DOCTORAL THESIS

Commutators of integral operators with functions in generalized Campanato spaces on Orlicz and Orlicz-Morrey spaces

オーリッツ空間およびオーリッツ・モリー空間上における積分作用素と一般化カンパナト空間に属する関数との交換子

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Abstract

Let $\mathbb{R}^n$ be the n-dimensional Euclidean space. Let $b \in \mathrm{BMO}(\mathbb{R}^n)$ and T be a Calderón-Zygmund singular integral operator. In 1976 Coifman, Rochberg and Weiss proved that the commutator $[b,T] = bT - Tb$ is bounded on $L^p(\mathbb{R}^n)$ $(1 < p < \infty)$, that is,

$$\|[b,T]f\|_{L^p} = \|bTf - T(bf)\|_{L^p} \le C\|b\|_{\mathrm{BMO}}\|f\|_{L^p},$$

where C is a positive constant independent of b and f. For the fractional integral operator I_α, Chanillo proved the boundedness of $[b, I_\alpha]$ in 1982. Coifman, Rochberg and Weiss and Chanillo also gave the necessary conditions for the boundedness. These results were extended to Orlicz spaces by Janson in 1978, and to Morrey spaces by Di Fazio and Ragusa in 1991.

In this paper we consider the commutators $[b,T]$ and $[b,I_\rho]$, where T is a Calderón-Zygmund operator, I_ρ is a generalized fractional integral operator and b is a function in generalized Campanato spaces. We consider the boundedness of $[b,T]$ and $[b,I_\rho]$ on Orlicz and Orlicz-Morrey spaces. Orlicz and Orlicz-Morrey spaces unify several function spaces, and the Campanato spaces unify BMO and Lipschitz spaces. Therefore, our results contain many previous results as corollaries.

Firstly, we consider generalized fractional integral operators I_ρ on Orlicz spaces. The operator I_ρ was introduced by Nakai in 2000 to extend the Hardy-Littlewood-Sobolev theorem to Orlicz spaces. We first investigate the commutator $[b, I_\rho]$ on Orlicz spaces $L^\Phi(\mathbb{R}^n)$. We prove the boundedness

$$\|[b,I_\rho]f\|_{L^\Psi} \le C\|b\|_{\mathcal{L}_{1,\psi}}\|f\|_{L^\Phi}, \quad f \in C^\infty_{\mathrm{comp}}(\mathbb{R}^n),$$

and use the density of $C^\infty_{\mathrm{comp}}(\mathbb{R}^n)$ in $L^\Phi(\mathbb{R}^n)$ to obtain the boundedness from $L^\Phi(\mathbb{R}^n)$ to $L^\Psi(\mathbb{R}^n)$, where $\mathcal{L}_{1,\psi}(\mathbb{R}^n)$ is the generalized Campanato space. To prove the

boundedness, we need a generalizetion of Young functions. We give the definition of generalizied Young functions and investigate their properties. We also prove that, if $[b, I_\rho]$ is bounded from $L^\Phi(\mathbb{R}^n)$ to $L^\Psi(\mathbb{R}^n)$, then b is in $\mathcal{L}_{1,\psi}(\mathbb{R}^n)$ and the norm $\|b\|_{\mathcal{L}_{1,\psi}}$ is dominated by the operator norm.

Next, we investigate $[b, T]$ and $[b, I_\rho]$ on Orlicz-Morrey spaces. We prove the boundedness

$$\|[b, T]f\|_{L^{(\Psi,\varphi)}} \le C\|b\|_{\mathcal{L}_{1,\psi}}\|f\|_{L^{(\Phi,\varphi)}},$$
$$\|[b, I_\rho]f\|_{L^{(\Psi,\varphi)}} \le C\|b\|_{\mathcal{L}_{1,\psi}}\|f\|_{L^{(\Phi,\varphi)}},$$

under suitable assumptions. In this case, we need to show the well-definedness of commutators carefully, since neither $C^\infty_{\mathrm{comp}}(\mathbb{R}^n)$ nor $L^\infty_{\mathrm{comp}}(\mathbb{R}^n)$ is always dense in Orlicz-Morrey spaces. We also prove that, if $[b, T]$ or $[b, I_\rho]$ is bounded from $L^{(\Phi,\varphi)}(\mathbb{R}^n)$ to $L^{(\Psi,\varphi)}(\mathbb{R}^n)$, then b is in $\mathcal{L}_{1,\psi}(\mathbb{R}^n)$ and the norm $\|b\|_{\mathcal{L}_{1,\psi}}$ is dominated by the operator norm.

To prove the boundedness of the commutators we need the generalized fractional maximal operator M_ρ and the sharp maximal operator $M^\sharp$. It is known that the usual fractional maximal operator M_α is dominated pointwise by the fractional integral operator I_α, that is, $M_\alpha f(x) \le C I_\alpha|f|(x)$ for all $x \in \mathbb{R}^n$. Then the boundedness of M_α follows from one of I_α. However, we need a better estimate on M_ρ than I_ρ to prove the boundedness of the commutators. In this paper we give a necessary and sufficient condition for the boundedness of M_ρ. We also prove the norm estimates of the commutators $[b, T]f$ and $[b, I_\rho]f$ by their sharp maximal operator $M^\sharp([b, T]f)$ and $M^\sharp([b, I_\rho]f)$, respectively. To do this we investigate the relation between Orlicz-Campanato and Orlicz-Morrey spaces. Moreover, we show the pointwise estimates of the sharp maximal operators by the combinations of the generalized fractional maximal operators. Finally, combining all of these results, we prove the boundedness of the commutators.

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Contents

Chapter 1

Introduction

1.1 Introduction

Let $\mathbb{R}^n$ be the n-dimensional Euclidean space. Let $b \in \mathrm{BMO}(\mathbb{R}^n)$ and T be a Calderón-Zygmund singular integral operator. In 1976 Coifman, Rochberg and Weiss [5] proved that the commutator $[b,T] = bT - Tb$ is bounded on $L^p(\mathbb{R}^n)$ $(1 < p < \infty)$, that is,

$$\|[b,T]f\|_{L^p} = \|bTf - T(bf)\|_{L^p} \le C\|b\|_{\mathrm{BMO}}\|f\|_{L^p},$$

where C is a positive constant independent of b and f. For the fractional integral operator I_α, Chanillo [3] proved the boundedness of $[b, I_\alpha]$ in 1982. Coifman, Rochberg and Weiss [5] and Chanillo [3] also gave the necessary conditions for the boundedness. These results were extended to Orlicz spaces by Janson [17] (1978) and to Morrey spaces by Di Fazio and Ragusa [8] (1991). For other extensions and generalization, see [1, 9, 11, 12, 14, 27, 25, 31, 42, 52, 53, 55, 56], etc.

In this paper we consider the commutators $[b,T]$ and $[b, I_\rho]$, where T is a Calderón-Zygmund operator, I_ρ is the generalized fractional integral operator and b is a function in generalized Campanato spaces. We consider the boundedness of $[b,T]$ and $[b, I_\rho]$ on Orlicz and Orlicz-Morrey spaces. The Orlicz and Orlicz-Morrey spaces unify several function spaces, and the Campanato spaces unify BMO and Lipschitz spaces. Therefore, our results contain many previous results as corollaries.

This paper is a systematic reconstruction of all results in [50, 51] and some results of [6]. Related results are in [19].

Firstly, we consider generalized fractional integral operators I_ρ on Orlicz spaces in Chapter 2. For a function $\rho : (0,\infty) \to (0,\infty)$, the generalized functional integral operator I_ρ is defined by

$$I_\rho f(x) = \int_{\mathbb{R}^n} \frac{\rho(|x-y|)}{|x-y|^n} f(y)\, dy, \quad x \in \mathbb{R}^n, \tag{1.1.1}$$

where we always assume that

$$\int_0^1 \frac{\rho(t)}{t}\, dt < \infty. \tag{1.1.2}$$

Condition (1.1.2) is needed for the integral in (1.1.1) to converge for bounded measurable functions f with compact support. See Lemma 2.6.1 also. In this paper we also assume that there exist positive constants C, K_1 and K_2 with $K_1 < K_2$ such that, for all $r > 0$,

$$\sup_{r \le t \le 2r} \rho(t) \le C \int_{K_1 r}^{K_2 r} \frac{\rho(t)}{t}\, dt. \tag{1.1.3}$$

The condition above was considered in [46].
If $\rho(r) = r^\alpha$, $0 < \alpha < n$, then I_ρ is the usual fractional integral operator I_α defined by

$$I_\alpha f(x) = \int_{\mathbb{R}^n} \frac{f(y)}{|x-y|^{n-\alpha}}\, dy, \quad x \in \mathbb{R}^n.$$

It is known as the Hardy-Littlewood-Sobolev theorem that I_α is bounded from $L^p(\mathbb{R}^n)$ to $L^q(\mathbb{R}^n)$, if $\alpha \in (0,n)$, $p,q \in (1,\infty)$ and $-n/p + \alpha = -n/q$. This boundedness was extended to Orlicz spaces by several authors, see [4, 10, 23, 43, 54, 57, 58], etc. Chanillo [3] considerd the commutator

$$[b, I_\alpha]f = bI_\alpha f - I_\alpha(bf),$$

with $b \in \mathrm{BMO}$ and proved that $[b, I_\alpha]$ has the same boundedness as I_α. The result was also extended to Orlicz spaces by Fu, Yang and Yuan [12] and Guliyev, Deringoz and Hasanov [14].

The operator I_ρ was introduced in [33] to extend the Hardy-Littlewood-Sobolev theorem to Orlicz spaces whose partial results were announced in [32]. For example, the generalized fractional integral I_ρ is bounded from $\exp L^p(\mathbb{R}^n)$ to $\exp L^q(\mathbb{R}^n)$, where

$$\rho(r) = \begin{cases} 1/(\log(1/r))^{\alpha+1} & \text{for small } r, \\ (\log r)^{\alpha-1} & \text{for large } r, \end{cases} \quad \alpha > 0, \tag{1.1.4}$$

$p, q \in (0, \infty)$, $-1/p + \alpha = -1/q$ and $\exp L^p(\mathbb{R}^n)$ is the Orlicz space $L^\Phi(\mathbb{R}^n)$ with

$$\Phi(r) = \begin{cases} 1/\exp(1/r^p) & \text{for small } r, \\ \exp(r^p) & \text{for large } r. \end{cases} \tag{1.1.5}$$

See also [34, 35, 36, 38, 41].

We first investigate the commutator $[b, I_\rho]$ on Orlicz spaces $L^\Phi(\mathbb{R}^n)$. We prove the boundedness

$$\|[b, I_\rho]f\|_{L^\Psi} \le C\|b\|_{\mathcal{L}_{1,\psi}}\|f\|_{L^\Phi}, \quad f \in C^\infty_{\rm comp}(\mathbb{R}^n),$$

and use the density of $C^\infty_{\rm comp}(\mathbb{R}^n)$ in $L^\Phi(\mathbb{R}^n)$, where $\mathcal{L}_{1,\psi}(\mathbb{R}^n)$ is the generalized Campanato space, see the next section for its definition. To prove the boundedness we need a generalizetion of Young functions. We give the definition of generalzied Young functions and investigate their properties. We also prove that, if $[b, I_\rho]$ is bounded from $L^\Phi(\mathbb{R}^n)$ to $L^\Psi(\mathbb{R}^n)$, then b is in $\mathcal{L}_{1,\psi}(\mathbb{R}^n)$ and the norm $\|b\|_{\mathcal{L}_{1,\psi}}$ is dominated by the operator norm.

Next, we investigate $[b, T]$ and $[b, I_\rho]$ on Orlicz-Morrey spaces in Chapter 3. We prove the boundedness

$$\|[b, T]f\|_{L^{(\Psi,\varphi)}} \le C\|b\|_{\mathcal{L}_{1,\psi}}\|f\|_{L^{(\Phi,\varphi)}},$$
$$\|[b, I_\rho]f\|_{L^{(\Psi,\varphi)}} \le C\|b\|_{\mathcal{L}_{1,\psi}}\|f\|_{L^{(\Phi,\varphi)}},$$

under suitable assumptions. In this case, we need to show the well-definedness of commutators carefully, since neither $C^\infty_{\rm comp}(\mathbb{R}^n)$ nor $L^\infty_{\rm comp}(\mathbb{R}^n)$ is always dense in Orlicz-Morrey spaces. We also prove that, if $[b, T]$ or $[b, I_\rho]$ is bounded from $L^{(\Phi,\varphi)}(\mathbb{R}^n)$ to $L^{(\Psi,\varphi)}(\mathbb{R}^n)$, then b is in $\mathcal{L}_{1,\psi}(\mathbb{R}^n)$ and the norm $\|b\|_{\mathcal{L}_{1,\psi}}$ is dominated by the operator norm.

We denote by $B(x, r)$ the open ball centered at $x \in \mathbb{R}^n$ and of radius r, that is,

$$B(x, r) = \{y \in \mathbb{R}^n : |y - x| < r\}.$$

For a measurable set $G \subset \mathbb{R}^n$, we denote by $|G|$ and χ_G the Lebesgue measure of G and the characteristic function of G, respectively. For a function $f \in L^1_{\rm loc}(\mathbb{R}^n)$ and a ball B, let

$$f_B = ⨍_B f = ⨍_B f(y)\,dy = \frac{1}{|B|}\int_B f(y)\,dy.$$

To prove the boundedness of commutators we need the sharp maximal operator $M^\sharp$ and the generalized fractional integral operator M_ρ. The sharp maximal operator $M^\sharp$ is defined by

$$M^\sharp f(x) = \sup_{B \ni x} \fint_B |f(y) - f_B|\, dy, \quad x \in \mathbb{R}^n, \tag{1.1.6}$$

where the supremum is taken over all balls B containing x. For a function $\rho : (0,\infty) \to (0,\infty)$, let

$$M_\rho f(x) = \sup_{B(z,r) \ni x} \rho(r) \fint_{B(z,r)} |f(y)|\, dy, \quad x \in \mathbb{R}^n, \tag{1.1.7}$$

where the supremum is taken over all balls $B(z,r)$ containing x. We do not assume the condition (1.1.2) or (1.1.3) on the definition of M_ρ. The operator M_ρ was studied in [48] on generalized Morrey spaces. If $\rho(r) = |B(0,r)|^{\alpha/n}$, then M_ρ is the usual fractional maximal operator M_α. If $\rho \equiv 1$, then M_ρ is the Hardy-Littlewood maximal operator M, that is,

$$Mf(x) = \sup_{B \ni x} \fint_B |f(y)|\, dy, \quad x \in \mathbb{R}^n.$$

It is known that the usual fractional maximal operator M_α is dominated pointwise by the fractional integral operator I_α, that is, $M_\alpha f(x) \le C I_\alpha |f|(x)$ for all $x \in \mathbb{R}^n$. Then the boundedness of M_α follows from one of I_α. However, we need a better estimate on M_ρ than I_ρ to prove the boundedness of the commutators. In this paper we give a necessary and sufficient condition of the boundedness of M_ρ.

The organization of this paper is as follows. In the next section in this chapter we give the definitions of the generalized Campanato spaces, generalized Young functions and Orlicz and Orlicz-Morrey spaces.

In Chapter 2 we give a necessary and sufficient condition for the boundedness of the commutator $[b, I_\rho]$ on Orlicz spaces. We first state the theorems and examples in Section 2.1. Next, we investigate the properties on generalized Young functions and Orlicz spaces in Section 2.2. Then we prove the boundedness of I_ρ and M_ρ on Orlicz spaces in Sections 2.3 and 2.4, respectively. Moreover, we investigate pointwise estimate by using the sharp maximal operator and the norm estimate by the sharp maximal operator in Section 2.5. Finally, using generalized Young functions and the results in Sections 2.2–2.5, we prove the necessary and sufficient condition for the boundedness of $[b, I_\rho]$ in Section 2.6.

In Chapter 3 we give necessary and sufficient conditions for the boundedness of the commutators $[b,T]$ and $[b,I_\rho]$ on Orlicz-Morrey spaces. We first state the theorems in Section 3.1. Next we give basic properties on generalized Young functions and Orlicz-Morrey spaces in Section 3.2. To prove the theorems we show the boundedness of the generalized fractional maximal operators on Orlicz-Morrey spaces in Section 3.3. In Section 3.4 we investigate the relation between Orlicz-Campanato and Orlicz-Morrey spaces and, using this relation, we show that, if $f_{B(0,r)} \to 0$ as $r \to \infty$, then

$$\|f\|_{L^{(\Phi,\varphi)}} \le C\|M^\sharp f\|_{L^{(\Phi,\varphi)}}. \tag{1.1.8}$$

In Section 3.5 we show the well-definedness of the commutators $[b,T]$ and $[b,I_\rho]$ for functions in Orlicz-Morrey spaces. Finally, using all of them, we prove the theorems in Sections 3.6, 3.7 and 3.8.

At the end of this section, we make some conventions. Throughout this paper, we always use C to denote a positive constant that is independent of the main parameters involved but whose value may differ from line to line. Constants with subscripts, such as C_p, are dependent on the subscripts. If $f \le Cg$, we then write $f \lesssim g$ or $g \gtrsim f$; and if $f \lesssim g \lesssim f$, we then write $f \sim g$.

1.2 Definitions

1.2.1 Generalized Campanato spaces

First we recall the definition of the generalized Campanato space.

Definition 1.2.1. For $p \in [1,\infty)$ and $\psi : (0,\infty) \to (0,\infty)$, let $\mathcal{L}_{p,\psi}(\mathbb{R}^n)$ be the set of all functions f such that the following functional is finite:

$$\|f\|_{\mathcal{L}_{p,\psi}(\mathbb{R}^n)} = \sup_{B=B(x,r)} \frac{1}{\psi(r)} \left(\fint_B |f(y) - f_B|^p \, dy \right)^{1/p},$$

where the supremum is taken over all balls $B(x,r)$ in $\mathbb{R}^n$.

Then $\|f\|_{\mathcal{L}_{p,\psi}(\mathbb{R}^n)}$ is a norm modulo constant functions and thereby $\mathcal{L}_{p,\psi}(\mathbb{R}^n)$ is a Banach space. If $p=1$ and $\psi \equiv 1$, then $\mathcal{L}_{p,\psi}(\mathbb{R}^n) = \mathrm{BMO}(\mathbb{R}^n)$. If $p=1$ and $\psi(r) = r^\alpha$ $(0 < \alpha \le 1)$, then $\mathcal{L}_{p,\psi}(\mathbb{R}^n)$ coincides with $\mathrm{Lip}_\alpha(\mathbb{R}^n)$ with equivalent norms.

Next, we say that a function $\theta : (0,\infty) \to (0,\infty)$ satisfies the doubling condition if there exists a positive constant C such that, for all $r, s \in (0,\infty)$,

$$\frac{1}{C} \le \frac{\theta(r)}{\theta(s)} \le C, \quad \text{if } \frac{1}{2} \le \frac{r}{s} \le 2. \tag{1.2.1}$$

We say that θ is almost increasing (resp. almost decreasing) if there exists a positive constant C such that, for all $r, s \in (0,\infty)$,

$$\theta(r) \le C\theta(s) \quad (\text{resp. } \theta(s) \le C\theta(r)), \quad \text{if } r < s. \tag{1.2.2}$$

It is known that, if ψ is almost increasing, then $\mathcal{L}_{p,\psi}(\mathbb{R}^n) = \mathcal{L}_{1,\psi}(\mathbb{R}^n)$ with equivalent norms for every $p \in (1,\infty)$, see [1, Corollary 4.3] and [40, Theorem 3.1].

1.2.2 Generalization of the Young function

We define a set $\overline{\mathit{\Phi}}$ of increasing functions $\Phi : [0,\infty] \to [0,\infty]$ and give some properties of functions in $\overline{\mathit{\Phi}}$.

For an increasing function $\Phi : [0,\infty] \to [0,\infty]$, let

$$a(\Phi) = \sup\{t \ge 0 : \Phi(t) = 0\}, \quad b(\Phi) = \inf\{t \ge 0 : \Phi(t) = \infty\}, \tag{1.2.3}$$

with convention $\sup\emptyset = 0$ and $\inf\emptyset = \infty$. Then $0 \le a(\Phi) \le b(\Phi) \le \infty$. Let $\overline{\mathit{\Phi}}$ be the set of all increasing functions $\Phi : [0,\infty] \to [0,\infty]$ such that

$$0 \le a(\Phi) < \infty, \quad 0 < b(\Phi) \le \infty, \tag{1.2.4}$$

$$\lim_{t\to+0} \Phi(t) = \Phi(0) = 0, \tag{1.2.5}$$

$$\Phi \text{ is left continuous on } [0, b(\Phi)), \tag{1.2.6}$$

$$\text{if } b(\Phi) = \infty, \text{ then } \lim_{t\to\infty} \Phi(t) = \Phi(\infty) = \infty, \tag{1.2.7}$$

$$\text{if } b(\Phi) < \infty, \text{ then } \lim_{t\to b(\Phi)-0} \Phi(t) = \Phi(b(\Phi)) \ (\le \infty). \tag{1.2.8}$$

In what follows, if an increasing and left continuous function $\Phi : [0,\infty) \to [0,\infty)$ satisfies (1.2.5) and $\lim_{t\to\infty} \Phi(t) = \infty$, then it will be always tacitly understood that $\Phi(\infty) = \infty$ and that $\Phi \in \overline{\mathit{\Phi}}$.

For $\Phi \in \overline{\mathit{\Phi}}$, we recall the generalized inverse of Φ in the sense of O'Neil [43, Definition 1.2].

Definition 1.2.2. For $\Phi \in \overline{\varPhi}$ and $u \in [0,\infty]$, let

$$\Phi^{-1}(u) = \begin{cases} \inf\{t \ge 0 : \Phi(t) > u\}, & u \in [0,\infty), \\ \infty, & u = \infty. \end{cases} \tag{1.2.9}$$

Let $\Phi \in \overline{\varPhi}$. Then Φ^{-1} is finite, increasing and right continuous on $[0,\infty)$ and positive on $(0,\infty)$. If Φ is bijective from $[0,\infty]$ to itself, then Φ^{-1} is the usual inverse function of Φ. Moreover, we have the following proposition, which is a generalization of Property 1.3 in [43].

Let $\Phi \in \overline{\varPhi}$. Then

$$\Phi(\Phi^{-1}(u)) \le u \le \Phi^{-1}(\Phi(u)) \quad \text{for all } u \in [0,\infty]. \tag{1.2.10}$$

For $\Phi, \Psi \in \overline{\varPhi}$, we write $\Phi \approx \Psi$ if there exists a positive constant C such that

$$\Phi(C^{-1}t) \le \Psi(t) \le \Phi(Ct) \quad \text{for all } t \in [0,\infty].$$

Meanwhile for functions $P, Q : [0,\infty] \to [0,\infty]$, we write $P \sim Q$ if there exists a positive constant C such that

$$C^{-1}P(t) \le Q(t) \le CP(t) \quad \text{for all } t \in [0,\infty].$$

Then, for $\Phi, \Psi \in \overline{\varPhi}$,

$$\Phi \approx \Psi \quad \Leftrightarrow \quad \Phi^{-1} \sim \Psi^{-1}. \tag{1.2.11}$$

For the proof see Lemma 2.2.2.

Next we recall the definition of the Young function and give its generalization.

Definition 1.2.3. A function $\Phi \in \overline{\varPhi}$ is called a Young function (or sometimes also called an Orlicz function) if Φ is convex on $[0, b(\Phi))$. Let $\varPhi_Y$ be the set of all Young functions. Let $\overline{\varPhi}_Y$ be the set of all $\Phi \in \overline{\varPhi}$ such that $\Phi \approx \Psi$ for some $\Psi \in \varPhi_Y$.

By the convexity, any Young function Φ is continuous on $[0, b(\Phi))$ and strictly increasing on $[a(\Phi), b(\Phi)]$. Hence Φ is bijective from $[a(\Phi), b(\Phi)]$ to $[0, \Phi(b(\Phi))]$. Moreover, Φ is absolutely continuous on any closed subinterval in $[0, b(\Phi))$. That is, its derivative Φ' exists a.e. and

$$\Phi(t) = \int_0^t \Phi'(s)\, ds, \quad t \in [0, b(\Phi)). \tag{1.2.12}$$

Definition 1.2.4. Let $\mathcal{Y}$ be the set of all Young functions such that $a(\Phi) = 0$ and $b(\Phi) = \infty$.

