-b_B|^q\,dx &\le \int_0^{|E|}(\tilde{b}^*(t))^q\,dt\\ &\le (A_3\nu_2\psi(B))^q\int_0^{|E|}\left(1+\log\frac{A_1{\nu_2}^n|B|}{t}\right)^q dt\\ &\le (A_3\nu_2\psi(B))^q A_1{\nu_2}^n|B|\int_0^{|E|/(A_1{\nu_2}^n|B|)}\left(1+\log\frac{1}{t}\right)^q dt.\end{aligned}$$

Since

$$\left(1+\log\frac{1}{t}\right)^q \le 2\frac{d}{dt}\left(t\left(1+\log\frac{1}{t}\right)^q\right), \quad 0 < t \le e^{-2q},$$

if $|E|/(A_1{\nu_2}^n|B|) \le e^{-2q}$, then

$$\int_0^{|E|/(A_1{\nu_2}^n|B|)}\left(1+\log\frac{1}{t}\right)^q dt \le \frac{2|E|}{A_1{\nu_2}^n|B|}\left(1+\log\frac{A_1{\nu_2}^n|B|}{|E|}\right)^q. \tag{5.2.18}$$

Combining (5.2.16), (5.2.17) and (5.2.18), we have

$$\left(\int_E |[b,T]f^B(y)|^q\,dy\right)^{1/q} \leq C\varphi(B)^{1/p}|B|^{1/q}\psi(B)\left(\frac{|E|}{|B|}\right)^{1/q}\left(1+\log\frac{A_1{\nu_2}^n|B|}{|E|}\right),$$

where C is dependent only on n, A_0, A_2, ν_1 and ν_2. Therefore, we can choose $\nu_4 \in (0,1)$ such that (5.2.9) holds whenever $|E|/|B| \leq \nu_4$. □

Lemma 5.2.6. *Let $p, q \in (1,\infty)$ and $\alpha \in (0,n)$. Assume that $\varphi \in \mathcal{G}^{\mathrm{dec}}$ and $\psi \in \mathcal{G}^{\mathrm{inc}}$. Assume also that ψ satisfies* (3.1.1) *and that $n-\alpha-n/q>0$. Let b be a real valued function and $\|b\|_{\mathcal{L}_{1,\psi}}=1$. For any ball B, define f^B by* (5.2.1). *Then, for any constants $\epsilon_0, \mu_0 \in (0,\infty)$, there exist constants $\nu_1, \nu_2 \in [2,\infty)$ $(\nu_1<\nu_2)$, $\nu_3 \in (0,\infty)$ and $\nu_4 \in (0,1)$ such that, for all balls B satisfying $\mathrm{MO}(b,B)/\psi(B) \geq \epsilon_0$, the following three inequalities hold:*

$$\left(\frac{1}{|B|}\int_{\nu_2 B\setminus\nu_1 B}\left|[b,I_\alpha]f^B(y)\right|^q dy\right)^{1/q} \geq \nu_3\varphi(B)^{1/p}|B|^{\alpha/n}\psi(B), \tag{5.2.19}$$

$$\left(\frac{1}{|B|}\int_{\mathbb{R}^n\setminus\nu_2 B}\left|[b,I_\alpha]f^B(y)\right|^q dy\right)^{1/q} \leq \frac{\nu_3}{4\mu_0}\varphi(B)^{1/p}|B|^{\alpha/n}\psi(B), \tag{5.2.20}$$

and, for any measurable set $E \subset \nu_2 B\setminus\nu_1 B$ satisfying $|E|/|B| \leq \nu_4$,

$$\left(\frac{1}{|B|}\int_E\left|[b,I_\alpha]f^B(y)\right|^q dy\right)^{1/q} \leq \frac{\nu_3}{4}\varphi(B)^{1/p}|B|^{\alpha/n}\psi(B). \tag{5.2.21}$$

Proof. Let $B=B(x,r)$ satisfy $\mathrm{MO}(b,B)/\psi(B) \geq \epsilon_0$. For $y \notin 2B$ and $z \in B$, we have

$$\frac{1}{(2|y-x|)^{n-\alpha}} \leq \frac{1}{|y-z|^{n-\alpha}} \leq \frac{1}{(|y-x|/2)^{n-\alpha}}.$$