Definition 1.2.5. (i) A function $\Phi \in \overline{\varPhi}$ is said to satisfy the Δ_2-condition, denoted by $\Phi \in \overline{\Delta}_2$, if there exists a constant $C > 0$ such that

$$\Phi(2t) \le C\Phi(t) \quad \text{for all } t > 0. \tag{1.2.13}$$

(ii) A function $\Phi \in \overline{\varPhi}$ is said to satisfy the ∇_2-condition, denoted by $\Phi \in \overline{\nabla}_2$, if there exists a constant $k > 1$ such that

$$\Phi(t) \le \frac{1}{2k}\Phi(kt) \quad \text{for all } t > 0. \tag{1.2.14}$$

(iii) Let $\Delta_2 = \varPhi_Y \cap \overline{\Delta}_2$ and $\nabla_2 = \varPhi_Y \cap \overline{\nabla}_2$.

Remark 1.2.1. (i) $\Delta_2 \subset \mathcal{Y}$ and $\overline{\nabla}_2 \subset \overline{\varPhi}_Y$ ([23, Lemma 1.2.3]).

(ii) Let $\Phi \in \overline{\varPhi}_Y$. Then $\Phi \in \overline{\Delta}_2$ if and only if $\Phi \approx \Psi$ for some $\Psi \in \Delta_2$, and, $\Phi \in \overline{\nabla}_2$ if and only if $\Phi \approx \Psi$ for some $\Psi \in \nabla_2$.

(iii) Let $\Phi \in \varPhi_Y$. Then Φ^{-1} satisfies the doubling condition by its concavity, that is,

$$\Phi^{-1}(u) \le \Phi^{-1}(2u) \le 2\Phi^{-1}(u) \quad \text{for all } u \in [0, \infty].$$

(iv) Let $\Phi \in \varPhi_Y$. Then $\Phi \in \Delta_2$ if and only if $t \mapsto \dfrac{\Phi(t)}{t^p}$ is almost decreasing for some $p \in [1, \infty)$.

(v) Let $\Phi \in \varPhi_Y$. Then $\Phi \in \nabla_2$ if and only if $t \mapsto \dfrac{\Phi(t)}{t^p}$ is almost increasing for some $p \in (1, \infty)$.

Definition 1.2.6. For a Young function Φ, its complementary function is defined by

$$\widetilde{\Phi}(t) = \begin{cases} \sup\{tu - \Phi(u) : u \in [0, \infty)\}, & t \in [0, \infty), \\ \infty, & t = \infty. \end{cases}$$

Then $\widetilde{\Phi}$ is also a Young function, and $(\Phi, \widetilde{\Phi})$ is called a complementary pair. For example, if $\Phi(t) = t^p/p$, then $\widetilde{\Phi}(t) = t^{p'}/p'$ for $p, p' \in (1, \infty)$ and $1/p + 1/p' = 1$. If $\Phi(t) = t$, then

$$\widetilde{\Phi}(t) = \begin{cases} 0, & t \in [0, 1], \\ \infty, & t \in (1, \infty]. \end{cases}$$

Let $(\Phi, \widetilde{\Phi})$ be a complementary pair of functions in $\mathit{\Phi}_Y$. Then the following inequality holds:

$$t \le \Phi^{-1}(t)\widetilde{\Phi}^{-1}(t) \le 2t \quad \text{for} \quad t \in [0,\infty], \tag{1.2.15}$$

which is [57, (1.3)].

1.2.3 Orlicz and Orlicz-Morrey spaces

We recall the definitions of Orlicz and Orlicz-Morrey spaces generalized by Young functions. The Orlicz space $L^\Phi(\mathbb{R}^n)$ is introduced by [44, 45]. For the theory of Orlicz spaces, see [22, 23, 26, 28, 47] for example. Orlicz-Morrey spaces were investigated in [36, 38, 39], etc.

For $\Phi \in \overline{\mathit{\Phi}}_Y$, we define the Orlicz space $L^\Phi(\mathbb{R}^n)$ and the weak Orlicz space $\mathrm{w}L^\Phi(\mathbb{R}^n)$. Let $L^0(\mathbb{R}^n)$ be the set of all complex valued measurable functions on $\mathbb{R}^n$.

Definition 1.2.7 (Orlicz and weak Orlicz spaces)**.** For a function $\Phi \in \overline{\mathit{\Phi}}_Y$, let

$$\begin{aligned}
L^\Phi(\mathbb{R}^n) &= \left\{ f \in L^0(\mathbb{R}^n) : \int_{\mathbb{R}^n} \Phi(\epsilon|f(x)|)\,dx < \infty \text{ for some } \epsilon > 0 \right\}, \\
\|f\|_{L^\Phi} &= \inf\left\{ \lambda > 0 : \int_{\mathbb{R}^n} \Phi\left(\frac{|f(x)|}{\lambda}\right) dx \le 1 \right\}, \\
\mathrm{w}L^\Phi(\mathbb{R}^n) &= \left\{ f \in L^0(\mathbb{R}^n) : \sup_{t\in(0,\infty)} \Phi(t)\, m(\epsilon f, t) < \infty \text{ for some } \epsilon > 0 \right\}, \\
\|f\|_{\mathrm{w}L^\Phi} &= \inf\left\{ \lambda > 0 : \sup_{t\in(0,\infty)} \Phi(t)\, m\left(\frac{f}{\lambda}, t\right) \le 1 \right\}, \\
&\text{where} \quad m(f,t) = |\{x \in \mathbb{R}^n : |f(x)| > t\}|.
\end{aligned}$$

Then $\|\cdot\|_{L^\Phi}$ and $\|\cdot\|_{\mathrm{w}L^\Phi}$ are quasi-norms and $L^\Phi(\mathbb{R}^n) \subset L^1_{\mathrm{loc}}(\mathbb{R}^n)$. If $\Phi \in \mathit{\Phi}_Y$, then $\|\cdot\|_{L^\Phi}$ is a norm and thereby $L^\Phi(\mathbb{R}^n)$ is a Banach space. For $\Phi, \Psi \in \overline{\mathit{\Phi}}_Y$, if $\Phi \approx \Psi$, then $L^\Phi(\mathbb{R}^n) = L^\Psi(\mathbb{R}^n)$ and $\mathrm{w}L^\Phi(\mathbb{R}^n) = \mathrm{w}L^\Psi(\mathbb{R}^n)$ with equivalent quasi-norms, respectively. Orlicz spaces are introduced by [44, 45]. For the theory of Orlicz spaces, see [22, 23, 26, 28, 47] for example.

We note that, for any Young function Φ, we have that

$$\sup_{t\in(0,\infty)} \Phi(t)\, m(f,t) = \sup_{t\in(0,\infty)} t\, m(\Phi(|f|), t),$$

and then

$$\|f\|_{\mathrm{w}L^\Phi} = \inf\left\{\lambda>0 : \sup_{t\in(0,\infty)} \Phi(t)\, m\!\left(\frac{f}{\lambda}, t\right) \le 1\right\}$$
$$= \inf\left\{\lambda>0 : \sup_{t\in(0,\infty)} t\, m\!\left(\Phi\left(\frac{|f|}{\lambda}\right), t\right) \le 1\right\}.$$

For the above equality, see [18, Proposition 4.2] for example.

The following theorem is known, see [23, Theorem 1.2.1] for example.

Theorem 1.2.1. *Let $\Phi \in \overline{\mathit{\Phi}}_Y$. Then M is bounded from $L^\Phi(\mathbb{R}^n)$ to $\mathrm{w}L^\Phi(\mathbb{R}^n)$, that is, there exists a positive constant C_0 such that, for all $f \in L^\Phi(\mathbb{R}^n)$,*

$$\|Mf\|_{\mathrm{w}L^\Phi} \le C_0 \|f\|_{L^\Phi}. \tag{1.2.16}$$

Moreover, if $\Phi \in \overline{\nabla}_2$, then M is bounded on $L^\Phi(\mathbb{R}^n)$, that is, there exists a positive constant C_0 such that, for all $f \in L^\Phi(\mathbb{R}^n)$,

$$\|Mf\|_{L^\Phi} \le C_0 \|f\|_{L^\Phi}. \tag{1.2.17}$$

See also [4, 20, 21] for the Hardy-Littlewood maximal operator on Orlicz spaces.

Remark 1.2.2. Let $\Phi \in \mathit{\Phi}_Y$. Then $\Phi \in \Delta_2$ if and only if $C^\infty_{\mathrm{comp}}(\mathbb{R}^n)$ is dense in $L^\Phi(\mathbb{R}^n)$, and, $\Phi \in \nabla_2$ if and only if the Hardy-Littlewood maximal operator M is bounded on $L^\Phi(\mathbb{R}^n)$.

In this paper we consider the following class of $\varphi : (0,\infty) \to (0,\infty)$.

Definition 1.2.8. (i) Let $\mathcal{G}^{\mathrm{dec}}$ be the set of all functions $\varphi : (0,\infty) \to (0,\infty)$ such that φ is almost decreasing and that $r \mapsto \varphi(r)r^n$ is almost increasing. That is, there exists a positive constant C such that, for all $r, s \in (0,\infty)$,

$$C\varphi(r) \ge \varphi(s), \quad \varphi(r)r^n \le C\varphi(s)s^n, \quad \text{if } r < s.$$

(ii) Let $\mathcal{G}^{\mathrm{inc}}$ be the set of all functions $\varphi : (0,\infty) \to (0,\infty)$ such that φ is almost increasing and that $r \mapsto \varphi(r)/r$ is almost decreasing. That is, there exists a positive constant C such that, for all $r, s \in (0,\infty)$,

$$\varphi(r) \le C\varphi(s), \quad C\varphi(r)/r \ge \varphi(s)/s, \quad \text{if } r < s.$$

If $\varphi \in \mathcal{G}^{\rm dec}$ or $\varphi \in \mathcal{G}^{\rm inc}$, then φ satisfies the doubling condition (1.2.1). Let $\psi : (0,\infty) \to (0,\infty)$. If $\psi \sim \varphi$ for some $\varphi \in \mathcal{G}^{\rm dec}$ (resp. $\mathcal{G}^{\rm inc}$), then $\psi \in \mathcal{G}^{\rm dec}$ (resp. $\mathcal{G}^{\rm inc}$).

Remark 1.2.3. Let $\varphi \in \mathcal{G}^{\rm dec}$. Then there exists $\tilde\varphi \in \mathcal{G}^{\rm dec}$ such that $\varphi \sim \tilde\varphi$ and that $\tilde\varphi$ is continuous and strictly decreasing, see [38, Proposition 3.4]. Moreover, if

$$\lim_{r\to 0}\varphi(r) = \infty, \quad \lim_{r\to\infty}\varphi(r) = 0, \tag{1.2.18}$$

then $\tilde\varphi$ is bijective from $(0,\infty)$ to itself.

Definition 1.2.9 (Orlicz-Morrey space)**.** For a Young function $\Phi : [0,\infty] \to [0,\infty]$, a function $\varphi : (0,\infty) \to (0,\infty)$ and a ball $B = B(a,r)$, let

$$\|f\|_{\Phi,\varphi,B} = \inf\left\{\lambda > 0 : \frac{1}{\varphi(r)} ⨍_B \Phi\left(\frac{|f(x)|}{\lambda}\right)dx \le 1\right\}. \tag{1.2.19}$$

For a ball $B = B(a,r)$, let $L^{(\Phi,\varphi)}(\mathbb{R}^n)$ be the set of all functions f such that the following functional is finite:

$$\|f\|_{L^{(\Phi,\varphi)}} = \sup_B \|f\|_{\Phi,\varphi,B}, \tag{1.2.20}$$

where the supremum is taken over all balls B in $\mathbb{R}^n$.

Let $\mu_B = \frac{dx}{|B|\varphi(r)}$. Then we have the following relation:

$$\|f\|_{\Phi,\varphi,B} = \|f\|_{L^\Phi(B,\mu_B)}. \tag{1.2.21}$$

Because of the relation (1.2.21), $\|\cdot\|_{L^{(\Phi,\varphi)}}$ is a quasi-norm, and thereby $L^{(\Phi,\varphi)}(\mathbb{R}^n)$ is a quasi-Banach space. If $\Phi \in \mathit{\Phi}_Y$, then $\|\cdot\|_{L^{(\Phi,\varphi)}}$ is a norm and thereby $L^{(\Phi,\varphi)}(\mathbb{R}^n)$ is a Banach space. If $\Phi \approx \Psi$ and $\varphi \sim \psi$, then $L^{(\Phi,\varphi)}(\mathbb{R}^n) = L^{(\Psi,\psi)}(\mathbb{R}^n)$ with equivalent quasi-norms.

Then $\|f\|_{L^{(\Phi,\varphi)}}$ is a norm and thereby $L^{(\Phi,\varphi)}(\mathbb{R}^n)$ is a Banach space. If $\varphi(r) = 1/r^n$, then $L^{(\Phi,\varphi)}(\mathbb{R}^n)$ coincides with the Orlicz space $L^\Phi(\mathbb{R}^n)$ equipped with the norm

$$\|f\|_{L^\Phi} = \inf\left\{\lambda > 0 : \int_{\mathbb{R}^n} \Phi\left(\frac{|f(x)|}{\lambda}\right) dx \le 1\right\}.$$

If $\Phi(t) = t^p$, $1 \le p < \infty$, then $L^{(\Phi,\varphi)}(\mathbb{R}^n)$ coincides with the generalized Morrey space $L^{(p,\varphi)}(\mathbb{R}^n)$ equipped with the norm

$$\|f\|_{L^{(p,\varphi)}} = \sup_{B=B(a,r)} \left(\frac{1}{\varphi(r)} ⨍_B |f(x)|^p\, dx\right)^{1/p}.$$

The Orlicz-Morrey space $L^{(\Phi,\varphi)}(\mathbb{R}^n)$ was first studied in [36]. For other kinds of Orlicz-Morrey spaces, see [6, 7, 15, 49], etc.

Chapter 2

Commutators on Orlicz spaces

2.1 Theorems and examples

The following theorem is an extension of the result in [33].

Theorem 2.1.1. *Let* $\rho : (0,\infty) \to (0,\infty)$ *satisfy* (1.1.2) *and* (1.1.3), *and let* $\Phi, \Psi \in \overline{\mathit{\Phi}}_Y$. *Assume that there exists a positive constant* A *such that, for all* $r \in (0,\infty)$,

$$\int_0^r \frac{\rho(t)}{t}\,dt\;\Phi^{-1}(1/r^n) + \int_r^\infty \frac{\rho(t)\,\Phi^{-1}(1/t^n)}{t}\,dt \le A\Psi^{-1}(1/r^n). \tag{2.1.1}$$

Then, for any positive constant C_0, *there exists a positive constant* C_1 *such that, for all* $f \in L^\Phi(\mathbb{R}^n)$ *with* $f \not\equiv 0$,

$$\Psi\left(\frac{|I_\rho f(x)|}{C_1\|f\|_{L^\Phi}}\right) \le \Phi\left(\frac{Mf(x)}{C_0\|f\|_{L^\Phi}}\right). \tag{2.1.2}$$

Consequently, I_ρ *is bounded from* $L^\Phi(\mathbb{R}^n)$ *to* $\mathrm{w}L^\Psi(\mathbb{R}^n)$. *Moreover, if* $\Phi \in \overline{\nabla}_2$, *then* I_ρ *is bounded from* $L^\Phi(\mathbb{R}^n)$ *to* $L^\Psi(\mathbb{R}^n)$.

Remark 2.1.1. We cannot replace $\int_0^r \frac{\rho(t)}{t}\,dt$ by $\rho(r)$ in (2.1.1), see [41, Section 5].

Here, we give some examples of the pair of (ρ, Φ, Ψ) which satisfies the assumptions in Theorem 2.1.1. For other examples, see [34]. See also [29] for the boundedness of I_ρ on the Orlicz space $L^\Phi(\Omega)$ with bounded domain $\Omega \subset \mathbb{R}^n$.

Example 2.1.1. If $\rho(r) = r^\alpha$, $\Phi(t) = t^p$ and $\Psi(t) = t^q$ with $p, q \in [1,\infty)$ and $0 < \alpha < n/p$, then

$$\int_0^r \frac{\rho(t)}{t}\,dt\;\Phi^{-1}(1/r^n) \sim \int_r^\infty \frac{\rho(t)\,\Phi^{-1}(1/t^n)}{t}\,dt \sim r^{\alpha-n/p} \quad\text{and}\quad \Psi^{-1}(1/r^n) = r^{-n/q}.$$

In this case,

$$\text{“(2.1.1)”} \quad \Leftrightarrow \quad r^{\alpha-n/p} \lesssim r^{-n/q},\ r\in(0,\infty) \quad \Leftrightarrow \quad \alpha - n/p = -n/q.$$

Therefore, the Hardy-Littlewood-Sobolev theorem is a corollary of Theorem 2.1.1.

Example 2.1.2. Let ρ and Φ be as in (1.1.4) and in (1.1.5), respectively, and let Ψ be as in (1.1.5) with q instead of p. Assume that $\alpha, p, q \in (0,\infty)$ and $-1/p+\alpha = -1/q$. Then

$$\int_0^r \frac{\rho(t)}{t}\,dt \sim \begin{cases} (\log(1/r))^{-\alpha} & \text{for small } r>0, \\ (\log r)^{\alpha} & \text{for large } r>0, \end{cases}$$

and

(2.1.3)

$$\Phi^{-1}(1/r^n) \sim \begin{cases} (\log(1/r))^{1/p}, \\ (\log r)^{-1/p}, \end{cases} \quad \Psi^{-1}(1/r^n) \sim \begin{cases} (\log(1/r))^{1/q} & \text{for small } r>0, \\ (\log r)^{-1/q} & \text{for large } r>0. \end{cases}$$

In this case we have

$$\begin{aligned} \int_0^r \frac{\rho(t)}{t}\,dt\ \Phi^{-1}(1/r^n) &\sim \int_r^\infty \frac{\rho(t)\,\Phi^{-1}(1/t^n)}{t}\,dt \\ &\sim \begin{cases} (\log(1/r))^{-\alpha+1/p} & \text{for small } r>0, \\ (\log r)^{\alpha-1/p} & \text{for large } r>0. \end{cases} \end{aligned}$$

Then the pair (ρ,Φ,Ψ) satisfies (2.1.1), that is, I_ρ is bounded from $\exp L^p(\mathbb{R}^n)$ to $\exp L^q(\mathbb{R}^n)$.

Example 2.1.3. Let $\alpha\in(0,n)$, $p,q\in[1,\infty)$ and $-n/p+\alpha=-n/q$. Let

$$\rho(r) = \begin{cases} r^\alpha & \text{for small } r>0, \\ e^{-r} & \text{for large } r>0. \end{cases}$$

Then

$$\int_0^r \frac{\rho(t)}{t}\,dt \sim \begin{cases} r^\alpha & \text{for small } r>0, \\ 1 & \text{for large } r>0. \end{cases}$$

(i) If $\Phi(r)=r^p$ and $\Psi(r)=\max(r^p,r^q)$, then (2.1.1) holds. In this case $L^\Phi(\mathbb{R}^n) = L^p(\mathbb{R}^n)$ and $L^\Psi(\mathbb{R}^n) = L^p(\mathbb{R}^n)\cap L^q(\mathbb{R}^n)$.

(ii) If $\Phi(r)=\max(0,r^p-1)$ and $\Psi(r)=\max(0,r^q-1)$, then (2.1.1) holds, since

$$\Phi^{-1}(u) \sim \begin{cases} 1 & \text{for small } u>0, \\ u^{1/p} & \text{for large } u>0, \end{cases} \quad \Phi^{-1}(1/r^n) \sim \begin{cases} r^{-n/p} & \text{for small } r>0, \\ 1 & \text{for large } r>0. \end{cases}$$

In this case $L^\Phi(\mathbb{R}^n) = L^p(\mathbb{R}^n)+L^\infty(\mathbb{R}^n)$ and $L^\Psi(\mathbb{R}^n) = L^q(\mathbb{R}^n)+L^\infty(\mathbb{R}^n)$.

A function $\Phi \in \mathcal{Y}$ is called an N-function if

$$\lim_{t\to+0}\frac{\Phi(t)}{t}=0, \quad \lim_{t\to\infty}\frac{\Phi(t)}{t}=\infty.$$

We say that a function $\theta : (0,\infty) \to (0,\infty)$ is almost increasing (resp. almost decreasing) if there exists a positive constant C such that, for all $r, s \in (0,\infty)$,

$$\theta(r) \le C\theta(s) \quad (\text{resp. } \theta(s) \le C\theta(r)), \quad \text{if } r < s. \tag{2.1.4}$$

Then we have the following corollary.

Corollary 2.1.2. *Let $1 < s < \infty$ and $\rho : (0,\infty) \to (0,\infty)$. Assume that ρ satisfies (1.1.2) and that $r \mapsto \rho(r)/r^{n/s-\epsilon}$ is almost decreasing for some positive constant ϵ. Then there exist an N-function Ψ and a positive constant C such that, for all $r > 0$,*

$$C^{-1}\Psi^{-1}\left(\frac{1}{r^n}\right) \le \frac{1}{r^{n/s}}\int_0^r \frac{\rho(t)}{t}\,dt \le C\Psi^{-1}\left(\frac{1}{r^n}\right). \tag{2.1.5}$$

Moreover, I_ρ is bounded from $L^s(\mathbb{R}^n)$ to $L^\Psi(\mathbb{R}^n)$.

In the above, (2.1.5) can be shown by the same way as the proof of [1, Theorem 3.5]. The boundedness of I_ρ from $L^s(\mathbb{R}^n)$ to $L^\Psi(\mathbb{R}^n)$ is proven by the following way. First note that ρ satisfies (1.1.3) by Remark 2.1.2 below. Let $\Phi(t) = t^s$. Then we have

$$\begin{aligned}\int_r^\infty \frac{\rho(t)\Phi^{-1}(1/t^n)}{t}\,dt &= \int_r^\infty \frac{\rho(t)/t^{n/s}}{t}\,dt \lesssim \frac{\rho(r)}{r^{n/s-\epsilon}}\int_r^\infty \frac{1}{t^{1+\epsilon}}\,dt \\ &\sim \frac{\rho(r)}{r^{n/s}} \lesssim \frac{1}{r^{n/s}}\int_0^r \frac{\rho(t)}{t}\,dt = \Phi^{-1}\left(\frac{1}{r^n}\right)\int_0^r \frac{\rho(t)}{t}\,dt,\end{aligned}$$

where we used (2.1.6) below for the last inequality. Combining this and (2.1.5), we have (2.1.1). Then we have the conclusion by Theorem 2.1.1.

Remark 2.1.2. If $r \mapsto \rho(r)/r^k$ is almost decreasing for some positive constant k, then ρ satisfies (1.1.3). Actually,

$$\sup_{r\le t\le 2r}\rho(t) \sim r^k \sup_{r\le t\le 2r}\frac{\rho(t)}{t^k} \lesssim r^k \int_{r/2}^r \frac{\rho(t)}{t^{k+1}}\,dt \sim \int_{r/2}^r \frac{\rho(t)}{t}\,dt. \tag{2.1.6}$$

Next we state the result on the operator M_ρ defined by (1.1.7) in which we don't assume (1.1.2) or (1.1.3).

Theorem 2.1.3. *Let $\rho : (0,\infty) \to (0,\infty)$, and let $\Phi, \Psi \in \overline{\varPhi}_Y$.*

(i) *Assume that there exists a positive constant A such that, for all $r \in (0,\infty)$,*

$$\left(\sup_{0<t\le r} \rho(t)\right) \Phi^{-1}(1/r^n) \le A\Psi^{-1}(1/r^n). \tag{2.1.7}$$

Then, for any positive constant C_0, there exists a positive constant C_1 such that, for all $f \in L^\Phi(\mathbb{R}^n)$ with $f \not\equiv 0$,

$$\Psi\left(\frac{M_\rho f(x)}{C_1\|f\|_{L^\Phi}}\right) \le \Phi\left(\frac{Mf(x)}{C_0\|f\|_{L^\Phi}}\right). \tag{2.1.8}$$

Consequently, M_ρ is bounded from $L^\Phi(\mathbb{R}^n)$ to $\mathrm{w}L^\Psi(\mathbb{R}^n)$. Moreover, if $\Phi \in \overline{\nabla}_2$, then M_ρ is bounded from $L^\Phi(\mathbb{R}^n)$ to $L^\Psi(\mathbb{R}^n)$.

(ii) *Conversely, if M_ρ is bounded from $L^\Phi(\mathbb{R}^n)$ to $\mathrm{w}L^\Psi(\mathbb{R}^n)$, then* (2.1.7) *holds for some A and all $r \in (0,\infty)$.*

Remark 2.1.3. Let $\rho : (0,\infty) \to (0,\infty)$, and let $\Phi, \Psi \in \overline{\mathit{\Phi}}_Y$.

(i) Let $\rho_1(r) = \sup_{0<t\le r} \rho(t)$. Then we conclude from the theorem above that I_ρ and I_{ρ_1} have the same boundedness, that is, we may assume that ρ is increasing.

(ii) Since Φ^{-1} is pseudo-concave, $u \mapsto \Phi^{-1}(u)/u$ is almost decreasing, and then $r \mapsto \Phi^{-1}(1/r^n)r^n$ is almost increasing. Therefore, from (2.1.7) it follows that $r \mapsto \rho(r)/r^n$ is dominated by the almost decreasing function $r \mapsto \frac{\Psi^{-1}(1/r^n)}{\Phi^{-1}(1/r^n)r^n}$.

Example 2.1.4. If $\rho(r) = r^\alpha$, $\Phi(t) = t^p$ and $\Psi(t) = t^q$ with $p, q \in [1,\infty)$ and $0 \le \alpha \le n/p$, then

$$\rho(r)\Phi^{-1}(1/r^n) \sim r^{\alpha-n/p} \quad \text{and} \quad \Psi^{-1}(1/r^n) = r^{-n/q}.$$

In this case,

$$\text{“(2.1.7)”} \quad \Leftrightarrow \quad r^{\alpha-n/p} \lesssim r^{-n/q},\ r \in (0,\infty) \quad \Leftrightarrow \quad \alpha - n/p = -n/q.$$

In this example, if $\alpha = 0$, then M_ρ is the Hardy-Littlewood maximal operator M and “(2.1.7)” $\Leftrightarrow p = q$. If $\alpha - n/p = 0$, then M_ρ is the fractional maximal operator M_α and it is bounded from $L^p(\mathbb{R}^n)$ to $L^\infty(\mathbb{R}^n)$, since we can take

$$\Psi(r) = \begin{cases} 0 & \text{for } r \in [0,1], \\ \infty & \text{for } r \in (1,\infty], \end{cases} \quad \text{and} \quad \Psi^{-1}(r) = \begin{cases} 1 & \text{for } r \in [0,\infty), \\ \infty & \text{for } r = \infty. \end{cases} \tag{2.1.9}$$

Example 2.1.5. Let Φ be as in (1.1.5), and let Ψ be as in (1.1.5) with q instead of p. Assume that $\alpha \in [0, \infty)$ and $p, q \in (0, \infty)$. Let

$$\rho(r) = \begin{cases} (\log(1/r))^{-\alpha} & \text{for small } r > 0, \\ (\log r)^{\alpha} & \text{for large } r > 0, \end{cases} \tag{2.1.10}$$

instead of (1.1.4). Here, we note that, if $0 \le \alpha \le 1$, then $\int_0^1 \frac{\rho(t)}{t}\,dt = \infty$, that is, I_ρ is not well-defined, while M_ρ is well-defined. Actually, M_ρ is bounded from $\exp L^p(\mathbb{R}^n)$ to $\exp L^q(\mathbb{R}^n)$, if $-1/p + \alpha = -1/q$ for any $\alpha \in [0, \infty)$, see (2.1.3) for the inverse functions of Φ and Ψ. Moreover, if $-1/p + \alpha = 0$, then M_ρ is bounded from $\exp L^p(\mathbb{R}^n)$ to $L^\infty(\mathbb{R}^n)$, since we can take Ψ as in (2.1.9).

Example 2.1.6. Assume that $\alpha, q \in [0, \infty)$ and $p \in (1, \infty)$. Let ρ be as in (2.1.10). Then M_ρ is bounded from $L^p(\mathbb{R}^n)$ to $L^p(\log L)^{p_1}(\mathbb{R}^n)$, if $p_1/p = \alpha$, where $L^p(\log L)^{p_1}(\mathbb{R}^n)$ is the Orlicz space $L^\Phi(\mathbb{R}^n)$ with

$$\Phi(r) = \begin{cases} r^p(\log(1/r))^{-p_1} & \text{for small } r > 0, \\ r^p(\log r)^{p_1} & \text{for large } r > 0. \end{cases}$$

In this case we have

$$\Phi^{-1}(1/r^n) \sim \begin{cases} r^{-n/p}(\log(1/r))^{-p_1/p} & \text{for small } r > 0, \\ r^{-n/p}(\log r)^{p_1/p} & \text{for large } r > 0. \end{cases} \tag{2.1.11}$$

In this example, if we take $p = 1$, then M_ρ is bounded from $L^1(\mathbb{R}^n)$ to $\mathrm{w}L^1(\log L)^\alpha(\mathbb{R}^n)$ which is the weak space of $L^1(\log L)^\alpha(\mathbb{R}^n)$.

Finally, we state the result on the commutator $[b, I_\rho]$. Let

$$\rho^*(r) = \int_0^r \frac{\rho(t)}{t}\,dt. \tag{2.1.12}$$

Theorem 2.1.4. *Let $\rho, \psi : (0, \infty) \to (0, \infty)$, and let $\Phi, \Psi \in \overline{\mathit{\Phi}}_Y$. Assume that ρ satisfies* (1.1.2). *Let $b \in L^1_{\mathrm{loc}}(\mathbb{R}^n)$.*

(i) *Let $\Phi, \Psi \in \overline{\Delta}_2 \cap \overline{\nabla}_2$. Assume that ψ be almost increasing and that $r \mapsto \rho(r)/r^{n-\epsilon}$ is almost decreasing for some $\epsilon \in (0, n)$. Assume also that there exists a positive constant A and $\Theta \in \overline{\nabla}_2$ such that, for all $r \in (0, \infty)$,*

$$\int_0^r \frac{\rho(t)}{t}\,dt\ \Phi^{-1}(1/r^n) + \int_r^\infty \frac{\rho(t)\,\Phi^{-1}(1/t^n)}{t}\,dt \le A\Theta^{-1}(1/r^n), \tag{2.1.13}$$

$$\psi(r)\Theta^{-1}(1/r^n) \le A\Psi^{-1}(1/r^n), \tag{2.1.14}$$

and that there exist a positive constant C_ρ such that, for all $r, s \in (0, \infty)$,

$$\left|\frac{\rho(r)}{r^n} - \frac{\rho(s)}{s^n}\right| \le C_\rho\,|r-s|\frac{\rho^*(r)}{r^{n+1}}, \quad \textit{if } \frac{1}{2} \le \frac{r}{s} \le 2. \tag{2.1.15}$$

If $b \in \mathcal{L}_{1,\psi}(\mathbb{R}^n)$, then $[b, I_\rho]$ is bounded from $L^\Phi(\mathbb{R}^n)$ to $L^\Psi(\mathbb{R}^n)$ and there exists a positive constant C such that, for all $f \in L^\Phi(\mathbb{R}^n)$,

$$\|[b, I_\rho]f\|_{L^\Psi} \le C\|b\|_{\mathcal{L}_{1,\psi}}\|f\|_{L^\Phi}. \tag{2.1.16}$$

(ii) *Conversely, assume that there exists a positive constant A such that, for all $r \in (0, \infty)$,*

$$\Psi^{-1}(1/r^n) \le A r^\alpha \psi(r) \Phi^{-1}(1/r^n).$$

If $[b, I_\alpha]$ is well-defined and bounded from $L^\Phi(\mathbb{R}^n)$ to $L^\Psi(\mathbb{R}^n)$, then b is in $\mathcal{L}_{1,\psi}(\mathbb{R}^n)$ and there exists a positive constant C, independent of b, such that

$$\|b\|_{\mathcal{L}_{1,\psi}} \le C\|[b, I_\alpha]\|_{L^\Phi \to L^\Psi},$$

where $\|[b, I_\alpha]\|_{L^\Phi \to L^\Psi}$ is the operator norm of $[b, I_\alpha]$ from $L^\Phi(\mathbb{R}^n)$ to $L^\Psi(\mathbb{R}^n)$.

Example 2.1.7. Let $\alpha \in (0, n)$, $\beta \in [0, 1]$ and $p, q \in (1, \infty)$, and, let

$$\rho(r) = r^\alpha, \;\; \psi(r) = r^\beta, \;\; \Phi(r) = r^p, \;\; \Psi(r) = r^q.$$

Assume that $-n/p + \alpha + \beta = -n/q$. Take $\Theta(r) = r^{\tilde{q}}$ with $-n/\tilde{q} = -n/p + \alpha$. Then (2.1.13), (2.1.14) and (2.1.15) hold, that is, $[b, I_\alpha]$ is bounded from $L^p(\mathbb{R}^n)$ to $L^q(\mathbb{R}^n)$, where $b \in \mathrm{Lip}_\beta(\mathbb{R}^n)$ if $\beta \in (0, 1]$, and $b \in \mathrm{BMO}(\mathbb{R}^n)$ if $\beta = 0$, which is Chanillo's result in [3].

Example 2.1.8. Let $\alpha \in (0, n)$ and $\alpha_1 \in (-\infty, \infty)$. Let $\beta \in (0, n)$ and $\beta_1 \in (-\infty, \infty)$, or, let $\beta = 0$ and $\beta_1 \in [0, \infty)$. Let

$$\rho(r) = \begin{cases} r^\alpha(\log(1/r))^{-\alpha_1}, \\ r^\alpha, \\ r^\alpha(\log r)^{\alpha_1}, \end{cases} \qquad \psi(r) = \begin{cases} r^\beta(\log(1/r))^{-\beta_1} & \text{for } r \in (0, 1/e), \\ r^\beta & \text{for } r \in [1/e, e], \\ r^\beta(\log r)^{\beta_1} & \text{for } r \in (e, \infty). \end{cases}$$

Then $\rho^* \sim \rho$ and $\rho'(t) \sim \rho(t)/t$. In this case ρ satisfies (2.1.15), since ρ is Lipschitz continuous on $[1/(2e), 2e]$, and, for $r, s \in (0, 1/e] \cup [e, \infty)$, there exists $\theta \in (0, 1)$ such that

$$\left|\frac{\rho(r)}{r^n} - \frac{\rho(s)}{s^n}\right| = |r-s|\left|\frac{d}{dt}\left(\frac{\rho(t)}{t^n}\right)\Big|_{t=(1-\theta)r+\theta s}\right| \lesssim |r-s|\frac{\rho(r)}{r^{n+1}}, \quad \text{if } \frac{1}{2} \le \frac{r}{s} \le 2.$$

Let $p, q \in (1, \infty)$ and $p_1, q_1 \in (-\infty, \infty)$, and let

$$\Phi(r) = \begin{cases} r^p (\log(1/r))^{-p_1}, \\ r^p (\log r)^{p_1}, \end{cases} \quad \Psi(r) = \begin{cases} r^q (\log(1/r))^{-q_1} & \text{for small } r > 0, \\ r^q (\log r)^{q_1} & \text{for large } r > 0. \end{cases}$$

For the inverse functions of Φ and Ψ, see (2.1.11). If

$$-n/p + \alpha + \beta = -n/\tilde{p} + \beta = -n/q, \quad p_1/p + \alpha_1 + \beta_1 = \tilde{p}_1/\tilde{p} + \beta_1 = q_1/q,$$

and

$$\Theta(r) = \begin{cases} r^{\tilde{p}} (\log(1/r))^{-\tilde{p}_1} & \text{for small } r > 0, \\ r^{\tilde{p}} (\log r)^{\tilde{p}_1} & \text{for large } r > 0, \end{cases}$$

then

$$\int_0^r \frac{\rho(t)}{t} \, dt \; \Phi^{-1}(1/r^n) \sim \int_r^\infty \frac{\rho(t) \, \Phi^{-1}(1/t^n)}{t} \, dt \sim \Theta^{-1}(r^{-n}),$$

and

$$\psi(r) \Theta^{-1}(r^{-n}) \sim \Psi^{-1}(r^{-n}) \sim \begin{cases} r^{-n/p+\alpha+\beta} (\log(1/r))^{-(p_1/p+\alpha_1+\beta_1)} & \text{for small } r > 0, \\ r^{-n/p+\alpha+\beta} (\log r)^{p_1/p+\alpha_1+\beta_1} & \text{for large } r > 0. \end{cases}$$

In this case $[b, I_\rho]$ is bounded from $L^p(\log L)^{p_1}(\mathbb{R}^n)$ to $L^q(\log L)^{q_1}(\mathbb{R}^n)$.

2.2 Properties on Young functions and Orlicz spaces

In this section we prepare some lemmas to prove our main results.

Proposition 2.2.1. *Let $\Phi \in \overline{\mathit{\Phi}}$. Then*

$$\Phi(\Phi^{-1}(u)) \le u \le \Phi^{-1}(\Phi(u)) \quad \textit{for all } u \in [0, \infty]. \tag{2.2.1}$$

Proof. First we show that, for all $t, u \in [0, \infty]$,

$$\Phi(t) \le u \;\Rightarrow\; t \le \Phi^{-1}(u). \tag{2.2.2}$$

If $\Phi(t) \le u$, then $\Phi(s) > u \Rightarrow \Phi(s) > \Phi(t) \Rightarrow s > t$ and

$$\{s \ge 0 : \Phi(s) > u\} \subset \{s \ge 0 : s > t\}.$$

Hence,

$$\Phi^{-1}(u) = \inf\{s \ge 0 : \Phi(s) > u\} \ge \inf\{s \ge 0 : s > t\} = t.$$

This shows (2.2.2). Now, letting $\Phi(t) = u$ and using (2.2.2), we have that $t \le \Phi^{-1}(u) = \Phi^{-1}(\Phi(t))$, which is the second inequality in (2.2.1).

Next we show that, for all $t \in (0, \infty]$ and $u \in [0, \infty]$,

$$\Phi(t) > u \;\Rightarrow\; t > \Phi^{-1}(u), \tag{2.2.3}$$

$$t \le \Phi^{-1}(u) \;\Rightarrow\; \Phi(t) \le u. \tag{2.2.4}$$

We only show (2.2.3), since (2.2.4) is the contraposition of (2.2.3), that is (2.2.4) is equivalent to (2.2.3). If $\Phi(t) > u$, then $\Phi(s) > u$ for some $s < t$ by the properties (1.2.6)–(1.2.8). By the definition of Φ^{-1} we have that $s \ge \Phi^{-1}(u)$. That is, $t > \Phi^{-1}(u)$, which shows (2.2.3). Now, if $\Phi^{-1}(u) = 0$, then the first inequality in (2.2.1) is true by (1.2.5). If $t = \Phi^{-1}(u) > 0$, then, using (2.2.4), we have that $\Phi(\Phi^{-1}(u)) = \Phi(t) \le u$, which is the first inequality in (2.2.1). □

Lemma 2.2.2. *Let $\Phi, \Psi \in \overline{\mathit{\Phi}}$, and let C be a fixed positive constant. Then*

$$\Phi(t) \le \Psi(Ct) \quad \textit{for all } t \in [0, \infty]$$

if and only if

$$\Psi^{-1}(u) \le C\Phi^{-1}(u) \quad \textit{for all } u \in [0, \infty].$$

Proof. As the conclusion can be obtained obviously if both Φ and Ψ are bijective, we prove it without the assumption. Let $\Phi(t) \le \Psi(Ct)$ for all $t \in [0, \infty]$. If $t = \Psi^{-1}(u)$, then by Proposition 2.2.1 we have that $\Psi(t) = \Psi(\Psi^{-1}(u)) \le u$ and that

$$\Psi^{-1}(u)/C = t/C \le \Phi^{-1}(\Phi(t/C)) \le \Phi^{-1}(\Psi(t)) \le \Phi^{-1}(u).$$

Conversely, let $\Psi^{-1}(u) \le C\Phi^{-1}(u)$ for all $u \in [0, \infty]$. If $u = \Psi(t)$, then by Proposition 2.2.1 we have $t \le \Psi^{-1}(\Psi(t)) = \Psi^{-1}(u)$ and

$$\Phi(t/C) \le \Phi(\Psi^{-1}(u)/C) \le \Phi(\Phi^{-1}(u)) \le u = \Psi(t).$$ □

Lemma 2.2.3. *Let $\Phi \in \mathit{\Phi}_Y$. For a measurable set $G \subset \mathbb{R}^n$ with finite measure,*

$$\|\chi_G\|_{L^\Phi} = \|\chi_G\|_{\mathrm{w}L^\Phi} = \frac{1}{\Phi^{-1}(1/|G|)}. \tag{2.2.5}$$

From (1.2.15) it follows that, for the characteristic function χ_B of the ball B,

$$\|\chi_B\|_{L^{\tilde{\Phi}}} = \frac{1}{\widetilde{\Phi}^{-1}(1/|B|)} \le |B|\Phi^{-1}(1/|B|). \tag{2.2.6}$$

Lemma 2.2.4 ([1]). *Let $k > 0$ and $\rho : (0, \infty) \to (0, \infty)$. Assume that ρ satisfies (1.1.2). Let ρ^* be as in (2.1.12). If $r \mapsto \rho(r)/r^k$ is almost decreasing, then $r \mapsto \rho^*(r)/r^k$ is also almost decreasing.*

Remark 2.2.1. Since ρ^* is increasing with respect to r, if $r \mapsto \rho(r)/r^k$ is almost decreasing for some $k > 0$, then we see that ρ^* satisfies the doubling condition, that is, there exists a positive constant C such that, for all $r \in (0, \infty)$,

$$\rho^*(r) \le \rho^*(2r) \le C\rho^*(r).$$

Lemma 2.2.5. *If $\Phi \in \Delta_2$, then its derivative Φ' satisfies*

$$\Phi'(2t) \le C_\Phi \Phi'(t), \quad a.e.\, t \in [0, \infty),$$

where the constant C_Φ is independent of t.

Proof. From the convexity of Φ and $\Phi(0) = 0$ it follows that its right derivative $\Phi'_+(t)$ exists for all $t \in [0, \infty)$ and it is increasing. By (1.2.12) we have

$$\Phi(t) = \int_0^t \Phi'(s)\, ds = \int_0^t \Phi'_+(s)\, ds,$$

since $\Phi' = \Phi'_+$ a.e. Then, for all $t \in (0, \infty)$,

$$\Phi'_+(2t) \le \frac{1}{t}\int_{2t}^{3t} \Phi'_+(s)\, ds \le \frac{1}{t}\Phi(3t) \le \frac{C_\Phi}{t}\Phi(t) \le C_\Phi \Phi'_+(t).$$

This shows the conclusion. □

Lemma 2.2.6. *If $\Phi \in \overline{\nabla}_2$, then $\Phi((\cdot)^\theta) \in \overline{\nabla}_2$ for some $\theta \in (0, 1)$.*

Proof. If $\Phi \in \overline{\nabla}_2$, then there exists a constant $k > 1$ such that

$$\Phi(t) \le \frac{1}{2k}\Phi(kt).$$

Take $\theta \in (0, 1)$ such that $k^{2(1/\theta - 1)} \le 2$. Then $k^2 \le (2k^2)^\theta$ and

$$\Phi(t^\theta) \le \frac{1}{2k}\Phi(kt^\theta) \le \frac{1}{(2k)^2}\Phi(k^2 t^\theta) \le \frac{1}{2(2k^2)}\Phi((2k^2 t)^\theta).$$

That is, $\Phi((\cdot)^\theta) \in \overline{\nabla}_2$. □

Remark 2.2.2. There exists $\Phi \in \nabla_2$ such that $\Phi((\cdot)^\theta) \notin \mathit{\Phi}_Y$ for any $\theta \in (0,1)$. Actually, let

$$\Phi(r) = \max(r^2, 3r-2) = \begin{cases} r^2, & 0 \le r \le 1, \\ 3r-2, & 1 < r < 2, \\ r^2, & 2 \le r. \end{cases}$$

Then Φ is convex and satisfies (1.2.14) with $k = 8$. However, $3r^\theta - 2$ is not convex for any $\theta \in (0,1)$.

Let $(\Phi, \widetilde{\Phi})$ be a complementary pair of functions in $\mathit{\Phi}_Y$. For the Orlicz spaces on a measure space (Ω, μ) we have the following generalized Hölder's inequality;

$$\int_\Omega |f(x)g(x)|\, d\mu(x) \le 2\|f\|_{L^\Phi(\Omega,\mu)} \|g\|_{L^{\widetilde{\Phi}}(\Omega,\mu)} \quad \text{for} \quad f \in L^\Phi(\Omega,\mu),\ g \in L^{\widetilde{\Phi}}(\Omega,\mu). \tag{2.2.7}$$

See [43].

2.3 Proof of Theorem 2.1.1

To prove Theorem 2.1.1 we may assume that $\Phi, \Psi \in \mathit{\Phi}_Y$ instead of $\Phi, \Psi \in \overline{\mathit{\Phi}}_Y$. Actually, if (2.1.1) holds for some $\Phi, \Psi \in \overline{\mathit{\Phi}}_Y$, then take $\Phi_1, \Psi_1 \in \mathit{\Phi}_Y$ with $\Phi \approx \Phi_1$ and $\Psi \approx \Psi_1$. Then, instead of Φ and Ψ, Φ_1 and Ψ_1 satisfy (2.1.1) for some positive constant A' by (1.2.11).

We need a couple of auxilary estimates. The following lemma was proved in [2, Lemma 2.1]:

Lemma 2.3.1. *There exist a constant $C > 0$ such that for all $x \in B(0, r/2)$ and $r > 0$,*

$$\int_0^{r/2} \frac{\rho(t)}{t} dt \le C I_\rho \chi_{B(0,r)}(x)$$

holds.

Proposition 2.3.2. *Let ρ satisfy* (1.1.3). *Define*

$$\tilde{\rho}(r) = \int_{k_1 r}^{k_2 r} \rho(s) \frac{ds}{s} \quad (r > 0). \tag{2.3.1}$$

Let $\tau : (0,\infty) \to (0,\infty)$ be a doubling function in the sense that $\tau(r) \sim \tau(s)$ if $0 < s \le r \le 2s$. Then, for each $r > 0$,

$$\sum_{j=-\infty}^{-1} \tilde{\rho}(2^j r) \lesssim \int_0^{k_2 r} \frac{\rho(s)}{s} ds, \tag{2.3.2}$$

$$\sum_{j=0}^{\infty} \tilde{\rho}(2^j r)\tau\big((2^j r)^{-n}\big) \lesssim \int_{k_1 r}^{\infty} \frac{\rho(s)}{s} \tau\big(s^{-n}\big) ds. \tag{2.3.3}$$

Proof. We invoke the overlapping property in [49] and by the doubling condition of τ we have

$$\begin{aligned}
\sum_{j=-\infty}^{-1} \tilde{\rho}(2^j r) &= \sum_{j=-\infty}^{-1} \int_{2^j k_1 r}^{2^j k_2 r} \rho(s) \frac{ds}{s} \\
&\le \int_0^{k_2 r} \left(\sum_{j=-\infty}^{-1} \chi_{[2^j k_1 r,\ 2^j k_2 r]}(s) \right) \frac{\rho(s)}{s} ds \\
&\lesssim \int_0^{k_2 r} \frac{\rho(s)}{s} ds
\end{aligned}$$

and

$$\begin{aligned}
\sum_{j=0}^{\infty} \tilde{\rho}(2^j r)\tau\big((2^j r)^{-n}\big) &= \int_{k_1 r}^{\infty} \left(\sum_{j=0}^{\infty} \chi_{[2^j k_1 r,\ 2^j k_2 r]}(s) \frac{\rho(s)}{s} \tau\big((2^j r)^{-n}\big) \right) ds \\
&\lesssim \int_{k_1 r}^{\infty} \left(\sum_{j=0}^{\infty} \chi_{[2^j k_1 r,\ 2^j k_2 r]}(s) \right) \frac{\rho(s)}{s} \tau\big(s^{-n}\big) ds \\
&\lesssim \int_{k_1 r}^{\infty} \frac{\rho(s)}{s} \tau\big(s^{-n}\big) ds.
\end{aligned}$$

□

To prove Theorem 2.1.1, we need the following estimate of Hedberg-type [16]:

Proposition 2.3.3. *Under the assumption of Theorem 2.1.1, for any positive constant C_0, there exists a positive constant C_1 such that, for all nonnegative functions $f \in L^\Phi(\mathbb{R}^n)$ with $f \ne 0$,*

$$I_\rho f(x) \le C_1 \|f\|_{L^\Phi} \Psi^{-1} \circ \Phi\Big(\frac{Mf(x)}{C_0 \|f\|_{L^\Phi}} \Big) \qquad (x \in \mathbb{R}^n). \tag{2.3.4}$$

Proof. Let $x \in \mathbb{R}^n$. Keeping in mind that $Mf(x) > 0$, we may assume

$$0 < \frac{Mf(x)}{C_0\|f\|_{L^\Phi}} < \infty, \quad 0 \le \Phi\left(\frac{Mf(x)}{C_0\|f\|_{L^\Phi}} \right) < \infty;$$

otherwise there is nothing to prove. If

$$\Phi\left(\frac{Mf(x)}{C_0\|f\|_{L^\Phi}}\right)=0,$$

then

$$\frac{Mf(x)}{C_0\|f\|_{L^\Phi}}\le \sup\{u\ge 0\,:\,\Phi(u)=0\}=\Phi^{-1}(0).$$

In this case

$$0<\Phi^{-1}(0)\int_0^\infty \frac{\rho(t)}{t}\,dt\le C\Psi^{-1}(0).$$

Hence

$$\begin{aligned}
I_\rho f(x) &\le C\sum_{j=-\infty}^{\infty}\frac{\tilde\rho(2^j)}{2^{jn}}\int_{|x-y|<2^j}|f(y)|\,dy\\
&\le C\left(\int_0^\infty \frac{\rho(s)}{s}\right)Mf(x)\\
&\le C\frac{\Psi^{-1}(0)}{\Phi^{-1}(0)}Mf(x)\\
&\le C\frac{1}{\Phi^{-1}(0)}\Psi^{-1}\left(\Phi\left(\frac{Mf(x)}{C_0\|f\|_{L^\Phi}}\right)\right)Mf(x)\\
&\le C\Psi^{-1}\left(\Phi\left(\frac{Mf(x)}{C_0\|f\|_{L^\Phi}}\right)\right)\|f\|_{L^\Phi}.
\end{aligned}$$

So, this case the result is valid.