From (5.2.3), (5.2.4), $\|b\|_{\mathcal{L}_{1,\psi}}=1$ and $\mathrm{MO}(b,B) \geq \psi(B)\epsilon_0$ it follows that, for $y \notin 2B$,

$$|I_\alpha((b-b_B)f^B)(y)| = \int_B \frac{(b(z)-b_B)f^B(z)}{|y-z|^{n-\alpha}}\,dz \leq \frac{\varphi(B)^{1/p}\psi(B)|B|}{(|y-x|/2)^{n-\alpha}}, \tag{5.2.22}$$

$$|I_\alpha((b-b_B)f^B)(y)| = \int_B \frac{(b(z)-b_B)f^B(z)}{|y-z|^{n-\alpha}}\,dz \geq \frac{\varphi(B)^{1/p}\psi(B)|B|}{(2|y-x|)^{n-\alpha}}\epsilon_0. \tag{5.2.23}$$

From (5.2.2) and (5.2.5) it follows that, for $y \notin 2B$,

$$
\begin{aligned}
(5.2.24)\quad |(b(y)-b_B)I_\alpha(f^B)(y)| &= \left|(b(y)-b_B)\int_B \frac{f^B(z)}{|y-z|^{n-\alpha}}\,dz\right| \\
&= \left|(b(y)-b_B)\int_B \left(\frac{f^B(z)}{|y-z|^{n-\alpha}} - \frac{f^B(z)}{|y-x|^{n-\alpha}}\right)dz\right| \\
&\le \frac{r|b(y)-b_B|}{(n-\alpha)(|y-x|/2)^{n-\alpha+1}}\int_B |f^B(z)|\,dz \\
&\le \frac{r|b(y)-b_B|\varphi(B)^{1/p}|B|}{(n-\alpha)(|y-x|/2)^{n-\alpha+1}}.
\end{aligned}
$$

Next, let $\kappa = n-\alpha-n/q > 0$. Then in a similar way to Step 3 in the proof of Lemma 5.2.5, instead of (5.2.14), we have that

$$
\begin{aligned}
(5.2.25)\quad &\left(\int_{\mathbb{R}^n\setminus 2^{j_0}B} |(b(y)-b_B)I_\alpha(f^B)(y)|^q\,dy\right)^{1/q} \\
&\qquad\qquad \le C_1(2^{j_0})^{-\kappa-\theta}\varphi(B)^{1/p}|B|^{\alpha/n+1/q}\psi(B),
\end{aligned}
$$

for some $\theta \in (0,1)$, where the constant C_1 is independent of B and j_0. Moreover, in a similar way to Steps 4 and 5 in the proof of Lemma 5.2.5, using (5.2.22)–(5.2.25), we have (5.2.19), (5.2.20) and (5.2.21). □

5.3 Proofs of the theorems

In this section, we prove Theorem 5.1.1 by using Theorem 4.1.1 and Lemma 5.2.5. We omit the proof of Theorem 5.1.2, since we can prove it in the same way as Theorem 5.1.1 by using Lemma 5.2.6 instead of Lemma 5.2.5.

Proof of Theorem 5.1.1. Since $[b,T]$ is compact from $L^{(p,\varphi)}(\mathbb{R}^n)$ to $L^{(q,\varphi)}(\mathbb{R}^n)$, then $b \in \mathcal{L}_{1,\psi}(\mathbb{R}^n)$ by Theorem 2.1.1 (ii). We may assume that $\|b\|_{\mathcal{L}_{1,\psi}} = 1$. Below we show that b must satisfy the conditions (i), (ii) and (iii) in Theorem 4.1.1.

Part 1. Firstly, we show that, if b does not satisfy the condition (i), then $[b,T]$ is not compact. Since b does not satisfy the condition (i), there exist $\epsilon_0 > 0$ and a sequence of balls $\{B_j\}_{j=1}^\infty = \{B(x_j,r_j)\}_{j=1}^\infty$ with $\lim_{j\to\infty} r_j = 0$ such that, for every j,

$$
(5.3.1)\qquad \frac{\mathrm{MO}(b,B_j)}{\psi(B_j)} > \epsilon_0.
$$

For every B_j, we define $f_j = f^{B_j}$ by (5.2.1). Then

$$\sup_j \|f_j\|_{L^{(p,\varphi)}} \le C$$

by Lemma 5.2.4. If we can choose a subsequence $\{f_{j(k)}\}_{k=1}^{\infty}$ such that $\{[b,T]f_{j(k)}\}_{k=1}^{\infty}$ has no any convergence subsequence in $L^{(q,\varphi)}(\mathbb{R}^n)$, then we have the conclusion.