If

$$\Phi\left(\frac{Mf(x)}{C_0\|f\|_{L^\Phi}}\right)>0,$$

choose $r\in(0,\infty)$ so that

$$r^{-n}=\Phi\left(\frac{Mf(x)}{C_0\|f\|_{L^\Phi}}\right).$$

We have

$$I_\rho f(x)\le C\left[\sum_{j=-\infty}^{-1}+\sum_{j=0}^{\infty}\frac{\tilde\rho(2^jr)}{(2^jr)^n}\int_{|x-y|<2^jr}f(y)dy\right]=C(\mathrm{I}+\mathrm{II})$$

for given $x\in\mathbb{R}^n$ and $r>0$.

Then from Proposition 2.3.2 with the doubling property of Φ^{-1} (Remark 1.2.1 (iii)),

$$\begin{aligned}
\mathrm{I} &\le C \sum_{j=-\infty}^{-1} \tilde{\rho}(2^j r) Mf(x) \le C \left(\int_0^{k_2 r} \frac{\rho(s)}{s} ds \right) Mf(x) \\
\mathrm{II} &\le C \sum_{j=0}^{\infty} \tilde{\rho}(2^j r) \Phi^{-1}\big((2^j r)^{-n}\big) \|f\|_{L^\Phi(B(x,2^j r))} \\
&\le C \|f\|_{L^\Phi} \int_{k_1 r}^{\infty} \Phi^{-1}\big(s^{-n}\big) \frac{\rho(s)}{s} ds.
\end{aligned}$$

Consequently, we have

$$I_\rho f(x) \lesssim \left(\int_0^{k_2 r} \frac{\rho(s)}{s} ds \right) Mf(x) + \|f\|_{L^\Phi} \int_{k_1 r}^{\infty} \Phi^{-1}\big(s^{-n}\big) \frac{\rho(s)}{s} ds.$$

Thus, by (2.1.1) and the doubling property of Φ^{-1} and Ψ^{-1}, we obtain

$$\begin{aligned}
I_\rho f(x) &\lesssim Mf(x) \frac{\Psi^{-1}((k_2 r)^{-n})}{\Phi^{-1}((k_2 r)^{-n})} + \|f\|_{L^\Phi} \, \Psi^{-1}((k_1 r)^{-n}) \\
&\lesssim Mf(x) \frac{\Psi^{-1}(r^{-n})}{\Phi^{-1}(r^{-n})} + \|f\|_{L^\Phi} \, \Psi^{-1}(r^{-n}).
\end{aligned}$$

Recall that $\Phi^{-1}(\Phi(r)) = r$ if $0 < \Phi(r) < \infty$. Thus $\Phi^{-1}(r^{-n}) = \dfrac{Mf(x)}{C_0 \|f\|_{L^\Phi}}$ and

$$I_\rho f(x) \lesssim \|f\|_{L^\Phi} \, \Psi^{-1}(r^{-n}) = \|f\|_{L^\Phi} \, \Psi^{-1} \left(\Phi \left(\frac{Mf(x)}{C_0 \|f\|_{L^\Phi}} \right) \right).$$

Therefore, we get (2.3.4). □

Now we move on to the proof of Theorem 2.1.1.

Proof of Theorem 2.1.1. Let C_0 be as in (1.2.16). Let f be a non-negative measurable function. Then by (1.2.16) and (2.3.4),

$$\begin{aligned}
&\sup_{r>0} \Psi(r)\, m\Big(\frac{I_\rho f(x)}{C_1 \|f\|_{L^\Phi}}, r\Big) = \sup_{r>0} r\, m\Big(\Psi\Big(\frac{I_\rho f(x)}{C_1 \|f\|_{L^\Phi}}\Big), r\Big) \\
\le &\sup_{r>0} r\, m\Big(\Phi\Big(\frac{Mf(x)}{C_0 \|f\|_{L^\Phi}}\Big), r\Big) \le \sup_{r>0} \Phi(r)\, m\Big(\frac{Mf(x)}{\|Mf\|_{\mathrm{W}L^\Phi}}, r\Big) \le 1,
\end{aligned}$$

i.e.

$$\|I_\rho f\|_{\mathrm{W}L^\Psi} \lesssim \|f\|_{L^\Phi}.$$

Assume in addition that $\Phi \in \nabla_2$, so that we have (1.2.17), by which we have

$$\begin{aligned}\int_{\mathbb{R}^n} \Psi\left(\frac{I_\rho f(x)}{C_1\|f\|_{L^\Phi}}\right) dx &\le \int_{\mathbb{R}^n} \Phi\left(\frac{Mf(x)}{C_0\|f\|_{L^\Phi}}\right) dx \\ &\le \int_{\mathbb{R}^n} \Phi\left(\frac{Mf(x)}{\|Mf\|_{L^\Phi}}\right) dx \le 1,\end{aligned}$$

i.e.

$$\|I_\rho f\|_{L^\Psi} \lesssim \|f\|_{L^\Phi}.$$

The proof is complete. □

2.4 Proof of Theorem 2.1.3

In this section we prove Theorem 2.1.3.

Proof of Theorem 2.1.3 (i). We may assume that $\Phi, \Psi \in \mathit{\Phi}_Y$ by (1.2.11). Let $f \in L^\Phi(\mathbb{R}^n)$. We may also assume that $\|f\|_{L^\Phi} = 1$ then $Mf(x) > 0$ for all $x \in \mathbb{R}^n$. For any $x \in \mathbb{R}^n$ and any ball $B = B(z, r) \ni x$, if

$$\Phi\left(\frac{Mf(x)}{C_0}\right) \ge \frac{1}{r^n},$$

then, by (2.2.7), $\|f\|_{L^\Phi} = 1$, (2.2.6), the doubling condition of Φ^{-1} and (2.1.7), we have

$$\begin{aligned}\rho(r) \fint_B |f| &\le 2\frac{\rho(r)}{|B|}\|\chi_B\|_{L^{\tilde{\Phi}}} \le 2\frac{\rho(r)}{|B|}|B|\Phi^{-1}\left(\frac{1}{|B|}\right) \\ &\lesssim \rho(r)\Phi^{-1}\left(\frac{1}{r^n}\right) \le A\Psi^{-1}\left(\frac{1}{r^n}\right) \le A\Psi^{-1}\left(\Phi\left(\frac{Mf(x)}{C_0}\right)\right).\end{aligned}$$

Conversely, if

$$\Phi\left(\frac{Mf(x)}{C_0}\right) \le \frac{1}{r^n},$$

then, choosing $t_0 \ge r$ such that

$$\Phi\left(\frac{Mf(x)}{C_0}\right) = \frac{1}{{t_0}^n},$$

and using (2.1.7) and (2.2.1), we have

$$\rho(r) \le \sup_{0<t\le t_0} \rho(t) \le A\frac{\Psi^{-1}\left(\Phi\left(\frac{Mf(x)}{C_0}\right)\right)}{\Phi^{-1}\left(\Phi\left(\frac{Mf(x)}{C_0}\right)\right)} \le A\frac{\Psi^{-1}\left(\Phi\left(\frac{Mf(x)}{C_0}\right)\right)}{\frac{Mf(x)}{C_0}},$$

which implies

$$\rho(r) \fint_B |f| \le AC_0 \frac{\Psi^{-1}\left(\Phi\left(\frac{Mf(x)}{C_0}\right)\right)}{Mf(x)} \fint_B |f| \le AC_0 \Psi^{-1}\left(\Phi\left(\frac{Mf(x)}{C_0}\right)\right).$$

Hence, we have

$$M_\rho f(x) \le C_1 \Psi^{-1}\left(\Phi\left(\frac{Mf(x)}{C_0}\right)\right),$$

which shows (2.1.8) by (2.2.1). □

To prove Theorem 2.1.3 (ii) we need the following lemma.

Lemma 2.4.1. *Let* $\rho : (0,\infty) \to (0,\infty)$. *Then, for all* $x \in \mathbb{R}^n$ *and* $r \in (0,\infty)$,

$$\left(\sup_{0<t\le r} \rho(t)\right) \chi_{B(0,r)}(x) \le (M_\rho \chi_{B(0,r)})(x). \tag{2.4.1}$$

Proof. Let $x \in B(0,r)$. If $t \le r$, then we can choose a ball $B(z,t)$ such that $x \in B(z,t) \subset B(0,r)$. Hence,

$$\rho(t) = \rho(t) \fint_{B(z,t)} \chi_{B(0,r)}(y)\, dy \le (M_\rho \chi_{B(0,r)})(x).$$

Therefore, we have (2.4.1). □

Proof of Theorem 2.1.3 (ii). By Lemma 2.4.1 and the boundedness of M_ρ from $L^\Phi(\mathbb{R}^n)$ to $\mathrm{w}L^\Psi(\mathbb{R}^n)$ we have

$$\left(\sup_{0<t\le r} \rho(t)\right) \|\chi_{B(0,r)}\|_{\mathrm{w}L^\Psi} \le \|M_\rho \chi_{B(0,r)}\|_{\mathrm{w}L^\Psi} \lesssim \|\chi_{B(0,r)}\|_{L^\Phi}.$$

Then, by Lemma 2.2.3 and the doubling condition of Φ^{-1} and Ψ^{-1} we have the conclusion. □

2.5 Sharp maximal operators

In this section, to prove Theorem 2.1.4, we prove two propositions involving the sharp maximal operator $M^\sharp$ defined by (1.1.6).

First we state the John-Nirenberg type theorem for the Campanato space $\mathcal{L}_{p,\psi}(\mathbb{R}^n)$, which is known by [40, Theorem 3.1] for spaces of homogeneous type. See also [1] for its proof in the case of $\mathbb{R}^n$.

Theorem 2.5.1. *Let $p \in (1,\infty)$ and $\psi : (0,\infty) \to (0,\infty)$. Assume that ψ is almost increasing. Then $\mathcal{L}_{p,\psi}(\mathbb{R}^n) = \mathcal{L}_{1,\psi}(\mathbb{R}^n)$ with equivalent norms.*

Proposition 2.5.2. *Assume that $\rho : (0,\infty) \to (0,\infty)$ satisfies* (1.1.2)*. Let $\rho^*(r)$ be as in* (2.1.12)*. Assume that ψ is almost increasing, that $r \mapsto \rho(r)/r^{n-\epsilon}$ is almost decreasing for some $\epsilon > 0$ and that the condition* (2.1.15) *holds. Then, for any $\eta \in (1,\infty)$, there exists a positive constant C such that, for all $b \in \mathcal{L}_{1,\psi}(\mathbb{R}^n)$, $f \in C^\infty_{\mathrm{comp}}(\mathbb{R}^n)$ and $x \in \mathbb{R}^n$,*

$$M^\sharp([b,I_\rho]f)(x) \le C\|b\|_{\mathcal{L}_{1,\psi}}\left(\left(M_{\psi^\eta}(|I_\rho f|^\eta)(x)\right)^{1/\eta} + \left(M_{(\rho^*\psi)^\eta}(|f|^\eta)(x)\right)^{1/\eta}\right).$$

To prove the proposition we need the following known lemma, for its proof, see Lemma 4.7 and Remark 4.1 in [1] for example.

Lemma 2.5.3 ([1, Lemma 4.7])**.** *Let $p \in [1,\infty)$ and $\psi \in \mathcal{G}^{\mathrm{inc}}$. Then there exists a positive constant C dependent only on n, p and ψ such that, for all $f \in \mathcal{L}_{1,\psi}(\mathbb{R}^n)$ and for all $x \in \mathbb{R}^n$ and $r, s \in (0,\infty)$,*

$$\left(\fint_{B(x,s)} |f(y) - f_{B(x,r)}|^p\,dy\right)^{1/p} \le C\int_r^s \frac{\psi(t)}{t}\,dt\,\|f\|_{\mathcal{L}_{1,\psi}}, \quad \text{if } 2r < s.$$

Remark 2.5.1. In Lemma 2.5.3 we also have

$$\left(\fint_{B(x,s)} |f(y) - f_{B(x,r)}|^p\,dy\right)^{1/p} \le C\left(\log_2\frac{s}{r}\right)\psi(s)\,\|f\|_{\mathcal{L}_{1,\psi}}, \quad \text{if } 2r < s,$$

since

$$\int_r^s \frac{\psi(t)}{t}\,dt \lesssim \int_r^s \frac{\psi(s)}{t}\,dt = \psi(s)\log\frac{s}{r}.$$

Proof of Proposition 2.5.2. For any ball $B = B(x,t)$, let $f = f_1 + f_2$ with $f_1 = f\chi_{2B}$, and let

$$\begin{aligned} F_1(y) &= (b(y) - b_{2B})I_\rho f(y), \\ F_2(y) &= I_\rho((b - b_{2B})f_1)(y), \\ F_3(y) &= I_\rho((b - b_{2B})f_2)(y) - C_B, \end{aligned}$$

for $y \in B$, where $C_B = I_\rho((b - b_{2B})f_2)(x)$ and

$$I_\rho((b - b_{2B})f_2)(y) = \int_{\mathbb{R}^n} \frac{\rho(|y-z|)}{|y-z|^n}(b(z) - b_{2B})f_2(z)\,dz, \quad y \in B.$$

Then we have

$$[b, I_\rho]f + C_B = [b - b_{2B}, I_\rho]f + C_B = F_1 - F_2 - F_3.$$

We show that

$$\text{(2.5.1)} \quad \fint_B |F_i(y)|\,dy \le C\|b\|_{\mathcal{L}_{1,\psi}} \left(\left(M_{\psi^\eta}(|I_\rho f|^\eta)(x)\right)^{1/\eta} + \left(M_{(\rho^*\psi)^\eta}(|f|^\eta)(x)\right)^{1/\eta} \right), \quad i = 1, 2, 3.$$

Then we have the conclusion.

Now, by Hölder's inequality with $1/\eta + 1/\eta' = 1$ and Theorem 2.5.1 we have

$$\begin{aligned}\fint_B |F_1(y)|\,dy &\le \left(\fint_B |b(y) - b_{2B}|^{\eta'}\,dy\right)^{1/\eta'} \left(\fint_B |I_\rho f(y)|^\eta\,dy\right)^{1/\eta} \\ &= \frac{1}{\psi(t)} \left(\fint_B |b(y) - b_{2B}|^{\eta'}\,dy\right)^{1/\eta'} \left(\psi(t)^\eta \fint_B |I_\rho f(y)|^\eta\,dy\right)^{1/\eta} \\ &\lesssim \|b\|_{\mathcal{L}_{1,\psi}} \left(M_{\psi^\eta}(|I_\rho f|^\eta)(x)\right)^{1/\eta}.\end{aligned}$$

Choose $v \in (1, \eta)$ such that $n/v - \epsilon/2 \ge n - \epsilon$. Then by the almost decreasingness of $r \mapsto \rho(r)/r^{n-\epsilon}$ we have the almost decreasingness of $r \mapsto \rho(r)/r^{n/v-\epsilon/2}$. Hence, from Corollary 2.1.2 it follows that there exists an N-function Ψ such that I_ρ is bounded from $L^v(\mathbb{R}^n)$ to $L^\Psi(\mathbb{R}^n)$. Let $\widetilde{\Psi}$ be the complementary function of Ψ. Then by the generalized Hölder's inequality (2.2.7), (2.2.6), (2.1.5) and the boundedness of I_ρ we have

$$\begin{aligned}\fint_B |F_2(y)|\,dy &\le \frac{2}{|B|}\|\chi_B\|_{L^{\tilde{\Psi}}(\mathbb{R}^n)}\|F_2\|_{L^\Psi(\mathbb{R}^n)} \\ &\lesssim \Psi^{-1}(1/|B|)\|(b - b_{2B})f_1\|_{L^v(\mathbb{R}^n)} \\ &\lesssim \frac{\rho^*(t)}{|B|^{1/v}}\|(b - b_{2B})f\|_{L^v(2B)}.\end{aligned}$$

Let $1/v = 1/u + 1/\eta$. Then by Hölder's inequality and Theorem 2.5.1 we have

$$\begin{aligned}&\fint_B |F_2(y)|\,dy \\ &\lesssim \rho^*(t) \left(\fint_{2B} |b(y) - b_{2B}|^u\,dy\right)^{1/u} \left(\fint_{2B} |f(y)|^\eta\,dy\right)^{1/\eta} \\ &\lesssim \frac{1}{\psi(2t)} \left(\fint_{2B} |b(y) - b_{2B}|^u\,dy\right)^{1/u} \left((\rho^*(2t)\psi(2t))^\eta \fint_{2B} |f(y)|^\eta\,dy\right)^{1/\eta} \\ &\lesssim \|b\|_{\mathcal{L}_{1,\psi}} \left(M_{(\rho^*\psi)^\eta}(|f|^\eta)(x)\right)^{1/\eta}.\end{aligned}$$

Finally, using the relation

$$\frac{1}{2} \le \frac{|y-z|}{|x-z|} \le 2 \quad \text{for } y \in B \text{ and } z \notin 2B$$

and (2.1.15), we have

$$\begin{aligned}
|F_3(y)| &= |I_\rho((b-b_{2B})f_2)(y) - I_\rho((b-b_{2B})f_2)(x)| \\
&= \left| \int_{\mathbb{R}^n} \left(\frac{\rho(|y-z|)}{|y-z|^n} - \frac{\rho(|x-z|)}{|x-z|^n} \right) (b(z)-b_{2B}) f_2(z)\, dz \right| \\
&\lesssim \int_{\mathbb{R}^n \setminus 2B} \frac{|x-y|\rho^*(|x-z|)}{|x-z|^{n+1}} |b(z)-b_{2B}||f(z)|\, dz \\
&= \sum_{j=0}^{\infty} \int_{2^{j+2}B \setminus 2^{j+1}B} \frac{|x-y|\rho^*(|x-z|)}{|x-z|^{n+1}} |b(z)-b_{2B}||f(z)|\, dz.
\end{aligned}$$

By the doubling condition of ρ^* (see Remark 2.2.1), Hölder's inequality and Lemma 2.5.3 we have

$$\begin{aligned}
&\int_{2^{j+2}B \setminus 2^{j+1}B} \frac{|x-y|\rho^*(|x-z|)}{|x-z|^{n+1}} |b(z)-b_{2B}||f(z)|\, dz \\
&\lesssim \frac{t\rho^*(2^{j+2}t)}{(2^{j+2}t)^{n+1}} \int_{2^{j+2}B \setminus 2^{j+1}B} |b(z)-b_{2B}||f(z)|\, dz \\
&\lesssim \frac{\rho^*(2^{j+2}t)}{2^{j+2}} \left(\fint_{2^{j+2}B} |b(z)-b_{2B}|^{\eta'}\, dz \right)^{1/\eta'} \left(\fint_{2^{j+2}B} |f(z)|^{\eta}\, dz \right)^{1/\eta} \\
&\le \frac{j+2}{2^{j+2}} \|b\|_{\mathcal{L}_{1,\psi}} \left((\rho^*(2^{j+2}t)\psi(2^{j+2}t))^{\eta} \fint_{2^{j+2}B} |f(z)|^{\eta}\, dz \right)^{1/\eta}.
\end{aligned}$$

Then

$$\begin{aligned}
|F_3(y)| &\lesssim \|b\|_{\mathcal{L}_{1,\psi}} \sum_{j=0}^{\infty} \frac{j+2}{2^{j+2}} \left((\rho^*(2^{j+2}t)\psi(2^{j+2}t))^{\eta} \fint_{2^{j+2}B} |f(z)|^{\eta}\, dz \right)^{1/\eta} \\
&\lesssim \|b\|_{\mathcal{L}_{1,\psi}} \left(M_{(\rho^*\psi)^\eta}(|f|^\eta)(x) \right)^{1/\eta},
\end{aligned}$$

which shows

$$\fint_B |F_3(y)|\, dy \lesssim \|b\|_{\mathcal{L}_{1,\psi}} \left(M_{(\rho^*\psi)^\eta}(|f|^\eta)(x) \right)^{1/\eta}.$$

Therefore, we have (2.5.1) and the conclusion. □

Next we define the dyadic maximal operator M^{dy}. We denote by $\mathcal{Q}^{\mathrm{dy}}$ the set of all dyadic cubes, that is,

$$\mathcal{Q}^{\mathrm{dy}} = \left\{ Q_{j,k} = \prod_{i=1}^{n} [2^{-j}k_i, 2^{-j}(k_i+1)) : j \in \mathbb{Z},\ k = (k_1, \ldots, k_n) \in \mathbb{Z}^n \right\}.$$

Then we define

$$M^{\mathrm{dy}} f(x) = \sup_{R \in \mathcal{Q}^{\mathrm{dy}},\, R \ni x} \fint_R |f(y)|\, dy, \quad x \in \mathbb{R}^n,$$

where the supremum is taken over all $R \in \mathcal{Q}^{\mathrm{dy}}$ containing x.

Next we prove the following proposition.

Proposition 2.5.4. *Let $\Phi \in \Delta_2$. If $M^{\mathrm{dy}} f \in L^\Phi(\mathbb{R}^n)$, then*

$$\|M^{\mathrm{dy}} f\|_{L^\Phi} \le C \|M^\sharp f\|_{L^\Phi}. \tag{2.5.2}$$

where C is a positive constant which is dependent only on n and Φ.

The following lemma is well known as the good lambda inequality, see [13, Theorem 3.4.4.] for example.

Lemma 2.5.5. *For all $\gamma > 0$, all $\lambda > 0$, and all locally integrable functions f on $\mathbb{R}^n$, the following estimate holds.*

$$|\{x \in \mathbb{R}^n : M^{\mathrm{dy}} f(x) > 2\lambda, M^\sharp f(x) \le \gamma\lambda\}| \le 2^n \gamma |\{x \in \mathbb{R}^n : M^{\mathrm{dy}} f(x) > \lambda\}|.$$

Proof of Proposition 2.5.4. For a positive real number N we set

$$I_N = \int_0^N \Phi'(\lambda) |\{x \in \mathbb{R}^n : M^{\mathrm{dy}} f(x) > \lambda\}|\, d\lambda.$$

We note that $I_N \le \int_{\mathbb{R}^n} \Phi(M^{\mathrm{dy}} f(x))\, dx < \infty$. By Lemma 2.2.5 we have

$$\begin{aligned} I_N &= 2 \int_0^{N/2} \Phi'(2\lambda) |\{x \in \mathbb{R}^n : M^{\mathrm{dy}} f(x) > 2\lambda\}|\, d\lambda \\ &\le 2C_\Phi \int_0^{N/2} \Phi'(\lambda) |\{x \in \mathbb{R}^n : M^{\mathrm{dy}} f(x) > 2\lambda\}|\, d\lambda. \end{aligned}$$

Then, using the good lambda inequality, we obtain the following sequence of inequalities:

$$\begin{aligned} I_N &\le 2C_\Phi \int_0^{N/2} \Phi'(\lambda) |\{x \in \mathbb{R}^n : M^{\mathrm{dy}} f(x) > 2\lambda\ , M^\sharp f(x) \le \gamma\lambda\ \}|\, d\lambda \\ &\qquad + 2C_\Phi \int_0^{N/2} \Phi'(\lambda) |\{x \in \mathbb{R}^n : M^\sharp f(x) > \gamma\lambda\}|\, d\lambda \\ &\le 2^{n+1} C_\Phi \gamma \int_0^{N/2} \Phi'(\lambda) |\{x \in \mathbb{R}^n : M^{\mathrm{dy}} f(x) > \lambda\}|\, d\lambda \\ &\qquad + 2C_\Phi \int_0^{N/2} \Phi'(\lambda) |\{x \in \mathbb{R}^n : M^\sharp f(x) > \gamma\lambda\}|\, d\lambda \\ &\le 2^{n+1} C_\Phi \gamma I_N + 2C_\Phi \frac{1}{\gamma} \int_0^{N\gamma/2} \Phi'(\lambda/\gamma) |\{x \in \mathbb{R}^n : M^\sharp f(x) > \lambda\}|\, d\lambda. \end{aligned}$$

At this point we choose γ such that $2^{n+1}C_\Phi\gamma = 1/2$. Since I_N is finite, we can substract from both sides of the inequality the quantity $I_N/2$ to obtain

$$\begin{aligned} I_N &\le 2^{n+4}C_\Phi^2 \int_0^{N/(2^{n+3}C_\Phi)} \Phi'(2^{n+2}C_\Phi\lambda)|\{x\in\mathbb{R}^n : M^\sharp f(x) > \lambda\}|\,d\lambda \\ &\le C_{n,\Phi}\int_0^\infty \Phi'(\lambda)|\{x\in\mathbb{R}^n : M^\sharp f(x)>\lambda\}|\,d\lambda, \end{aligned}$$

where $C_{n,\Phi}$ is a constant dependent only on n and Φ, from which we obtain

$$\int_{\mathbb{R}^n} \Phi(M^{\mathrm{dy}} f(x))\,dx \le C_{n,\Phi}\int_{\mathbb{R}^n}\Phi(M^\sharp f(x))\,dx.$$

This shows (2.5.2). □

2.6 Proof of Theorem 2.1.4

We first note that, for $\theta \in (0,\infty)$,

$$\||g|^\theta\|_{L^\Phi} = \left(\|g\|_{L^{\Phi((\cdot)^\theta)}}\right)^\theta. \qquad (2.6.1)$$

Lemma 2.6.1. *Under the assumption in Theorem 2.1.4 (i), if $f \in L^\infty_{\mathrm{comp}}(\mathbb{R}^n)$, then $I_\rho f \in L^\Psi(\mathbb{R}^n)$.*

Proof. If $f\in L^\infty_{\mathrm{comp}}(\mathbb{R}^n)$, then $f\in L^\Phi(\mathbb{R}^n)$, since $L^\infty_{\mathrm{comp}}(\mathbb{R}^n)\subset L^\Phi(\mathbb{R}^n)$. By (2.1.13) and Theorem 2.1.1 I_ρ is bounded from $L^\Phi(\mathbb{R}^n)$ to $L^\Theta(\mathbb{R}^n)$. Then $I_\rho f$ is in $L^\Theta(\mathbb{R}^n)$. On the other hand, since $r\mapsto \rho(r)/r^{n-\epsilon}$ is almost decreasing, if the support of f is in $B(0,R)$, then

$$|I_\rho f(x)| \le \|f\|_{L^\infty}\int_{B(0,R)} \frac{\rho(|x-y|)}{|x-y|^{n-\epsilon}}\,dy \lesssim \|f\|_{L^\infty}\int_0^R \frac{\rho(t)}{t^{1-\epsilon}}\,dt < \infty.$$

Then $I_\rho f$ is in $L^\Theta(\mathbb{R}^n)\cap L^\infty(\mathbb{R}^n)$.

Next, by (2.1.14) and the almost increasingness of ψ we have

$$\Theta^{-1}(1/r^n) \lesssim \frac{\Psi^{-1}(1/r^n)}{\psi(r)} \lesssim \frac{\Psi^{-1}(1/r^n)}{\psi(1)} \quad \text{for} \quad r\ge 1,$$

and then

$$\Theta^{-1}(u)\lesssim \Psi^{-1}(u) \quad \text{for} \quad u \le 1.$$

Hence, we conclude that

$$\Psi(t) \le \begin{cases} \Theta(Ct), & t\le 1,\\ \infty, & t>1,\end{cases}$$

which shows that $L^\Theta(\mathbb{R}^n)\cap L^\infty(\mathbb{R}^n)\subset L^\Psi(\mathbb{R}^n)$. □

Proof of Theorem 2.1.4 (i). We may assume that $\Phi, \Psi \in \Delta_2 \cap \nabla_2$ and $\Theta \in \nabla_2$. We may also assume that b is real valued, since the commutator $[b, I_\rho]f$ is linear with respect to b and $\|\Re(b)\|_{\mathcal{L}_{1,\psi}}, \|\Im(b)\|_{\mathcal{L}_{1,\psi}} \le \|b\|_{\mathcal{L}_{1,\psi}}$. Let

$$b_k(x) = \begin{cases} k, & \text{if } b(x) > k, \\ b(x), & \text{if } -k \le b(x) \le k, \\ -k, & \text{if } b(x) < -k. \end{cases}$$

Then $b_k \in L^\infty(\mathbb{R}^n)$ and $\|b_k\|_{\mathcal{L}_{1,\psi}} \le (9/4)\|b\|_{\mathcal{L}_{1,\psi}}$. For $f \in C^\infty_{\mathrm{comp}}(\mathbb{R}^n)$, $b_k f$ lies in $L^\infty_{\mathrm{comp}}(\mathbb{R}^n)$, thus $I_\rho(b_k f)$ lies in $L^\Psi(\mathbb{R}^n)$ by Lemma 2.6.1. Likewise, $b_k I_\rho f$ also lies in $L^\Psi(\mathbb{R}^n)$. Since $\Psi \in \nabla_2$, $M^{\mathrm{dy}}([b, I_\rho]f)$ is also in $L^\Psi(\mathbb{R}^n)$. From this fact and Propositions 2.5.2 and 2.5.4 it follows that

$$\begin{aligned} \|[b_k, I_\rho]f\|_{L^\Psi} &\le \|M^{\mathrm{dy}}([b_k, I_\rho]f)\|_{L^\Psi} \lesssim \|M^\sharp([b_k, I_\rho]f)\|_{L^\Psi} \\ &\lesssim \|b\|_{\mathcal{L}_{1,\psi}} \left(\left\| \left(M_{\psi^\eta}(|I_\rho f|^\eta)\right)^{1/\eta} \right\|_{L^\Psi} + \left\| \left(M_{(\rho^*\psi)^\eta}(|f|^\eta)\right)^{1/\eta} \right\|_{L^\Psi} \right), \end{aligned}$$

here, we can choose $\eta \in (1, \infty)$ such that $\Phi((\cdot)^{1/\eta})$, $\Psi((\cdot)^{1/\eta})$ and $\Theta((\cdot)^{1/\eta})$ are in $\overline{\nabla}_2$ by Lemma 2.2.6. We show that

$$\left\| \left(M_{\psi^\eta}(|I_\rho f|^\eta)\right)^{1/\eta} \right\|_{L^\Psi} + \left\| \left(M_{(\rho^*\psi)^\eta}(|f|^\eta)\right)^{1/\eta} \right\|_{L^\Psi} \lesssim \|f\|_{L^\Phi},$$

where we note that ψ^η and $(\rho^*\psi)^\eta$ are almost increasing.

By Theorems 2.1.1 and 2.1.3 we see that I_ρ is bounded from $L^\Phi(\mathbb{R}^n)$ to $L^\Theta(\mathbb{R}^n)$ and M_{ψ^η} is bounded from $L^{\Theta((\cdot)^{1/\eta})}(\mathbb{R}^n)$ to $L^{\Psi((\cdot)^{1/\eta})}(\mathbb{R}^n)$, respectively. Then, using (2.6.1), we have

$$\begin{aligned} \left\| \left(M_{\psi^\eta}(|I_\rho f|^\eta)\right)^{1/\eta} \right\|_{L^\Psi} &= \left(\|M_{\psi^\eta}(|I_\rho f|^\eta)\|_{L^{\Psi((\cdot)^{1/\eta})}}\right)^{1/\eta} \\ &\lesssim \left(\||I_\rho f|^\eta\|_{L^{\Theta((\cdot)^{1/\eta})}}\right)^{1/\eta} = \|I_\rho f\|_{L^\Theta} \lesssim \|f\|_{L^\Phi}. \end{aligned}$$

From (2.1.13) and (2.1.14) it follows that

$$(\rho^*(r)\psi(r))^\eta \left(\Phi^{-1}(1/r^n)\right)^\eta \le A^{2\eta} \left(\Psi^{-1}(1/r^n)\right)^\eta.$$

By using Theorem 2.1.3, we have the boundedness of $M_{(\rho^*\psi)^\eta}$ from $L^{\Phi((\cdot)^{1/\eta})}$ to $L^{\Psi((\cdot)^{1/\eta})}$. That is,

$$\begin{aligned} \left\| \left(M_{(\rho^*\psi)^\eta}(|f|^\eta)\right)^{1/\eta} \right\|_{L^\Psi} &= \left(\left\|M_{(\rho^*\psi)^\eta}(|f|^\eta)\right\|_{L^{\Psi((\cdot)^{1/\eta})}}\right)^{1/\eta} \\ &\lesssim \left(\||f|^\eta\|_{L^{\Phi((\cdot)^{1/\eta})}}\right)^{1/\eta} = \|f\|_{L^\Phi}. \end{aligned}$$

Therefore, we obtain

$$\|[b_k, I_\rho]f\|_{L^\Psi} \lesssim \|b\|_{\mathcal{L}_{1,\psi}} \|f\|_{L^\Phi} \quad \text{for all } f \in C^\infty_{\text{comp}}(\mathbb{R}^n).$$

By the standard argument (see [13, p. 240] for example) we deduce that, for some subsequence of integers k_j, $[b_{k_j}, I_\rho]f \to [b, I_\rho]f$ a.e. Letting $j \to \infty$ and using Fatou's lemma, we have

$$\|[b, I_\rho]f\|_{L^\Psi} \lesssim \|b\|_{\mathcal{L}_{1,\psi}} \|f\|_{L^\Phi} \quad \text{for all } f \in C^\infty_{\text{comp}}(\mathbb{R}^n).$$

Since $C^\infty_{\text{comp}}(\mathbb{R}^n)$ is dense in $L^\Phi(\mathbb{R}^n)$ (see Remark 1.2.2), it follows that the commutator admits a bounded extension on $L^\Phi(\mathbb{R}^n)$ that satisfies (2.1.16). □

Proof of Theorem 2.1.4 (ii). We use the method by Janson [17]. Since $|z|^{n-\alpha}$ is infinitely differentiable in an open set, we may choose $z_0 \neq 0$ and $\delta > 0$ such that $|z|^{n-\alpha}$ can be expressed in the neighborhood $|z - z_0| < 2\delta$ as an absolutely convergent Fourier series, $|z|^{n-\alpha} = \sum a_j e^{iv_j \cdot z}$. (The exact form of the vectors v_j is irrelevant.)

Set $z_1 = z_0/\delta$. If $|z - z_1| < 2$, we have the expansion

$$|z|^{n-\alpha} = \delta^{-n+\alpha}|\delta z|^{n-\alpha} = \delta^{-n+\alpha} \sum a_j e^{iv_j \cdot \delta z}.$$

Choose now any ball $B = B(x_0, r)$. Set $y_0 = x_0 - rz_1$ and $B' = B(y_0, r)$. Then, if $x \in B$ and $y \in B'$,

$$\left|\frac{x-y}{r} - z_1\right| \leq \left|\frac{x-x_0}{r}\right| + \left|\frac{y-y_0}{r}\right| < 2.$$

Denote $\operatorname{sgn}(f(x) - f_{B'})$ by $s(x)$. Then

$$\begin{aligned}
&\int_B |b(x) - b_{B'}|\, dx = \int_B (b(x) - b_{B'}) s(x)\, dx = \frac{1}{|B'|}\int_B \int_{B'} (b(x) - b(y)) s(x)\, dy\, dx \\
&= \frac{1}{|B'|}\int_{\mathbb{R}^n}\int_{\mathbb{R}^n} (b(x) - b(y)) \frac{r^{n-\alpha}\left|\frac{x-y}{r}\right|^{n-\alpha}}{|x-y|^{n-\alpha}} s(x)\chi_B(x)\chi_{B'}(y)\, dy\, dx \\
&= \frac{r^{n-\alpha}\delta^{-n+\alpha}}{|B'|}\int_{\mathbb{R}^n}\int_{\mathbb{R}^n} \frac{b(x) - b(y)}{|x-y|^{n-\alpha}} \sum a_j e^{iv_j \cdot \delta \frac{x-y}{r}} s(x)\chi_B(x)\chi_{B'}(y)\, dy\, dx.
\end{aligned}$$

Here, we set $C = \delta^{-n+\alpha}|B(0,1)|^{-1}$ and

$$g_j(y) = e^{-iv_j \cdot \delta \frac{y}{r}}\chi_{B'}(y), \quad h_j(x) = e^{iv_j \cdot \delta \frac{x}{r}} s(x)\chi_B(x).$$

Then

$$\begin{aligned}\int_B |b(x) - b_{B'}|\,dx &= Cr^{-\alpha}\sum a_j \int_{\mathbb{R}^n}\int_{\mathbb{R}^n} \frac{b(x)-b(y)}{|x-y|^{n-\alpha}} g_j(y)h_j(x)\,dy\,dx \\ &= Cr^{-\alpha}\sum a_j \int_{\mathbb{R}^n} ([b, I_\alpha]g_j)(x)h_j(x)\,dx \\ &\le Cr^{-\alpha}\sum |a_j| \int_{\mathbb{R}^n} |([b, I_\alpha]g_j)(x)||h_j(x)|\,dx \\ &= Cr^{-\alpha}\sum |a_j| \int_B |([b, I_\alpha]g_j)(x)|\,dx \\ &\le 2Cr^{-\alpha}\sum |a_j|\|\chi_B\|_{L^{\tilde{\Psi}}}\|[b, I_\alpha]g_j\|_{L^\Psi} \\ &\le 2Cr^{-\alpha}\|[b, I_\alpha]\|_{L^\Phi\to L^\Psi}|B|\Psi^{-1}(|B|^{-1})\sum |a_j|\|g_j\|_{L^\Phi}.\end{aligned}$$

Since $\|g_j\|_{L^\Phi} = \|\chi_{B'}\|_{L^\Phi} = 1/\Phi^{-1}(|B'|^{-1}) \sim 1/\Phi^{-1}(r^{-n})$, we have

$$\frac{1}{\psi(B)} \fint_B |b(x) - b_{B'}|\,dx \lesssim \|[b, I_\alpha]\|_{L^\Phi\to L^\Psi} \frac{\Psi^{-1}(r^{-n})}{r^\alpha \psi(B)\Phi^{-1}(r^{-n})} \lesssim \|[b, I_\alpha]\|_{L^\Phi\to L^\Psi}.$$

That is, $\|b\|_{\mathcal{L}^{(1,\psi)}} \lesssim \|[b, I_\alpha]\|_{L^\Phi\to L^\Psi}$ and we have the conclusion. □

Chapter 3

Commutators on Orlicz-Morrey spaces

3.1 Theorems

First we recall the definition of Calderón-Zygmund operators following [61]. Let Ω be the set of all increasing functions $\omega : (0,\infty) \to (0,\infty)$ such that $\int_0^1 \frac{\omega(t)}{t} dt < \infty$.

Definition 3.1.1 (standard kernel)**.** Let $\omega \in \Omega$. A continuous function $K(x,y)$ on $\mathbb{R}^n \times \mathbb{R}^n \setminus \{(x,x) \in \mathbb{R}^{2n}\}$ is said to be a standard kernel of type ω if the following conditions are satisfied:

$$|K(x,y)| \le \frac{C}{|x-y|^n} \quad \text{for} \quad x \ne y, \tag{3.1.1}$$

$$|K(x,y) - K(x,z)| + |K(y,x) - K(z,x)| \le \frac{C}{|x-y|^n}\,\omega\left(\frac{|y-z|}{|x-y|}\right) \quad \text{for} \quad 2|y-z| < |x-y|. \tag{3.1.2}$$

Definition 3.1.2 (Calderón-Zygmund operator)**.** Let $\omega \in \Omega$. A linear operator T from $\mathcal{S}(\mathbb{R}^n)$ to $\mathcal{S}'(\mathbb{R}^n)$ is said to be a Calderón-Zygmund operator of type ω, if T is bounded on $L^2(\mathbb{R}^n)$ and there exists a standard kernel K of type ω such that, for $f \in C^\infty_{\mathrm{comp}}(\mathbb{R}^n)$,

$$Tf(x) = \int_{\mathbb{R}^n} K(x,y) f(y)\, dy, \quad x \notin \operatorname{supp} f. \tag{3.1.3}$$

Remark 3.1.1. If $x \notin \operatorname{supp} f$, then $K(x,y)$ is continuous on $\operatorname{supp} f$ with respect to y. Therefore, if (3.1.3) holds for $f \in C^\infty_{\mathrm{comp}}(\mathbb{R}^n)$, then (3.1.3) holds for $f \in L^1_{\mathrm{comp}}(\mathbb{R}^n)$.

It was known by [61, Theorem 2.4] that any Calderón-Zygmund operator of type $\omega \in \Omega$ is bounded on $L^p(\mathbb{R}^n)$ for $1 < p < \infty$. This result was extended to Orlicz-Morrey spaces $L^{(\Phi,\varphi)}(\mathbb{R}^n)$ by [39] as the following: Assume that $\varphi \in \mathcal{G}^{\rm dec}$ and that there exists a positive constant C such that, for all $r \in (0,\infty)$,

$$\int_r^\infty \frac{\varphi(t)}{t}\,dt \le C\varphi(r). \tag{3.1.4}$$

Let $\Phi \in \Delta_2 \cap \nabla_2$. For $f \in L^{(\Phi,\varphi)}(\mathbb{R}^n)$, we define Tf on each ball B by

$$Tf(x) = T(f\chi_{2B})(x) + \int_{\mathbb{R}^n\setminus 2B} K(x,y)f(y)\,dy, \quad x \in B. \tag{3.1.5}$$

Then the first term in the right hand side is well-defined, since $f\chi_{2B} \in L^\Phi(\mathbb{R}^n)$, and the integral of the second term converges absolutely. Moreover, $Tf(x)$ is independent of the choice of the ball containing x. By this definition we can show that T is a bounded operator on $L^{(\Phi,\varphi)}(\mathbb{R}^n)$, see [39].

For functions f in Orlicz-Morrey spaces, we define $[b,T]f$ on each ball B by

$$[b,T]f(x) = [b,T](f\chi_{2B})(x) + \int_{\mathbb{R}^n\setminus 2B} (b(x)-b(y))K(x,y)f(y)\,dy, \quad x \in B, \tag{3.1.6}$$

see Remark 3.5.1 for its well-definedness. Then we have the following theorem.

Theorem 3.1.1. *Let $\Phi,\Psi \in \overline{\mathit{\Phi}}_Y$, $\varphi \in \mathcal{G}^{\rm dec}$ and $\psi \in \mathcal{G}^{\rm inc}$. Let T be a Calderón-Zygmund operator of type $\omega \in \Omega$.*

(i) *Let $\Phi,\Psi \in \overline{\Delta}_2 \cap \overline{\nabla}_2$ and $\int_0^1 \frac{\omega(t)\log(1/t)}{t}\,dt < \infty$. Assume that φ satisfies (3.1.4) and that there exists a positive constant C_0 such that, for all $r \in (0,\infty)$,*

$$\psi(r)\Phi^{-1}(\varphi(r)) \le C_0\Psi^{-1}(\varphi(r)). \tag{3.1.7}$$

If $b \in \mathcal{L}_{1,\psi}(\mathbb{R}^n)$, then $[b,T]f$ in (3.1.6) is well-defined for all $f \in L^{(\Phi,\varphi)}(\mathbb{R}^n)$ and there exists a positive constant C, independent of b and f, such that

$$\|[b,T]f\|_{L^{(\Psi,\varphi)}} \le C\|b\|_{\mathcal{L}_{1,\psi}}\|f\|_{L^{(\Phi,\varphi)}}.$$

(ii) *Conversely, assume that there exists a positive constant C_0 such that, for all $r \in (0,\infty)$,*

$$C_0\psi(r)\Phi^{-1}(\varphi(r)) \ge \Psi^{-1}(\varphi(r)). \tag{3.1.8}$$

If T is a convolution type such that

$$Tf(x) = p.v. \int_{\mathbb{R}^n} K(x-y)f(y)\,dy$$

with homogeneous kernel K satisfying $K(x) = |x|^{-n}K(x/|x|)$, $\int_{S^{n-1}} K = 0$, $K \in C^\infty(S^{n-1})$ and $K \not\equiv 0$, and if $[b,T]$ is bounded from $L^{(\Phi,\varphi)}(\mathbb{R}^n)$ to $L^{(\Psi,\varphi)}(\mathbb{R}^n)$, then b is in $\mathcal{L}_{1,\psi}(\mathbb{R}^n)$ and there exists a positive constant C, independent of b, such that

$$\|b\|_{\mathcal{L}_{1,\psi}} \le C\|[b,T]\|_{L^{(\Phi,\varphi)} \to L^{(\Psi,\varphi)}},$$

where $\|[b,T]\|_{L^{(\Phi,\varphi)} \to L^{(\Psi,\varphi)}}$ is the operator norm of $[b,T]$ from $L^{(\Phi,\varphi)}(\mathbb{R}^n)$ to $L^{(\Psi,\varphi)}(\mathbb{R}^n)$.

Remark 3.1.2. From the theorem above we have the following several corollaries.

(i) Take $\Phi(t) = t^p$. Then we have the result for generalized Morrey spaces $L^{(p,\varphi)}(\mathbb{R}^n)$. This case is known by [1, Theorem 2.1], which is an extension of Di Fazio and Ragusa [8, Theorem 1].

(ii) Take $\varphi(r) = 1/r^n$. Then we have the result for Orlicz spaces $L^\Phi(\mathbb{R}^n)$. This case is an extension of Janson [17, Theorem].

(iii) Take $\Phi(t) = \Psi(t) = t^p$, $\varphi(r) = 1/r^n$ and $\psi \equiv 1$. Then $L^{(\Phi,\varphi)}(\mathbb{R}^n) = L^{(\Psi,\varphi)}(\mathbb{R}^n) = L^p(\mathbb{R}^n)$ and $\mathcal{L}_{1,\psi}(\mathbb{R}^n) = \mathrm{BMO}(\mathbb{R}^n)$. This case is the result by Coifman, Rochberg and Weiss [5].

To state the result on the commutator $[b, I_\rho]$ we first recall the boundedness of I_ρ on the Orlicz-Morrey spaces. Let $\Phi, \Psi \in \overline{\varPhi}_Y$ and $\varphi \in \mathcal{G}^{\mathrm{dec}}$. If $\Phi \in \overline{\nabla}_2$ and

$$\int_0^r \frac{\rho(t)}{t}\,dt\;\Phi^{-1}(\varphi(r)) + \int_r^\infty \frac{\rho(t)\,\Phi^{-1}(\varphi(t))}{t}\,dt \lesssim \Psi^{-1}(\varphi(r))$$

holds for all $r \in (0,\infty)$, then I_ρ is bounded from $L^{(\Phi,\varphi)}(\mathbb{R}^n)$ to $L^{(\Psi,\varphi)}(\mathbb{R}^n)$, see [38, Theorem 7.3]. More precisely, in [38, Theorem 7.3] the author assumed that Φ and Ψ are bijective, but it can be extended to $\Phi, \Psi \in \overline{\varPhi}_Y$ by the boundedness of I_ρ from $L^\Phi(\mathbb{R}^n)$ to $L^\Psi(\mathbb{R}^n)$ with $\Phi \in \overline{\nabla}_2$ as we did in Theorem 2.1.1.

Now we state the result on the commutator $[b, I_\rho]$. For the well-definedness of $[b, I_\rho]$ on $L^{(\Phi,\varphi)}(\mathbb{R}^n)$, see Remark 3.5.2.

Theorem 3.1.2. *Let* $\Phi, \Psi \in \overline{\varPhi}_Y$, $\varphi \in \mathcal{G}^{\mathrm{dec}}$, $\psi \in \mathcal{G}^{\mathrm{inc}}$ *and* $\rho : (0,\infty) \to (0,\infty)$. *Assume that* ρ *satisfies* (1.1.2) *and* (1.1.3).