Now, for the constant ϵ_0 in (5.3.1), let ν_i $(i = 1, 2, 3, 4)$ be the constants defined by Lemma 5.2.5. By $\lim_{j\to\infty} r_j = 0$ and the assumption (5.1.1) we may choose a subsequence $\{B_{j(k)}\}$ such that

$$\frac{|B_{j(k+1)}|}{|B_{j(k)}|} < \frac{\nu_4}{{\nu_2}^n} \tag{5.3.2}$$

and

$$\varphi(B_{j(k+1)})^{1/p}\psi(B_{j(k+1)})|B_{j(k+1)}|^{1/q} \le \mu_0\varphi(B_{j(k)})^{1/p}\psi(B_{j(k)})|B_{j(k)}|^{1/q}. \tag{5.3.3}$$

Then the subsequence $\{f_{j(k)}\}$ associated with $\{B_{j(k)}\}$ is just what we request. Namely, there exists a positive constant δ such that, for any $k, \ell \in \mathbb{N}$ with $k < \ell$,

$$\|[b,T]f_{j(k)} - [b,T]f_{j(\ell)}\|_{L^{(q,\varphi)}} \ge \delta. \tag{5.3.4}$$

In fact, for fixed $k, \ell \in \mathbb{N}$ with $k < \ell$, denote

$$G = \nu_2 B_{j(k)} \setminus \nu_1 B_{j(k)}, \quad E = G \cap \nu_2 B_{j(\ell)}.$$

Then by (5.3.2) we have

$$\frac{|E|}{|B_{j(k)}|} \le \frac{|\nu_2 B_{j(\ell)}|}{|B_{j(k)}|} < \nu_4.$$

From the relation $G \setminus E = G \setminus \nu_2 B_{j(\ell)} \subset \nu_2 B_{j(k)} \cap (\nu_2 B_{j(\ell)})^{\complement}$ it follows that

(5.3.5)

$$\begin{aligned}\left(\int_G |[b,T]f_{j(k)}|^q\,dx - \int_E |[b,T]f_{j(k)}|^q\,dx\right)^{\frac{1}{q}} &= \left(\int_{G\setminus \nu_2 B_{j(\ell)}} |[b,T]f_{j(k)}|^q\,dx\right)^{\frac{1}{q}} \\ &\le \left(\int_{\nu_2 B_{j(k)}} |[b,T]f_{j(k)} - [b,T]f_{j(\ell)}|^q\,dx\right)^{\frac{1}{q}} + \left(\int_{(\nu_2 B_{j(\ell)})^{\complement}} |[b,T]f_{j(\ell)}|^q\,dx\right)^{\frac{1}{q}}.\end{aligned}$$

By (5.2.7), (5.2.8), (5.2.9) and (5.3.3) we have

$$\int_G |[b,T]f_{j(k)}|^q\,dx \ge \left(\nu_3\,\varphi(B_{j(k)})^{1/p}\psi(B_{j(k)})\right)^q |B_{j(k)}|, \tag{5.3.6}$$

$$\begin{aligned}\left(\int_{(\nu_2 B_{j(\ell)})^{\complement}} |[b,T]f_{j(\ell)}|^q\,dx\right)^{\frac{1}{q}} &\le \frac{\nu_3}{4\mu_0}\varphi(B_{j(\ell)})^{1/p}\psi(B_{j(\ell)})|B_{j(\ell)}|^{1/q}\\ &\le \frac{\nu_3}{4}\varphi(B_{j(k)})^{1/p}\psi(B_{j(k)})|B_{j(k)}|^{1/q},\end{aligned} \tag{5.3.7}$$

$$\int_E |[b,T]f_{j(k)}|^q\,dx \le \left(\frac{\nu_3}{4}\varphi(B_{j(k)})^{1/p}\psi(B_{j(k)})\right)^q |B_{j(k)}|. \tag{5.3.8}$$