(i) *Let* $\Phi, \Psi \in \overline{\Delta}_2 \cap \overline{\nabla}_2$. *Assume that* φ *satisfies* (3.1.4) *and that* $r \mapsto \rho(r)/r^{n-\epsilon}$ *is almost decreasing for some* $\epsilon \in (0,n)$. *Assume also that there exist positive constants* C_ρ, C_0, C_1 *and a function* $\Theta \in \overline{\nabla}_2$ *such that, for all* $r, s \in (0,\infty)$,

$$C_\rho \frac{\rho(r)}{r^{n-\epsilon}} \ge \frac{\rho(s)}{s^{n-\epsilon}}, \quad \text{if } r < s, \tag{3.1.9}$$

$$\left| \frac{\rho(r)}{r^n} - \frac{\rho(s)}{s^n} \right| \le C_\rho \, |r-s| \frac{1}{r^{n+1}} \int_0^r \frac{\rho(t)}{t}\, dt, \quad \text{if } \frac{1}{2} \le \frac{r}{s} \le 2, \tag{3.1.10}$$

$$\int_0^r \frac{\rho(t)}{t}\, dt \; \Phi^{-1}(\varphi(r)) + \int_r^\infty \frac{\rho(t)\, \Phi^{-1}(\varphi(t))}{t}\, dt \le C_0 \Theta^{-1}(\varphi(r)), \tag{3.1.11}$$

$$\psi(r)\Theta^{-1}(\varphi(r)) \le C_1 \Psi^{-1}(\varphi(r)). \tag{3.1.12}$$

If $b \in \mathcal{L}_{1,\psi}(\mathbb{R}^n)$, *then* $[b, I_\rho]f$ *is well-defined for all* $f \in L^{(\Phi,\varphi)}(\mathbb{R}^n)$ *and there exists a positive constant* C, *independent of* b *and* f, *such that*

$$\|[b, I_\rho]f\|_{L^{(\Psi,\varphi)}} \le C \|b\|_{\mathcal{L}_{1,\psi}} \|f\|_{L^{(\Phi,\varphi)}}. \tag{3.1.13}$$

(ii) *Conversely, assume that* $0 < \alpha < n$ *and that there exists a positive constant* C_0 *such that, for all* $r \in (0,\infty)$,

$$\Psi^{-1}(\varphi(r)) \le C_0 r^\alpha \psi(r) \Phi^{-1}(\varphi(r)).$$

If $[b, I_\alpha]$ *is bounded from* $L^{(\Phi,\varphi)}(\mathbb{R}^n)$ *to* $L^{(\Psi,\varphi)}(\mathbb{R}^n)$, *then* b *is in* $\mathcal{L}_{1,\psi}(\mathbb{R}^n)$ *and there exists a positive constant* C, *independent of* b, *such that*

$$\|b\|_{\mathcal{L}_{1,\psi}} \le C \|[b, I_\alpha]\|_{L^{(\Phi,\varphi)} \to L^{(\Psi,\varphi)}},$$

where $\|[b, I_\alpha]\|_{L^{(\Phi,\varphi)} \to L^{(\Psi,\varphi)}}$ *is the operator norm of* $[b, I_\alpha]$ *from* $L^{(\Phi,\varphi)}(\mathbb{R}^n)$ *to* $L^{(\Psi,\varphi)}(\mathbb{R}^n)$.

Remark 3.1.3. From the theorem above we have the following several corollaries.

(i) Take $\Phi(t) = t^p$. Then we have the result for generalized Morrey spaces $L^{(p,\varphi)}(\mathbb{R}^n)$. This case is known by [1, Theorem 2.2].

(ii) Take $\varphi(r) = 1/r^n$. Then we have the result for Orlicz spaces $L^\Phi(\mathbb{R}^n)$. This case is known by Theorem 2.1.4.

(iii) Take $\rho(r) = r^\alpha$, $\Phi(t) = t^p$, $\Psi(t) = t^q$, $\varphi(r) = 1/r^n$ and $\psi \equiv 1$. Then $L^{(\Phi,\varphi)}(\mathbb{R}^n) = L^p(\mathbb{R}^n)$, $L^{(\Psi,\varphi)}(\mathbb{R}^n) = L^q(\mathbb{R}^n)$ and $\mathcal{L}_{1,\psi}(\mathbb{R}^n) = \mathrm{BMO}(\mathbb{R}^n)$. This case is the result by Chanillo [3].

For the case $\psi \in \mathcal{G}^{\mathrm{dec}}$, we have the following theorems.

Theorem 3.1.3. *Let* $\Phi, \Psi \in \overline{\nabla}_2$, $\Phi_0 \in \overline{\Delta}_2$ *and* $\varphi, \psi, \theta \in \mathcal{G}^{\mathrm{dec}}$. *Assume that*

$$\Phi_0^{-1}(t\psi(r))\Phi^{-1}(t\varphi(r)) \lesssim \Psi^{-1}(t\theta(r)) \tag{3.1.14}$$

for all $r, t \in (0, \infty)$. *Assume also that* φ, ψ, θ *satisfy* (3.1.4). *Let* T *be a Calderón-Zygmund operator of type* $\omega \in \Omega$. *If* $b \in \mathcal{L}^{(\Phi_0,\psi)}(\mathbb{R}^n)$, *then* $[b, T]f$ *is well-defined for all* $f \in L^{(\Phi,\varphi)}(\mathbb{R}^n)$ *and there exists a positive constant* C, *independent of* b *and* f, *such that*

$$\|[b, T]f\|_{L^{(\Psi,\theta)}} \le C\|b\|_{\mathcal{L}^{(\Phi_0,\psi)}}\|f\|_{L^{(\Phi,\varphi)}}.$$

Theorem 3.1.4. *Let* $\Phi \in \overline{\nabla}_2$, $\Phi_0 \in \overline{\Delta}_2$, $\Psi \in \overline{\mathit{\Phi}}_Y$ *and* $\varphi \in \mathcal{G}^{\mathrm{dec}}$. *Assume that* ρ *satisfies* (1.1.2) *and* (1.1.3) *and that* φ *satisfies* (3.1.4). *Assume also that there exist* $\Psi_0 \in \overline{\nabla}_2$ *and* $\Theta \in \overline{\mathit{\Phi}}_Y$ *such that* $\Phi^{-1}\Phi_0^{-1} \sim \Psi_0^{-1}$, $\Phi_0^{-1}\Theta^{-1} \lesssim \Psi^{-1}$ *and* (3.1.11). *If* $b \in \mathcal{L}^{(\Phi_0,\varphi)}(\mathbb{R}^n)$, *then* $[b, I_\rho]f$ *is well-defined for all* $f \in L^{(\Phi,\varphi)}(\mathbb{R}^n)$ *and there exists a positive constant* C, *independent of* b *and* f, *such that*

$$\|[b, I_\rho]f\|_{L^{(\Psi,\varphi)}} \le C\|b\|_{\mathcal{L}^{(\Phi_0,\varphi)}}\|f\|_{L^{(\Phi,\varphi)}}.$$

At the end of this section we note that, to prove the theorems, we may assume that $\Phi, \Psi \in \mathit{\Phi}_Y$ instead of $\Phi, \Psi \in \overline{\mathit{\Phi}}_Y$. For example, if Φ and Ψ satisfy (3.1.7) and $\Phi \approx \Phi_1$, $\Psi \approx \Psi_1$, then Φ_1 and Ψ_1 also satisfy (3.1.7) by the relation (1.2.11). Moreover, $L^{(\Phi,\varphi)}(\mathbb{R}^n) = L^{(\Phi_1,\varphi)}(\mathbb{R}^n)$ and $L^{(\Psi,\varphi)}(\mathbb{R}^n) = L^{(\Psi_1,\varphi)}(\mathbb{R}^n)$ with equivalent quasi-norms.

3.2 Properties on Young functions and Orlicz-Morrey spaces

Let $\Phi \in \mathit{\Phi}_Y$, $\varphi : (0, \infty) \to (0, \infty)$ and $B = B(a, r) \subset \mathbb{R}^n$, and let $\mu_B = dx/(|B|\varphi(r))$. Then by the relation (1.2.21) and (2.2.5) we have

$$\|\chi_B\|_{\Phi,\varphi,B} = \|\chi_B\|_{L^\Phi(B,\mu_B)} = \frac{1}{\Phi^{-1}(1/\mu_B(B))} = \frac{1}{\Phi^{-1}(\varphi(r))}. \tag{3.2.1}$$

Moreover, by the relation (1.2.21) and (2.2.7) we have

$$\frac{1}{|B|\varphi(r)}\int_B |f(x)g(x)|\,dx \le 2\|f\|_{\Phi,\varphi,B}\|g\|_{\widetilde{\Phi},\varphi,B}. \tag{3.2.2}$$

Lemma 3.2.1. *Let $\Phi \in \mathit{\Phi}_Y$ and $\varphi \in \mathcal{G}^{\rm dec}$. Then there exists a constant $C \ge 1$ such that, for any ball $B = B(a,r)$,*

$$\frac{1}{\Phi^{-1}(\varphi(r))} \le \|\chi_B\|_{L^{(\Phi,\varphi)}} \le \frac{C}{\Phi^{-1}(\varphi(r))}. \tag{3.2.3}$$

Proof. Fix a ball $B = B(a,r)$. By (3.2.1) we have

$$\frac{1}{\Phi^{-1}(\varphi(r))} = \|\chi_B\|_{\Phi,\varphi,B} \le \|\chi_B\|_{L^{(\Phi,\varphi)}}.$$

To show the second inequality in (3.2.3), let $\lambda = 1/\Phi^{-1}(\varphi(r))$. Then it is enough to show that, for some $C \ge 1$ and for all balls $B' = B(b,r')$ with $B \cap B' \ne \emptyset$,

$$\frac{1}{\varphi(r')}\fint_{B'} \Phi\left(\frac{\chi_B(x)}{C\lambda}\right) dx \le 1. \tag{3.2.4}$$

If $B' \subset 3B$, then $\varphi(r') \gtrsim \varphi(3r) \sim \varphi(r)$. Hence

$$\frac{1}{\varphi(r')|B'|}\int_{B'} \Phi\left(\frac{\chi_B(x)}{\lambda}\right) dx \le \frac{1}{\varphi(r')}\Phi\left(\frac{1}{\lambda}\right) \lesssim \frac{1}{\varphi(r)}\Phi\left(\frac{1}{\lambda}\right) \le 1.$$

In the above we used (2.2.1) for the last inequality. If $B' \cap (3B)^{\complement} \ne \emptyset$ and $B' \cap B \ne \emptyset$, then $3B' \supset B$. Hence $\varphi(r')|B'| \sim \varphi(3r')|3B'| \gtrsim \varphi(r)|B|$ and

$$\frac{1}{\varphi(r')|B'|}\int_{B'} \Phi\left(\frac{\chi_B(x)}{\lambda}\right) dx \lesssim \frac{1}{\varphi(r)|B|}\int_B \Phi\left(\frac{1}{\lambda}\right) dx \le 1.$$

Then, by the convexity of Φ we have (3.2.4). □

Lemma 3.2.2. *Let $\Phi \in \mathit{\Phi}_Y$, $\varphi : (0,\infty) \to (0,\infty)$ and $B = B(a,r) \subset \mathbb{R}^n$. Then*

$$\fint_B |f(x)|\,dx \le 2\Phi^{-1}(\varphi(r))\|f\|_{\Phi,\varphi,B}. \tag{3.2.5}$$

Moreover, if $\Phi \in \nabla_2$, then there exists $p \in (1,\infty)$ such that

$$\left(\fint_B |f(y)|^p dy\right)^{1/p} \le C\Phi^{-1}(\varphi(r))\|f\|_{\Phi,\varphi,B},$$

where the constant C is independent of f and $B = B(a,r)$.

Proof. By (3.2.2), (3.2.1) and (1.2.15) we have

$$\begin{aligned} \fint_B |f(x)|\,dx &\le 2\varphi(r)\|f\|_{\Phi,\varphi,B}\|\chi_B\|_{\widetilde{\Phi},\varphi,B} \\ &= \frac{2\varphi(r)}{\widetilde{\Phi}^{-1}(\varphi(r))}\|f\|_{\Phi,\varphi,B} \\ &\le 2\Phi^{-1}(\varphi(r))\|f\|_{\Phi,\varphi,B}. \end{aligned}$$

Next we assume that $\Phi \in \nabla_2$. Then by Lemma 2.2.6 we can take $\theta \in (0,1)$ such that $\Phi((\cdot)^\theta) \in \overline{\nabla}_2$. Let $\Phi_\theta \in \nabla_2$ such that $\Phi_\theta \approx \Phi\left((\cdot)^\theta\right)$. Then ${\Phi_\theta}^{-1} \sim (\Phi^{-1})^{1/\theta}$. Let $p = 1/\theta$. Then $\||f|^p\|_{\Phi_\theta,\varphi,B} \sim (\|f\|_{\Phi,\varphi,B})^p$. Using (3.2.5), we have

$$\left(\fint_B |f(y)|^p dy\right)^{1/p} \le \left(2{\Phi_\theta}^{-1}(\varphi(r))\||f|^p\|_{\Phi_\theta,\varphi,B}\right)^{1/p} \sim \Phi^{-1}(\varphi(r))\|f\|_{\Phi,\varphi,B}. \qquad \square$$

Lemma 3.2.3. *Let $\Phi \in \Delta_2$ and $\varphi \in \mathcal{G}^{\mathrm{dec}}$. If φ satisfies* (3.1.4)*, then there exists a positive constant C such that, for all $r \in (0,\infty)$,*

$$\int_r^\infty \frac{\Phi^{-1}(\varphi(t))}{t}\,dt \le C\Phi^{-1}(\varphi(r)). \tag{3.2.6}$$

Proof. By Remark 1.2.1 (iv) we see that $t \to \dfrac{\Phi^{-1}(t)}{t^p}$ is almost increasing for some $p \in (0,1]$. From (3.1.4) it follows that

$$\int_r^\infty \frac{\varphi(t)^p}{t}\,dt \le C_p\,\varphi(r)^p,$$

for some $C_p > 0$, see [40, Lemma 7.1]. Then

$$\begin{aligned} \int_r^\infty \frac{\Phi^{-1}(\varphi(t))}{t}\,dt &= \int_r^\infty \frac{\Phi^{-1}(\varphi(t))}{\varphi(t)^p}\frac{\varphi(t)^p}{t}\,dt \\ &\lesssim \frac{\Phi^{-1}(\varphi(r))}{\varphi(r)^p}\int_r^\infty \frac{\varphi(t)^p}{t}\,dt \le C_p\Phi^{-1}(\varphi(r)). \end{aligned}$$

This shows the conclusion. $\square$

Lemma 3.2.4 ([38, Theorem 4.1])**.** *Let $\Phi_i \in \mathit{\Phi}_Y$ and $\varphi_i \in \mathcal{G}^{\mathrm{dec}}$, $i = 1,2,3$. Assume that*

$$\Phi_1^{-1}(t\varphi_1(r))\Phi_3^{-1}(t\varphi_3(r)) \le C\Phi_2^{-1}(t\varphi_2(r))$$

for all $r,t \in (0,\infty)$. Then

$$\|fg\|_{L^{(\Phi_2,\varphi_2)}} \le 2C\|f\|_{L^{(\Phi_1,\varphi_1)}}\|g\|_{L^{(\Phi_3,\varphi_3)}}.$$

3.3 Fractional maximal operators

It is well known that the Hardy-Littlewood maximal operator M is bounded on $L^p(\mathbb{R}^n)$ if $1 < p \le \infty$. This boundedness was extended to Orlicz-Morrey spaces by [38, Theorem 6.1]. Namely, if Φ is bijective and in ∇_2 and $\varphi \in \mathcal{G}^{\mathrm{dec}}$, then M is bounded on $L^{(\Phi,\varphi)}(\mathbb{R}^n)$. This result is valid for any $\Phi \in \overline{\nabla}_2$ by the modular inequality

$$\int_{\mathbb{R}^n} \Phi(Mf(x))\, dx \le \int_{\mathbb{R}^n} \Phi(C|f(x)|)\, dx$$

in [23, Theorem 1.2.1].

For the operator M_ρ we prove the following theorem.

Theorem 3.3.1. *Let $\Phi, \Psi \in \overline{\mathit{\Phi}}_Y$, $\varphi \in \mathcal{G}^{\mathrm{dec}}$ and $\rho : (0,\infty) \to (0,\infty)$. Assume that $\lim_{r\to\infty} \varphi(r) = 0$ or that $\Psi^{-1}(t)/\Phi^{-1}(t)$ is almost decreasing on $(0,\infty)$. If there exists a positive constant A such that, for all $r \in (0,\infty)$,*

$$\left(\sup_{0<t\le r} \rho(t)\right) \Phi^{-1}(\varphi(r)) \le A\Psi^{-1}(\varphi(r)), \tag{3.3.1}$$

then, for any positive constant C_0, there exists a positive constant C_1 such that, for all $f \in L^{(\Phi,\varphi)}(\mathbb{R}^n)$ with $f \not\equiv 0$,

$$\Psi\left(\frac{M_\rho f(x)}{C_1\|f\|_{L^{(\Phi,\varphi)}}}\right) \le \Phi\left(\frac{Mf(x)}{C_0\|f\|_{L^{(\Phi,\varphi)}}}\right), \quad x \in \mathbb{R}^n. \tag{3.3.2}$$

Consequently, if $\Phi \in \overline{\nabla}_2$, then M_ρ is bounded from $L^{(\Phi,\varphi)}(\mathbb{R}^n)$ to $L^{(\Psi,\varphi)}(\mathbb{R}^n)$.

Remark 3.3.1. If ρ is almost increasing or if $\Psi^{-1}(t)/\Phi^{-1}(t)$ is almost decreasing, then the inequality $\rho(r)\Phi^{-1}(\varphi(r)) \lesssim \Psi^{-1}(\varphi(r))$ implies (3.3.1).

Proof of Theorem 3.3.1. We may assume that $\Phi, \Psi \in \mathit{\Phi}_Y$. We may also assume that φ is continuous and strictly decreasing, see Remark 1.2.3. Let $f \in L^{(\Phi,\varphi)}(\mathbb{R}^n)$, and fix $x \in \mathbb{R}^n$. To prove (3.3.2) we may assume that $\|f\|_{L^{(\Phi,\varphi)}} = 1$ and that $0 < Mf(x) < \infty$.

We show that, for any ball $B = B(a,r)$ containing x,

$$\rho(r) \fint_B |f| \le C_1\Psi^{-1}\left(\Phi\left(\frac{Mf(x)}{C_0}\right)\right). \tag{3.3.3}$$

Then we have the pointwise estimate

$$\Psi\left(\frac{M_\rho f(x)}{C_1}\right) \le \Phi\left(\frac{Mf(x)}{C_0}\right),$$

which is the conclusion.

To show (3.3.3), we consider two cases:

$$\Phi\left(\frac{Mf(x)}{C_0}\right) \ge \varphi(r) \quad \text{or} \quad \Phi\left(\frac{Mf(x)}{C_0}\right) \le \varphi(r).$$

If $\Phi\left(\frac{Mf(x)}{C_0}\right) \ge \varphi(r)$, then by (3.2.5) and $\|f\|_{\Phi,\varphi,B} \le 1$, we have

$$\rho(r) \fint_B |f| \le 2\rho(r)\Phi^{-1}(\varphi(r)).$$

Combining this inequality with (3.3.1) we have

$$\rho(r) \fint_B |f| \le 2A\Psi^{-1}(\varphi(r)) \le 2A\Psi^{-1}\left(\Phi\left(\frac{Mf(x)}{C_0}\right)\right).$$

Conversely, let $\Phi\left(\frac{Mf(x)}{C_0}\right) \le \varphi(r)$. If $\lim_{r\to\infty} \varphi(r) = 0$ then we can choose $t_0 \in [r, \infty)$ such that

$$\Phi\left(\frac{Mf(x)}{C_0}\right) = \varphi(t_0).$$

Using (3.3.1) and (2.2.1), we have

$$\rho(r) \le \sup_{0<t\le t_0} \rho(t) \le A\frac{\Psi^{-1}(\varphi(t_0))}{\Phi^{-1}(\varphi(t_0))} = A\frac{\Psi^{-1}\left(\Phi\left(\frac{Mf(x)}{C_0}\right)\right)}{\Phi^{-1}\left(\Phi\left(\frac{Mf(x)}{C_0}\right)\right)} \le A\frac{\Psi^{-1}\left(\Phi\left(\frac{Mf(x)}{C_0}\right)\right)}{\frac{Mf(x)}{C_0}}.$$

If $\Psi^{-1}(t)/\Phi^{-1}(t)$ is almost decreasing, then $\Phi\left(\frac{Mf(x)}{C_0}\right) \le \varphi(r)$ implies that

$$\rho(r) \le A\frac{\Psi^{-1}(\varphi(r))}{\Phi^{-1}(\varphi(r))} \lesssim \frac{\Psi^{-1}\left(\Phi\left(\frac{Mf(x)}{C_0}\right)\right)}{\Phi^{-1}\left(\Phi\left(\frac{Mf(x)}{C_0}\right)\right)} \le \frac{\Psi^{-1}\left(\Phi\left(\frac{Mf(x)}{C_0}\right)\right)}{\frac{Mf(x)}{C_0}}.$$

In any way we have

$$\rho(r) \fint_B |f| \le AC_0\frac{\Psi^{-1}\left(\Phi\left(\frac{Mf(x)}{C_0}\right)\right)}{Mf(x)} \fint_B |f| \le AC_0\Psi^{-1}\left(\Phi\left(\frac{Mf(x)}{C_0}\right)\right).$$

Then we have (3.3.3) and the proof is complete. □

3.4 Orlicz-Campanato spaces and relations to Orlicz-Morrey spaces

In this section we define Orlicz-Campanato spaces and investigate their relations to Orlicz-Morrey spaces.

Definition 3.4.1 (Orlicz-Campanato space)**.** For $\Phi \in \overline{\mathit{\Phi}}_Y$ and $\varphi : (0, \infty) \to (0, \infty)$, let

$$\mathcal{L}^{(\Phi,\varphi)}(\mathbb{R}^n) = \left\{ f \in L^1_{\mathrm{loc}}(\mathbb{R}^n) : \|f\|_{\mathcal{L}^{(\Phi,\varphi)}} < \infty \right\},$$
$$\|f\|_{\mathcal{L}^{(\Phi,\varphi)}} = \sup_B \|f - f_B\|_{\Phi,\varphi,B},$$

where the supremum is taken over all balls B in $\mathbb{R}^n$ and $\|f\|_{\Phi,\varphi,B}$ is as in (1.2.19).

Then $\|\cdot\|_{\mathcal{L}^{(\Phi,\varphi)}}$ is a quasi-norm modulo constant functions and thereby $\mathcal{L}^{(\Phi,\varphi)}(\mathbb{R}^n)$ is a quasi-Banach space. If $\Phi \in \mathit{\Phi}_Y$, then $\|\cdot\|_{L^{(\Phi,\varphi)}(\mathbb{R}^n)}$ is a norm modulo constant functions and thereby $\mathcal{L}^{(\Phi,\varphi)}(\mathbb{R}^n)$ is a Banach space. If $\Phi \approx \Psi$ and $\varphi \sim \psi$, then $\mathcal{L}^{(\Phi,\varphi)}(\mathbb{R}^n) = \mathcal{L}^{(\Psi,\psi)}(\mathbb{R}^n)$ with equivalent quasi-norms.

If $\Phi(r) = r^p$ $(1 \le p < \infty)$, then we denote $\mathcal{L}^{(\Phi,\varphi)}(\mathbb{R}^n)$ by $\mathcal{L}^{(p,\varphi)}(\mathbb{R}^n)$, which coincides with $\mathcal{L}_{p,\varphi^p}(\mathbb{R}^n)$ defined by Definition 1.2.1.

In this section we prove the following two theorems. Let $\mathcal{C}$ be the set of all constant functions. The first theorem is an extension of [30, Theorem 2.1] and [37, Theorem 2.1].

Theorem 3.4.1. *Let $\Phi \in \overline{\mathit{\Phi}}_Y$ and $\varphi \in \mathcal{G}^{\mathrm{dec}}$. Assume that $\Phi \in \overline{\Delta}_2$ and that φ satisfies* (3.1.4)*. Then*

$$\mathcal{L}^{(\Phi,\varphi)}(\mathbb{R}^n)/\mathcal{C} = L^{(\Phi,\varphi)}(\mathbb{R}^n) \quad \textit{and} \quad \|f\|_{\mathcal{L}^{(\Phi,\varphi)}} \sim \|f - \lim_{r\to\infty} f_{B(0,r)}\|_{L^{(\Phi,\varphi)}}.$$

More precisely, for every $f \in \mathcal{L}^{(\Phi,\varphi)}(\mathbb{R}^n)$, $f_{B(0,r)}$ converges as $r \to \infty$, and the mapping $f \mapsto f - \lim_{r\to\infty} f_{B(0,r)}$ is bijective and bicontinuous from $\mathcal{L}^{(\Phi,\varphi)}(\mathbb{R}^n)/\mathcal{C}$ to $L^{(\Phi,\varphi)}(\mathbb{R}^n)$. In this case $\lim_{r\to\infty} f_{B(a,r)} = \lim_{r\to\infty} f_{B(0,r)}$ for all $a \in \mathbb{R}^n$.

Theorem 3.4.2. *Let $\Phi \in \overline{\mathit{\Phi}}_Y$ and $\varphi \in \mathcal{G}^{\mathrm{dec}}$. If $\Phi \in \overline{\Delta}_2$, then there exists a positive constant C such that, for all $f \in L^1_{\mathrm{loc}}(\mathbb{R}^n)$,*

$$\|f\|_{\mathcal{L}^{(\Phi,\varphi)}} \le C\|M^\sharp f\|_{L^{(\Phi,\varphi)}}. \tag{3.4.1}$$

Moreover, if $\Phi \in \overline{\nabla}_2$ and φ satisfies (3.1.4)*, then*

$$C^{-1}\|f\|_{\mathcal{L}^{(\Phi,\varphi)}} \le \|M^\sharp f\|_{L^{(\Phi,\varphi)}} \le C\|f\|_{\mathcal{L}^{(\Phi,\varphi)}}. \tag{3.4.2}$$

By Theorems 3.4.1 and 3.4.2 we have the following corollary.

Corollary 3.4.3. *Let $\Phi \in \overline{\mathit{\Phi}}_Y$ and $\varphi \in \mathcal{G}^{\mathrm{dec}}$. Assume that $\Phi \in \overline{\Delta}_2$ and that φ satisfies* (3.1.4)*. Then there exist a positive constant C such that, for any $f \in L^{(\Phi,\varphi)}(\mathbb{R}^n)$ satisfying $\lim_{r\to\infty} f_{B(0,r)} = 0$,*

$$\|f\|_{L^{(\Phi,\varphi)}} \le C\|M^\sharp f\|_{L^{(\Phi,\varphi)}}. \tag{3.4.3}$$

Moreover, if $\Phi \in \overline{\nabla}_2$, then

$$C^{-1}\|f\|_{L^{(\Phi,\varphi)}} \le \|M^\sharp f\|_{L^{(\Phi,\varphi)}} \le C\|f\|_{L^{(\Phi,\varphi)}}.$$

To prove the theorems we prepare several lemmas.

Lemma 3.4.4. *Let $\Phi \in \overline{\mathit{\Phi}}_Y$ and $\varphi : (0,\infty) \to (0,\infty)$. Then, for any two balls B_1 and B_2 such that $B_1 \subset B_2$,*

$$|f_{B_1} - f_{B_2}| \le 2\frac{|B_2|}{|B_1|}\Phi^{-1}(\varphi(r_2))\|f\|_{\mathcal{L}^{(\Phi,\varphi)}}, \tag{3.4.4}$$

where r_2 is the radius of B_2.

Proof. By (3.2.5) we have

$$\begin{aligned}
|f_{B_1} - f_{B_2}| &\le \frac{1}{|B_1|}\int_{B_1} |f(x) - f_{B_2}|\,dx \\
&\le \frac{|B_2|}{|B_1|} \fint_{B_2} |f(x) - f_{B_2}|\,dx \\
&\le 2\frac{|B_2|}{|B_1|}\Phi^{-1}(\varphi(r_2))\|f\|_{\mathcal{L}^{(\Phi,\varphi)}}.
\end{aligned}$$

□

Lemma 3.4.5. *Let $\Phi \in \overline{\mathit{\Phi}}_Y$ and $\varphi : (0,\infty) \to (0,\infty)$. Assume that φ satisfies the doubling condition. Then there exists a positive constant C such that, for any $f \in \mathcal{L}^{(\Phi,\varphi)}(\mathbb{R}^n)$ and for any two balls $B(a,r)$ and $B(b,s)$ satisfying $B(a,r) \subset B(b,s)$,*

$$|f_{B(a,r)} - f_{B(b,s)}| \le C\int_r^{2s} \frac{\Phi^{-1}(\varphi(t))}{t}\,dt\,\|f\|_{\mathcal{L}^{(\Phi,\varphi)}}. \tag{3.4.5}$$

Proof. Let $f \in \mathcal{L}^{(\Phi,\varphi)}(\mathbb{R}^n)$. Take balls $B_j = B(a_j, 2^j r)$, $j = 0,1,2,\dots$, such that

$$B(a,r) = B_0 \subset B_1 \subset B_2 \subset \cdots \subset B_{k-1} \subset B(b,s) \subset B_k.$$

Then, by (3.4.4) and the doubling condition of $\Phi^{-1}(\varphi(\cdot))$ we have

$$\begin{aligned}|f_{B(a,r)}-f_{B(b,s)}| &\le |f_{B_0}-f_{B_1}|+|f_{B_1}-f_{B_2}|+\cdots+|f_{B_{k-1}}-f_{B(b,s)}|\\ &\le 2^{n+1}\sum_{j=1}^{k-1}\Phi^{-1}(\varphi(2^j r))\|f\|_{\mathcal{L}^{(\Phi,\varphi)}}+2\frac{|B(b,s)|}{|B_{k-1}|}\Phi^{-1}(\varphi(s))\|f\|_{\mathcal{L}^{(\Phi,\varphi)}}\\ &\lesssim \sum_{j=1}^{k-1}\int_{2^{j-1}r}^{2^j r}\frac{\Phi^{-1}(\varphi(t))}{t}dt\,\|f\|_{\mathcal{L}^{(\Phi,\varphi)}}+\int_s^{2s}\frac{\Phi^{-1}(\varphi(t))}{t}dt\,\|f\|_{\mathcal{L}^{(\Phi,\varphi)}}\\ &\lesssim \int_r^{2s}\frac{\Phi^{-1}(\varphi(t))}{t}dt\,\|f\|_{\mathcal{L}^{(\Phi,\varphi)}}.\end{aligned}$$

This shows the conclusion. □

Lemma 3.4.6. *Let $\Phi\in\overline{\varPhi}_Y$ and $\varphi\in\mathcal{G}^{\mathrm{dec}}$. If $\int_1^\infty \frac{\Phi^{-1}(\varphi(t))}{t}\,dt<\infty$, then, for every $f\in\mathcal{L}^{(\Phi,\varphi)}(\mathbb{R}^n)$, there exists a constant $\sigma(f)$ such that $\sigma(f)=\lim_{r\to\infty} f_{B(a,r)}$ for all $a\in\mathbb{R}^n$.*

Proof. Let $f\in\mathcal{L}^{(\Phi,\varphi)}(\mathbb{R}^n)$. By (3.4.5) we see that

$$|f_{B(0,r)}-f_{B(0,s)}|\le C\int_r^{2s}\frac{\Phi^{-1}(\varphi(t))}{t}\,dt\,\|f\|_{\mathcal{L}^{(\Phi,\varphi)}}\to 0 \quad \text{as } r,s\to\infty \text{ with } r<s.$$

Hence $f_{B(0,r)}$ converges as r tends to infinity by Cauchy's test. Let $\sigma(f)=\lim_{r\to\infty} f_{B(0,r)}$. If $|a|\le r$, then $B(a,r)\subset B(0,2r)$. From (3.4.4) it follows that

$$\begin{aligned}|f_{B(a,r)}-\sigma(f)| &\le |f_{B(a,r)}-f_{B(0,2r)}|+|f_{B(0,2r)}-\sigma(f)|\\ &\le 2^{n+1}\|f\|_{\mathcal{L}^{(\Phi,\varphi)}}\Phi^{-1}(\varphi(2r))+|f_{B(0,2r)}-\sigma(f)|\to 0 \quad \text{as } r\to\infty,\end{aligned}$$

since $\Phi^{-1}(\varphi(2r))\to 0$ as $r\to\infty$ by the assumption. □

Remark 3.4.1. If $\Phi\in\Delta_2$ and φ satisfies (3.1.4), then $\int_1^\infty \frac{\Phi^{-1}(\varphi(t))}{t}\,dt<\infty$ by Lemma 3.2.3.

Proof of Theorem 3.4.1. We may assume that $\Phi\in\Delta_2$. Let $f\in\mathcal{L}^{(\Phi,\varphi)}(\mathbb{R}^n)$. Then by the definition of $\mathcal{L}^{(\Phi,\varphi)}(\mathbb{R}^n)$, for any ball $B=B(a,r)$,

$$\frac{1}{\varphi(r)}\fint_B \Phi\left(\frac{|f(x)-f_B|}{\|f\|_{\mathcal{L}^{(\Phi,\varphi)}}}\right)dx\le 1.$$

Letting $s\to\infty$ in (3.4.5) and using Lemma 3.4.6 with Remark 3.4.1, we have

$$|f_B-\sigma(f)|\lesssim\int_r^\infty\frac{\Phi^{-1}(\varphi(t))}{t}dt\,\|f\|_{\mathcal{L}^{(\Phi,\varphi)}}.$$

By Lemma 3.2.3 we have

$$|f_B - \sigma(f)| \le C\Phi^{-1}(\varphi(r))\, \|f\|_{\mathcal{L}^{(\Phi,\varphi)}}$$

for some $C \ge 1$ independent of f. Then by (2.2.1) we have

$$\Phi\left(\frac{|f_B - \sigma(f)|}{C\|f\|_{\mathcal{L}^{(\Phi,\varphi)}}}\right) \le \Phi(\Phi^{-1}(\varphi(r))) \le \varphi(r).$$

By the convexity of Φ we have

$$\begin{aligned}
&\frac{1}{\varphi(r)} \fint_B \Phi\left(\frac{|f(x) - \sigma(f)|}{2C\|f\|_{\mathcal{L}^{(\Phi,\varphi)}}}\right) dx \\
&\le \frac{1}{\varphi(r)} \fint_B \frac{1}{2}\left\{\Phi\left(\frac{|f(x) - f_B|}{C\|f\|_{\mathcal{L}^{(\Phi,\varphi)}}}\right) + \Phi\left(\frac{|f_B - \sigma(f)|}{C\|f\|_{\mathcal{L}^{(\Phi,\varphi)}}}\right)\right\} dx \le 1.
\end{aligned}$$

This means that $f - \sigma(f) \in L^{(\Phi,\varphi)}(\mathbb{R}^n)$ and that

$$\|f - \sigma(f)\|_{L^{(\Phi,\varphi)}} \le 2C\|f\|_{\mathcal{L}^{(\Phi,\varphi)}}.$$

Conversely, let $f \in L^{(\Phi,\varphi)}(\mathbb{R}^n)$. Then by (3.2.5) we have, for any ball $B = B(a,r)$,

$$|f_B| \le \fint_B |f(x)|\, dx \le 2\Phi^{-1}(\varphi(r))\|f\|_{L^{(\Phi,\varphi)}}. \tag{3.4.6}$$

Since $\Phi^{-1}(\varphi(r)) \to 0$ as $r \to \infty$ by the assumption, we conclude that $\sigma(f) = \lim_{r\to\infty} f_{B(a,r)} = 0$ by Lemma 3.4.6. Moreover, from (3.4.6) and (3.2.1) it follows that

$$\|f_B\|_{\Phi,\varphi,B} = |f_B|\|1\|_{\Phi,\varphi,B} \le 2\Phi^{-1}(\varphi(r))\|f\|_{L^{(\Phi,\varphi)}} \frac{1}{\Phi^{-1}(\varphi(r))} = 2\|f\|_{L^{(\Phi,\varphi)}}.$$

Then

$$\|f - f_B\|_{\Phi,\varphi,B} \le \|f\|_{\Phi,\varphi,B} + \|f_B\|_{\Phi,\varphi,B} \le 3\|f\|_{L^{(\Phi,\varphi)}}.$$

This shows that $f \in \mathcal{L}^{(\Phi,\varphi)}(\mathbb{R}^n)$ and

$$\|f\|_{\mathcal{L}^{(\Phi,\varphi)}} \le 3\|f\|_{L^{(\Phi,\varphi)}} = 3\|f - \sigma(f)\|_{L^{(\Phi,\varphi)}}.$$

The proof is complete. □

To prove Theorem 3.4.2 we define local versions of the dyadic maximal operator and the dyadic sharp maximal operator. For any cube $Q \subset \mathbb{R}^n$ centered at $a \in \mathbb{R}^n$

and with side length $2r > 0$, we denote by $\mathcal{Q}^{\rm dy}(Q)$ the set of all dyadic cubes with respect to Q, that is,

$$\mathcal{Q}^{\rm dy}(Q) = \left\{ Q_{j,k} = a + \prod_{i=1}^{n} [2^{-j} k_i r, 2^{-j}(k_i+1)r) : j \in \mathbb{Z},\ k = (k_1, \cdots, k_n) \in \mathbb{Z}^n \right\}.$$

For any cube $Q \subset \mathbb{R}^n$, let

$$M_Q^{\rm dy} f(x) = \sup_{R \in \mathcal{Q}^{\rm dy}(Q),\, x \in R \subset Q} \fint_R |f(y)|\, dy,$$

$$M_Q^{\sharp,\rm dy} f(x) = \sup_{R \in \mathcal{Q}^{\rm dy}(Q),\, x \in R \subset Q} \fint_R |f(y) - f_R|\, dy.$$

Then we have the following lemma.

Lemma 3.4.7. *Let $\Phi \in \Delta_2$ and $\Phi(2t) \le C_\Phi \Phi(t)$ for all $t \in [0, \infty]$ and some $C_\Phi \ge 1$. Then there exists a positive constant $C_{n,\Phi}$ such that, for any $f \in L^1_{\rm loc}(\mathbb{R}^n)$ and any cube Q,*

$$\int_Q \Phi\left(M_Q^{\rm dy} f(x)\right) dx \le C_{n,\Phi} \int_Q \Phi\left(M_Q^{\sharp,\rm dy} f(x)\right) dx + 2C_\Phi \Phi\left(|f|_Q\right) |Q|, \tag{3.4.7}$$

and

$$\int_Q \Phi\left(M_Q^{\rm dy}(f(x) - f_Q)\right) dx \le (C_{n,\Phi} + 2C_\Phi) \int_Q \Phi\left(M_Q^{\sharp,\rm dy} f(x)\right) dx. \tag{3.4.8}$$

To prove Lemma 3.4.7 we use the following local version good λ inequality:

Lemma 3.4.8 (Tsutsui [59], Komori-Furuya [24])**.** *Let $f \in L^1_{\rm loc}(\mathbb{R}^n)$. Then, for any cube Q, $0 < \gamma \le 1$ and $\lambda > |f|_Q$, we have*

$$\begin{aligned} &\left|\left\{x \in Q :\ M_Q^{\rm dy} f(x) > 2\lambda, M_Q^{\sharp,\rm dy} f(x) \le \gamma\lambda\right\}\right| \\ &\qquad \le 2^n \gamma \left|\left\{x \in Q :\ M_Q^{\rm dy} f(x) > \lambda\right\}\right|. \end{aligned} \tag{3.4.9}$$

Proof of Lemma 3.4.7. For $N > 0$, let

$$I_N = \int_0^N \Phi'(\lambda) \left|\left\{x \in Q : M_Q^{\rm dy} f(x) > \lambda\right\}\right| d\lambda.$$

If $N > 2|f|_Q$, then

$$\begin{aligned} I_N &= \int_0^{2|f|_Q} + \int_{2|f|_Q}^{N} \Phi'(\lambda) \left|\{x \in Q : M_Q^{\rm dy} f(x) > \lambda\}\right| d\lambda \\ &\le \Phi\left(2|f|_Q\right) |Q| + 2 \int_{|f|_Q}^{N/2} \Phi'\left(2\lambda\right) \left|\{x \in Q : M_Q^{\rm dy} f(x) > 2\lambda\}\right| d\lambda. \end{aligned}$$

By the doubling conditions of Φ and Φ' and the good-λ inequality (3.4.9), we have

$$\begin{aligned} I_N &\le 2C_{\Phi'}\bigg(2^n\gamma\int_{|f|_Q}^{N/2}\Phi'(\lambda)\left|\{x\in Q: M_Q^{\mathrm{dy}}f(x)>\lambda\}\right|d\lambda \\ &\qquad + \int_{|f|_Q}^{N/2}\Phi'(\lambda)\left|\{x\in Q: M_Q^{\sharp,\mathrm{dy}}f(x)>\gamma\lambda\}\right|d\lambda\bigg) + C_\Phi\Phi\left(|f|_Q\right)|Q| \\ &\le 2^{n+1}\gamma C_{\Phi'}I_N + \frac{2C_{\Phi'}}{\gamma}\int_0^{N\gamma/2}\Phi'\left(\frac{\lambda}{\gamma}\right)\left|\{x\in Q: M_Q^{\sharp,\mathrm{dy}}f(x)>\lambda\}\right|d\lambda \\ &\qquad\qquad + C_\Phi\Phi\left(|f|_Q\right)|Q|. \end{aligned}$$

At this point we pick a γ such that $2^{n+1}\gamma C_{\Phi'} = 1/2$, then

$$I_N \le C_{n,\Phi}\int_0^\infty \Phi'(\lambda)\left|\{x\in Q: M_Q^{\sharp,\mathrm{dy}}f(x)>\lambda\}\right|d\lambda + 2C_\Phi\Phi\left(|f|_Q\right)|Q|.$$

Letting $N\to\infty$, we deduce (3.4.7). Next, substitute $f-f_Q$ for f in (3.4.7). Then

$$\begin{aligned} &\int_Q \Phi\left(M_Q^{\mathrm{dy}}(f(x)-f_Q)\right)dx \\ &\le C_{n,\Phi}\int_Q\Phi\left(M_Q^{\sharp,\mathrm{dy}}f(x)\right)dx + 2C_\Phi\Phi\left(|f-f_Q|_Q\right)|Q| \\ &\le C_{n,\Phi}\int_Q\Phi\left(M_Q^{\sharp,\mathrm{dy}}f(x)\right)dx + 2C_\Phi\Phi\left(\min_{x\in Q}M_Q^{\sharp,\mathrm{dy}}f(x)\right)|Q| \\ &\le (C_{n,\Phi}+2C_\Phi)\int_Q\Phi\left(M_Q^{\sharp,\mathrm{dy}}f(x)\right)dx, \end{aligned}$$

which is (3.4.8). □

Proof of Theorem 3.4.2. To prove (3.4.1) we may assume that $\|M^\sharp f\|_{L^{(\Phi,\varphi)}} = 1$. Then it is enough to prove that there exists a positive constant C' such that, for all balls $B = B(a,r)$,

$$\frac{1}{|B|\varphi(r)}\int_B\Phi\left(\frac{|f(x)-f_B|}{C'}\right)dx\le 1. \tag{3.4.10}$$

Take the cube Q such that $B\subset Q\subset\sqrt{n}B$. By Jensen's inequality we have

$$\Phi\left(|f_Q-f_B|\right)\le\Phi\left(⨍_B|f(x)-f_Q|dx\right)\le ⨍_B\Phi\left(|f(x)-f_Q|\right)dx.$$

Then

$$\begin{aligned} \int_B\Phi\left(\frac{|f(x)-f_B|}{2}\right)dx &\le \frac12\int_B\Big(\Phi\left(|f(x)-f_Q|\right)+\Phi\left(|f_B-f_Q|\right)\Big)dx \\ &\le \int_B\Phi\left(|f(x)-f_Q|\right)dx\le\int_Q\Phi\left(|f(x)-f_Q|\right)dx. \end{aligned} \tag{3.4.11}$$

By (3.4.8) and the fact that $M_Q^{\sharp,\mathrm{dy}} f \le C_n M^\sharp f$ for some positive constant C_n, we have

$$\int_Q \Phi\left(|f(x)-f_Q|\right) dx \le \int_Q \Phi\left(M_Q^{\mathrm{dy}}(f(x)-f_Q)\right) dx$$
$$\le (C_{n,\Phi}+2C_\Phi)\int_Q \Phi\left(M_Q^{\sharp,\mathrm{dy}} f(x)\right) dx$$
$$\le (C_{n,\Phi}+2C_\Phi)\int_{\sqrt{n}B} \Phi\left(C_n M^\sharp f(x)\right) dx. \tag{3.4.12}$$

Take $C_{n,\varphi}\ge 1$ such that $|\sqrt{n}B|\varphi(\sqrt{n}r)\le C_{n,\varphi}|B|\varphi(r)$. Then, from (3.4.11) and (3.4.12) it follows that

$$\frac{1}{|B|\varphi(r)}\int_B \Phi\left(\frac{|f(x)-f_B|}{2}\right) dx \le \frac{C_{n,\varphi}(C_{n,\Phi}+2C_\Phi)}{|\sqrt{n}B|\varphi(\sqrt{n}r)}\int_{\sqrt{n}B}\Phi\left(C_n M^\sharp f(x)\right) dx,$$

which shows that

$$\frac{1}{|B|\varphi(r)}\int_B \Phi\left(\frac{|f(x)-f_B|}{2C_{n,\varphi}(C_{n,\Phi}+2C_\Phi)C_n}\right) dx$$
$$\le \frac{1}{|\sqrt{n}B|\varphi(\sqrt{n}r)}\int_{\sqrt{n}B}\Phi\left(M^\sharp f(x)\right) dx \le 1.$$

Therefore we have (3.4.10).

Next, we add the assumptions that $\Phi\in\overline{\Delta}_2$ and that φ satisfies (3.1.4). Then the Hardy-Littlewood maximal operator M is bounded on $L^{(\Phi,\varphi)}(\mathbb{R}^n)$ and then

$$\|M^\sharp f\|_{L^{(\Phi,\varphi)}} \le 2\|Mf\|_{L^{(\Phi,\varphi)}} \le C\|f\|_{L^{(\Phi,\varphi)}}. \tag{3.4.13}$$

To prove the second inequality in (3.4.2) we may assume that $f\in\mathcal{L}^{(\Phi,\varphi)}(\mathbb{R}^n)$. By Theorem 3.4.1 we see that $f_{B(0,r)}$ converges as $r\to\infty$. Setting $\sigma(f)=\lim_{r\to\infty} f_{B(0,r)}$, we have $\|f-\sigma(f)\|_{L^{(\Phi,\varphi)}}\le C\|f\|_{\mathcal{L}^{(\Phi,\varphi)}}$. Substituting $f-\sigma(f)$ for f into (3.4.13), we have

$$\|M^\sharp f\|_{L^{(\Phi,\varphi)}} \le C\|f-\sigma(f)\|_{L^{(\Phi,\varphi)}} \le C\|f\|_{\mathcal{L}^{(\Phi,\varphi)}},$$

which shows the conclusion. □

3.5 Well-definedness of the commutators

In this section we prove that the commutators $[b,T]f$ and $[b,I_\rho]f$ are well-defined for all $b\in\mathcal{L}_{1,\psi}(\mathbb{R}^n)$ and $f\in L^{(\Phi,\varphi)}(\mathbb{R}^n)$.