Combining (5.3.5)–(5.3.8), we have

$$\begin{aligned}&\left({\nu_3}^q-\left(\nu_3/4\right)^q\right)^{1/q}\varphi(B_{j(k)})^{1/p}\psi(B_{j(k)})|B_{j(k)}|^{1/q}\\ &\quad\le\left(\int_{\nu_2 B_{j(k)}}|[b,T]f_{j(k)}-[b,T]f_{j(\ell)}|^q\,dx\right)^{\frac{1}{q}}+\frac{\nu_3}{4}\varphi(B_{j(k)})^{1/p}\psi(B_{j(k)})|B_{j(k)}|^{1/q},\end{aligned}$$

which shows

$$\delta_0\,\varphi(B_{j(k)})^{1/p}\psi(B_{j(k)})|B_{j(k)}|^{1/q}\le\left(\int_{\nu_2 B_{j(k)}}|[b,T]f_{j(k)}-[b,T]f_{j(\ell)}|^q\,dx\right)^{\frac{1}{q}},$$

where $\delta_0=\left({\nu_3}^q-\left(\nu_3/4\right)^q\right)^{1/q}-\nu_3/4>0$. Thus, using (2.1.8) and the almost decreasingness of φ, we have

$$\left(\frac{1}{\varphi(\nu_2 B_{j(k)})}⨍_{\nu_2 B_{j(k)}}|[b,T]f_{j(k)}-[b,T]f_{j(\ell)}|^q\,dx\right)^{\frac{1}{q}}\ge\delta,$$

where δ is independent on m and ℓ, which shows (5.3.4).

Part 2. Secondly, we show that, if b does not satisfy the condition (ii), then $[b,T]$ is not compact. Since b does not satisfy the condition (ii), there exist $\epsilon_0>0$ and a sequence of balls $\{B_j\}_{j=1}^{\infty}=\{B(x_j,r_j)\}_{j=1}^{\infty}$ with $\lim_{j\to\infty} r_j=\infty$ such that, for every j,

$$\frac{\mathrm{MO}(b,B_j)}{\psi(B_j)}>\epsilon_0.$$

For every B_j, we define $f_j=f^{B_j}$ by (5.2.1). Then

$$\sup_j\|f_j\|_{L^{(p,\varphi)}}\le C$$

by Lemma 5.2.4. By $\lim_{j\to 0} r_j = \infty$ and the assumption (5.1.2) we may choose a subsequence $\{B_{j(k)}\}_{k=1}^{\infty}$ such that

$$\frac{|B_{j(k)}|}{|B_{j(k+1)}|} < \frac{\nu_4}{{\nu_2}^n}$$

and

$$\varphi(B_{j(k)})^{1/p}\psi(B_{j(k)})|B_{j(k)}|^{1/q} \le \mu_0\varphi(B_{j(k+1)})^{1/p}\psi(B_{j(k+1)})|B_{j(k+1)}|^{1/q}.$$

Then, in a similar way to Step 1 we conclude that there exists a positive constant δ such that, for all $k, \ell \in \mathbb{N}$ with $k < \ell$,

$$\left(\frac{1}{\varphi(\nu_2 B_{j(\ell)})} \fint_{\nu_2 B_{j(\ell)}} |[b,T]f_{j(\ell)} - [b,T]f_{j(k)}|^q \, dx\right)^{\frac{1}{q}} \ge \delta.$$

That is, $[b,T]$ is not compact.

Part 3. Finally, we show that, if b does not satisfy the condition (iii), then $[b,T]$ is not compact. Since b does not satisfy the condition (iii), there exist $\epsilon_0 > 0$ and a sequence of balls $\{B_j\}_{j=1}^{\infty} = \{B(x_j, r)\}_{j=1}^{\infty}$ with $\lim_{j\to\infty} |x_j| = \infty$ such that, for every j,

$$\frac{\mathrm{MO}(b, B_j)}{\psi(B_j)} > \epsilon_0.$$

By $\lim_{j\to 0} |x_j| = \infty$ and the assumption (5.1.3) we may choose a subsequence $\{B_{j(k)}\}_{k=1}^{\infty}$ such that $\nu_2 B_{j(k)} \cap \nu_2 B_{j(k+1)} = \emptyset$ and

$$\varphi(B_{j(k+1)})^{1/p}\psi(B_{j(k+1)})|B_{j(k+1)}|^{1/q} \le \mu_0\varphi(B_{j(k)})^{1/p}\psi(B_{j(k)})|B_{j(k)}|^{1/q}.$$

Then, in a similar way to Step 1 we conclude that $[b,T]$ is not compact. $\square$

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