Lemma 3.5.1. *Let* $\Phi \in \mathit{\Phi}_Y$ *and* $\varphi \in \mathcal{G}^{\mathrm{dec}}$. *Let* K *be a standard kernel satisfying* (3.1.1). *Then there exists a positive constant* C *such that, for all* $f \in L^{(\Phi,\varphi)}(\mathbb{R}^n)$ *and all balls* $B = B(z, r)$,

$$\int_{\mathbb{R}^n \setminus 2B} |K(x,y)f(y)|\, dy \le C \int_{2r}^{\infty} \frac{\Phi^{-1}(\varphi(t))}{t}\, dt\, \|f\|_{L^{(\Phi,\varphi)}}, \quad x \in B.$$

Proof. If $x \in B$ and $y \notin 2B$, then $|z-y|/2 \le |x-y| \le 3|z-y|/2$. From (3.1.1) it follows that $|K(x,y)| \lesssim |x-y|^{-n} \sim |z-y|^{-n}$. Then

$$\int_{\mathbb{R}^n \setminus 2B} |K(x,y)||f(y)|\, dy \lesssim \int_{\mathbb{R}^n \setminus 2B} \frac{|f(y)|}{|z-y|^n}\, dy = \sum_{j=1}^{\infty} \int_{2^{j+1}B \setminus 2^j B} \frac{|f(y)|}{|z-y|^n}\, dy.$$

By (3.2.5), Hölder's inequality and the doubling condition of φ we have

$$\begin{aligned}\int_{2^{j+1}B \setminus 2^j B} \frac{|f(y)|}{|z-y|^n}\, dy \lesssim ⨍_{2^{j+1}B} |f(y)|\, dy &\lesssim \Phi^{-1}\left(\varphi(2^{j+1}r)\right) \|f\|_{L^{(\Phi,\varphi)}} \\ &\lesssim \int_{2^j r}^{2^{j+1}r} \frac{\Phi^{-1}(\varphi(t))}{t}\, dt\, \|f\|_{L^{(\Phi,\varphi)}}.\end{aligned}$$

Therefore, we have the conclusion. □

Lemma 3.5.2. *Let* $\Phi \in \nabla_2$, $\varphi \in \mathcal{G}^{\mathrm{dec}}$, $\psi \in \mathcal{G}^{\mathrm{inc}}$ *and* K *be a standard kernel satisfying* (3.1.1). *Then there exists a positive constant* C *such that, for all* $b \in \mathcal{L}_{1,\psi}(\mathbb{R}^n)$, *all* $f \in L^{(\Phi,\varphi)}(\mathbb{R}^n)$ *and all balls* $B = B(z, r)$,

$$\begin{aligned}&\int_{\mathbb{R}^n \setminus 2B} |(b(y)-b_B)K(x,y)f(y)|\, dy \\ &\qquad \le C \int_r^{\infty} \frac{\psi(t)}{t} \left(\int_t^{\infty} \frac{\Phi^{-1}(\varphi(u))}{u}\, du \right) dt\, \|b\|_{\mathcal{L}_{1,\psi}} \|f\|_{L^{(\Phi,\varphi)}}, \quad x \in B.\end{aligned}$$

Proof. If $x \in B$ and $y \notin 2B$, then $|z-y|/2 \le |x-y| \le 3|z-y|/2$. From (3.1.1) it follows that $|K(x-y)| \lesssim |x-y|^{-n} \sim |z-y|^{-n}$. Then

$$\begin{aligned}\int_{\mathbb{R}^n \setminus 2B} |(b(y)-b_B)K(x,y)f(y)|\, dy &\lesssim \int_{\mathbb{R}^n \setminus 2B} \frac{|(b(y)-b_B)f(y)|}{|z-y|^n}\, dy \\ &= \sum_{j=1}^{\infty} \int_{2^{j+1}B \setminus 2^j B} \frac{|(b(y)-b_B)f(y)|}{|z-y|^n}\, dy.\end{aligned}$$

By Lemma 3.2.2 we can find $p \in (1,\infty)$ such that

$$\left(⨍_{2^{j+1}B} |f(y)|^p dy \right)^{1/p} \lesssim \Phi^{-1}\left(\varphi(2^{j+1}r)\right) \|f\|_{L^{(\Phi,\varphi)}}.$$

By Hölder's inequality, Lemma 2.5.3 and the doubling condition of ψ and φ we have

$$\begin{aligned}
&\int_{2^{j+1}B\setminus 2^jB} \frac{|(b(y)-b_B)f(y)|}{|z-y|^n}\,dy \\
&\sim \frac{1}{(2^{j+1}r)^n}\int_{2^{j+1}B\setminus 2^jB} |(b(y)-b_B)f(y)|\,dy \\
&\lesssim \left(\fint_{2^{j+1}B} |b-b_B|^{p'}\,dy\right)^{1/p'} \left(\fint_{2^{j+1}B} |f(y)|^p\,dy\right)^{1/p} \\
&\lesssim \int_r^{2^{j+1}r} \frac{\psi(t)}{t}\,dt\,\Phi^{-1}(\varphi(2^{j+1}r))\,\|b\|_{\mathcal{L}_{1,\psi}}\|f\|_{L^{(\Phi,\varphi)}} \\
&\lesssim \int_{2^jr}^{2^{j+1}r} \left(\int_r^u \frac{\psi(t)}{t}\,dt\right)\frac{\Phi^{-1}(\varphi(u))}{u}\,du\,\|b\|_{\mathcal{L}_{1,\psi}}\|f\|_{L^{(\Phi,\varphi)}}.
\end{aligned}$$

Therefore,

$$\begin{aligned}
&\int_{\mathbb{R}^n\setminus 2B} |(b(y)-b_B)K(x,y)f(y)|\,dy \\
&\lesssim \int_r^{\infty} \left(\int_r^u \frac{\psi(t)}{t}\,dt\right)\frac{\Phi^{-1}(\varphi(u))}{u}\,du\,\|b\|_{\mathcal{L}_{1,\psi}}\|f\|_{L^{(\Phi,\varphi)}} \\
&= \int_r^{\infty} \frac{\psi(t)}{t}\left(\int_t^{\infty} \frac{\Phi^{-1}(\varphi(u))}{u}\,du\right)dt\,\|b\|_{\mathcal{L}_{1,\psi}}\|f\|_{L^{(\Phi,\varphi)}}.
\end{aligned}$$

This is the conclusion. □

Remark 3.5.1. Under the assumption in Theorem 3.1.1 (i), let $b \in \mathcal{L}_{1,\psi}(\mathbb{R}^n)$ and $f \in L^{(\Phi,\varphi)}(\mathbb{R}^n)$. Since $\Phi \in \overline{\Delta}_2$, there exists $p \in (1,\infty)$ such that $t^p \lesssim \Phi(t)$ for $t \geq 1$, see Remark 1.2.1 (v). Then $L^{(\Phi,\varphi)}(\mathbb{R}^n) \subset L^{\Phi}_{\mathrm{loc}}(\mathbb{R}^n) \subset L^{p}_{\mathrm{loc}}(\mathbb{R}^n)$, which implies $f \in L^{p}_{\mathrm{loc}}(\mathbb{R}^n)$ and $bf \in L^{p_1}_{\mathrm{loc}}(\mathbb{R}^n)$ for all $p_1 \in (1,p)$ by Theorem 2.5.1. Hence, $T(f\chi_{2B})$ and $T(bf\chi_{2B})$ are well-defined for any ball $B = B(z,r)$. By (3.1.4), (3.1.7) and Lemma 3.2.3 we have

$$\begin{aligned}
\text{(3.5.1)}\quad &\int_r^{\infty} \frac{\psi(t)}{t}\left(\int_t^{\infty} \frac{\Phi^{-1}(\varphi(u))}{u}\,du\right)dt \\
&\lesssim \int_r^{\infty} \frac{\psi(t)\Phi^{-1}(\varphi(t))}{t}\,dt \lesssim \int_r^{\infty} \frac{\Psi^{-1}(\varphi(t))}{t}\,dt \lesssim \Psi^{-1}(\varphi(r)).
\end{aligned}$$

Then, by Lemmas 3.5.1 and 3.5.2, the integrals

$$\int_{\mathbb{R}^n\setminus 2B} |K(x,y)f(y)|\,dy \quad \text{and} \quad \int_{\mathbb{R}^n\setminus 2B} |K(x,y)b(y)f(y)|\,dy$$

are finite. That is, we can write

$$[b,T]f(x) = [b,T](f\chi_{2B})(x) + \int_{\mathbb{R}^n\setminus 2B} (b(x)-b(y))K(x,y)f(y)\,dy, \quad x \in B.$$

Moreover, if $x \in B_1 \cap B_2$, then, taking B_3 such that $B_1 \cup B_2 \subset B_3$, we have

$$\begin{aligned}
&\left([b,T](f\chi_{2B_i})(x) + \int_{\mathbb{R}^n\setminus 2B_i} (b(x)-b(y))K(x,y)f(y)\,dy\right) \\
&- \left([b,T](f\chi_{2B_3})(x) + \int_{\mathbb{R}^n\setminus 2B_3} (b(x)-b(y))K(x,y)f(y)\,dy\right) \\
&= -[b,T](f\chi_{2B_3\setminus 2B_i})(x) + \int_{2B_3\setminus 2B_i} (b(x)-b(y))K(x,y)f(y)\,dy = 0,
\end{aligned}$$

by (3.1.3). That is,

$$\begin{aligned}
&[b,T](f\chi_{2B_1})(x) + \int_{\mathbb{R}^n\setminus 2B_1} (b(x)-b(y))K(x,y)f(y)\,dy \\
&= [b,T](f\chi_{2B_2})(x) + \int_{\mathbb{R}^n\setminus 2B_2} (b(x)-b(y))K(x,y)f(y)\,dy, \quad x \in B_1 \cap B_2.
\end{aligned}$$

This shows that $[b,T]f(x)$ in (3.1.6) is independent of the choice of the ball B containing x.

Lemma 3.5.3. *Under the assumption of Theorem 3.1.1 (i), there exists a positive constant C such that, for all $b \in \mathcal{L}_{1,\psi}(\mathbb{R}^n)$, all $f \in L^{(\Phi,\varphi)}(\mathbb{R}^n)$ and all balls $B = B(z,r)$,*

$$⨍_B \left(\int_{\mathbb{R}^n\setminus 2B} |(b(x)-b(y))K(x,y)f(y)|\,dy\right) dx \le C\Psi^{-1}(\varphi(B))\,\|b\|_{\mathcal{L}_{1,\psi}}\|f\|_{L^{(\Phi,\varphi)}}.$$

Proof. For $x \in B$, let

$$\begin{aligned}
G_1(x) &= |b(x)-b_B| \int_{\mathbb{R}^n\setminus 2B} |K(x,y)f(y)|\,dy, \\
G_2(x) &= \int_{\mathbb{R}^n\setminus 2B} |(b(y)-b_B)K(x,y)f(y)|\,dy.
\end{aligned}$$

Then

$$\left|\int_{\mathbb{R}^n\setminus 2B} (b(x)-b(y))K(x,y)f(y)\,dy\right| \le G_1(x) + G_2(x).$$

Using Lemmas 3.5.1 and 3.5.2, we have

$$\int_{\mathbb{R}^n\setminus 2B} |K(x,y)||f(y)|\,dy \lesssim \int_{2r}^{\infty} \frac{\Phi^{-1}(\varphi(t))}{t}\,dt\,\|f\|_{L^{(\Phi,\varphi)}}, \quad x \in B, \tag{3.5.2}$$

and

$$(3.5.3)\quad \int_{\mathbb{R}^n\setminus 2B} |b(y)-b_B||K(x,y)||f(y)|\,dy \lesssim \int_r^\infty \frac{\psi(t)}{t}\left(\int_t^\infty \frac{\Phi^{-1}(\varphi(u))}{u}\,du\right)dt\,\|b\|_{\mathcal{L}_{1,\psi}}\|f\|_{L^{(\Phi,\varphi)}}, \quad x\in B.$$

Then, using (3.5.2), (3.2.6) and (3.1.7), we have

$$\begin{aligned} ⨍_B G_1(x)\,dx &\lesssim ⨍_B |b(x)-b_B|\,dx\,\Phi^{-1}(\varphi(r))\|f\|_{L^{(\Phi,\varphi)}} \\ &\lesssim \psi(r)\Phi^{-1}(\varphi(r))\|b\|_{\mathcal{L}_{1,\psi}}\|f\|_{L^{(\Phi,\varphi)}} \\ &\lesssim \Psi^{-1}(\varphi(r))\|b\|_{\mathcal{L}_{1,\psi}}\|f\|_{L^{(\Phi,\varphi)}}. \end{aligned}$$

Using (3.5.3) and (3.5.1), we also have

$$⨍_B G_2(x)\,dx \lesssim \Psi^{-1}(\varphi(r))\|b\|_{\mathcal{L}_{1,\psi}}\|f\|_{L^{(\Phi,\varphi)}}.$$

Then we have the conclusion. □

Lemma 3.5.4. *Let $\Phi\in\mathit{\Phi}_Y$ and $\varphi\in\mathcal{G}^{\rm dec}$. Assume that ρ satisfies* (1.1.2) *and* (1.1.3). *Then there exists a positive constant C such that, for all $f\in L^{(\Phi,\varphi)}(\mathbb{R}^n)$ and all balls $B(x,r)$,*

$$\int_{\mathbb{R}^n\setminus B(x,r)} \frac{\rho(|x-y|)}{|x-y|^n}|f(y)|\,dy \le C\int_{K_1 r}^\infty \frac{\rho(t)\Phi^{-1}(\varphi(t))}{t}\,dt\,\|f\|_{L^{(\Phi,\varphi)}},$$

where K_1 is the constant in (1.1.3).

Proof. Let $B=B(x,r)$. Then

$$\int_{\mathbb{R}^n\setminus B(x,r)} \frac{\rho(|x-y|)}{|x-y|^n}|f(y)|\,dy = \sum_{j=0}^\infty \int_{2^{j+1}B\setminus 2^jB} \frac{\rho(|x-y|)}{|x-y|^n}|f(y)|\,dy.$$

By (1.1.3), (3.2.5), Hölder's inequality and the doubling condition of φ we have

$$\begin{aligned} &\int_{2^{j+1}B\setminus 2^jB} \frac{\rho(|x-y|)}{|x-y|^n}|f(y)|\,dy \lesssim \frac{\sup_{2^jr\le t\le 2^{j+1}r}\rho(t)}{(2^{j+1}r)^n}\int_{2^{j+1}B\setminus 2^jB}|f(y)|\,dy \\ &\lesssim \int_{K_12^jr}^{K_22^jr}\frac{\rho(t)}{t}\,dt\;\Phi^{-1}(\varphi(2^{j+1}r))\|f\|_{L^{(\Phi,\varphi)}} \lesssim \int_{K_12^jr}^{K_22^jr}\frac{\rho(t)\Phi^{-1}(\varphi(t))}{t}\,dt\,\|f\|_{L^{(\Phi,\varphi)}}. \end{aligned}$$

Therefore, we have the conclusion. □

Lemma 3.5.5. *Let $\Phi \in \nabla_2$, $\varphi \in \mathcal{G}^{\rm dec}$ and $\psi \in \mathcal{G}^{\rm inc}$. Assume that ρ satisfies* (1.1.2) *and* (1.1.3). *Then there exists a positive constant C such that, for all $b \in \mathcal{L}_{1,\psi}(\mathbb{R}^n)$, all $f \in L^{(\Phi,\varphi)}(\mathbb{R}^n)$ and all balls $B(x,r)$,*

$$\int_{\mathbb{R}^n \setminus B(x,r)} |b(y) - b_{B(x,r)}| \frac{\rho(|x-y|)}{|x-y|^n} |f(y)|\, dy \\ \le C \int_{K_1 r}^{\infty} \frac{\psi(t)}{t} \left(\int_t^{\infty} \frac{\rho(u)\Phi^{-1}(\varphi(u))}{u}\, du \right) dt\, \|b\|_{\mathcal{L}_{1,\psi}} \|f\|_{L^{(\Phi,\varphi)}},$$

where K_1 is the constant in (1.1.3).

Proof. Let $B = B(x,r)$. Then

$$\int_{\mathbb{R}^n \setminus B(x,r)} |b(y) - b_B| \frac{\rho(|x-y|)}{|x-y|^n} |f(y)|\, dy \\ = \sum_{j=0}^{\infty} \int_{2^{j+1}B \setminus 2^j B} |b(y) - b_B| \frac{\rho(|x-y|)}{|x-y|^n} |f(y)|\, dy.$$

By Lemma 3.2.2 we can find $p \in (1,\infty)$ such that

$$\left(\fint_{2^{j+1}B} |f(y)|^p dy \right)^{1/p} \lesssim \Phi^{-1}\left(\varphi(2^{j+1}r)\right) \|f\|_{L^{(\Phi,\varphi)}}.$$

By (1.1.3), (3.2.5), Hölder's inequality, Lemma 2.5.3 and the doubling condition of ψ and φ we have

$$\begin{aligned} &\int_{2^{j+1}B \setminus 2^j B} |b(y) - b_B| \frac{\rho(|x-y|)}{|x-y|^n} |f(y)|\, dy \\ &\lesssim \frac{\sup_{2^j r \le u \le 2^{j+1} r} \rho(u)}{(2^{j+1}r)^n} \int_{2^{j+1}B \setminus 2^j B} |b(y) - b_B| |f(y)|\, dy \\ &\lesssim \int_{K_1 2^j r}^{K_2 2^j r} \frac{\rho(u)}{u}\, du \left(\fint_{2^{j+1}B} |b - b_B|^{p'}\, dy \right)^{1/p'} \left(\fint_{2^{j+1}B} |f(y)|^p\, dy \right)^{1/p} \\ &\lesssim \int_{K_1 2^j r}^{K_2 2^j r} \frac{\rho(u)}{u}\, du \int_r^{2^{j+1}r} \frac{\psi(t)}{t}\, dt\, \Phi^{-1}(\varphi(2^{j+1}r)) \|b\|_{\mathcal{L}_{1,\psi}} \|f\|_{L^{(\Phi,\varphi)}} \\ &\lesssim \int_{K_1 2^j r}^{K_2 2^j r} \left(\int_{K_1 r}^{u} \frac{\psi(t)}{t}\, dt \right) \frac{\rho(u)\Phi^{-1}(\varphi(u))}{u}\, du\, \|b\|_{\mathcal{L}_{1,\psi}} \|f\|_{L^{(\Phi,\varphi)}}. \end{aligned}$$

Therefore,

$$\begin{aligned}
&\int_{\mathbb{R}^n\setminus B} |b(y)-b_B| \frac{\rho(|x-y|)}{|x-y|^n}|f(y)|\,dy \\
&\lesssim \int_{K_1 r}^{\infty} \left(\int_{K_1 r}^{u} \frac{\psi(t)}{t}\,dt\right) \frac{\rho(u)\Phi^{-1}(\varphi(u))}{u}\,du\,\|b\|_{\mathcal{L}_{1,\psi}}\|f\|_{L^{(\Phi,\varphi)}} \\
&= \int_{K_1 r}^{\infty} \frac{\psi(t)}{t} \left(\int_{t}^{\infty} \frac{\rho(u)\Phi^{-1}(\varphi(u))}{u}\,du\right) dt\,\|b\|_{\mathcal{L}_{1,\psi}}\|f\|_{L^{(\Phi,\varphi)}}.
\end{aligned}$$

This is the conclusion. □

Remark 3.5.2. Under the assumption in Theorem 3.1.2 (i), let $b \in \mathcal{L}_{1,\psi}(\mathbb{R}^n)$ and $f \in L^{(\Phi,\varphi)}(\mathbb{R}^n)$. Then f is in $L^p_{\mathrm{loc}}(\mathbb{R}^n)$ and bf is in $L^{p_1}_{\mathrm{loc}}(\mathbb{R}^n)$ for all $p_1 < p$ by the same way as in Remark 3.5.1. Since $\frac{\rho(|y|)}{|y|^n}$ is integrable near the origin with respect to y, $I_\rho(|f|\chi_{2B})$ and $I_\rho(|bf|\chi_{2B})$ are well-defined for any ball $B = B(x,r)$. By (3.1.11) and (3.1.12) we have

$$\int_{K_1 r}^{\infty} \frac{\rho(t)\Phi^{-1}(\varphi(t))}{t}\,dt \lesssim \Theta^{-1}(\varphi(K_1 r)) \lesssim \Theta^{-1}(\varphi(r)), \tag{3.5.4}$$

and

$$\begin{aligned}
&\int_{K_1 r}^{\infty} \frac{\psi(t)}{t} \left(\int_{t}^{\infty} \frac{\rho(u)\Phi^{-1}(\varphi(u))}{u}\,du\right) dt \\
&\qquad \lesssim \int_{K_1 r}^{\infty} \frac{\psi(t)\Theta^{-1}(\varphi(t))}{t}\,dt \lesssim \int_{K_1 r}^{\infty} \frac{\Psi^{-1}(\varphi(t))}{t}\,dt \lesssim \Psi^{-1}(\varphi(r)).
\end{aligned} \tag{3.5.5}$$

Then, by Lemmas 3.5.4 and 3.5.5, the integrals

$$\int_{\mathbb{R}^n\setminus 2B} \frac{\rho(|x-y|)}{|x-y|^n}|f(y)|\,dy \quad \text{and} \quad \int_{\mathbb{R}^n\setminus 2B} \frac{\rho(|x-y|)}{|x-y|^n}|b(y)f(y)|\,dy$$

converge. That is, the integrals

$$\int_{\mathbb{R}^n} \frac{\rho(|x-y|)}{|x-y|^n} f(y)\,dy \quad \text{and} \quad \int_{\mathbb{R}^n} \frac{\rho(|x-y|)}{|x-y|^n} b(y)f(y)\,dy$$

converge absolutely a.e. x and we can write

$$[b, I_\rho]f(x) = \int_{\mathbb{R}^n} (b(x)-b(y)) \frac{\rho(|x-y|)}{|x-y|^n} f(y)\,dy, \quad \text{a.e. } x.$$

Lemma 3.5.6. *Under the assumption of Theorem 3.1.2 (i), there exists a positive constant C such that, for all $b \in \mathcal{L}_{1,\psi}(\mathbb{R}^n)$, all $f \in L^{(\Phi,\varphi)}(\mathbb{R}^n)$ and all balls $B = B(z,r)$,*

$$\fint_B \left(\int_{\mathbb{R}^n \setminus 2B} \left| (b(x)-b(y)) \frac{\rho(|x-y|)}{|x-y|^n} f(y) \right| dy \right) dx \le C \Psi^{-1}(\varphi(B)) \, \|b\|_{\mathcal{L}_{1,\psi}} \|f\|_{L^{(\Phi,\varphi)}}.$$

Proof. For $x \in B$, let

$$G_1(x) = |b(x) - b_B| \int_{\mathbb{R}^n \setminus 2B} \frac{\rho(|x-y|)}{|x-y|^n} |f(y)| \, dy,$$
$$G_2(x) = \int_{\mathbb{R}^n \setminus 2B} |b(y) - b_B| \frac{\rho(|x-y|)}{|x-y|^n} |f(y)| \, dy.$$

Then

$$\left| \int_{\mathbb{R}^n \setminus 2B} (b(x)-b(y)) \frac{\rho(|x-y|)}{|x-y|^n} f(y) \, dy \right| \le G_1(x) + G_2(x).$$

Using this estimate and a similar way to Lemmas 3.5.4 and 3.5.5, we have, for all $x \in B$,

$$G_1(x) \lesssim |b(x) - b_B| \int_{K_1 r}^{\infty} \frac{\rho(t)\Phi^{-1}(\varphi(t))}{t} \, dt \, \|f\|_{L^{(\Phi,\varphi)}},$$
$$G_2(x) \lesssim C \int_{K_1 r}^{\infty} \frac{\psi(t)}{t} \left(\int_t^{\infty} \frac{\rho(u)\Phi^{-1}(\varphi(u))}{u} \, du \right) dt \, \|b\|_{\mathcal{L}_{1,\psi}} \|f\|_{L^{(\Phi,\varphi)}}.$$

Then, using (3.5.4) and (3.5.5) also, we have

$$\begin{aligned}
\fint_B G_1(x) \, dx &\lesssim \fint_B |b(x) - b_B| \, dx \, \Theta^{-1}(\varphi(r)) \|f\|_{L^{(\Phi,\varphi)}} \\
&\lesssim \psi(r)\Theta^{-1}(\varphi(r)) \|b\|_{\mathcal{L}_{1,\psi}} \|f\|_{L^{(\Phi,\varphi)}} \\
&\lesssim \Psi^{-1}(\varphi(r)) \|b\|_{\mathcal{L}_{1,\psi}} \|f\|_{L^{(\Phi,\varphi)}},
\end{aligned}$$

and

$$\fint_B G_2(x) \, dx \lesssim \Psi^{-1}(\varphi(r)) \|b\|_{\mathcal{L}_{1,\psi}} \|f\|_{L^{(\Phi,\varphi)}}.$$

Then we have the conclusion. □

3.6 Proof of Theorem 3.1.1

We use the following proposition. We omit its proof because the proof method is almost the same as [1, Proposition 5.1].

Proposition 3.6.1. *Let T be a Calderón-Zygmund operator of type ω. Let $\psi \in \mathcal{G}^{\rm inc}$. Assume that ω and ψ satisfy the same assumption in Theorem 3.1.1. Then, for any $\eta \in (1,\infty)$, there exists a positive constant C such that, for all $b \in \mathcal{L}_{1,\psi}(\mathbb{R}^n)$, $f \in L^{(\Phi,\varphi)}(\mathbb{R}^n)$ and $x \in \mathbb{R}^n$,*

$$M^\sharp([b,T]f)(x) \le C\|b\|_{\mathcal{L}_{1,\psi}}\left(\left(M_{\psi^\eta}(|Tf|^\eta)(x)\right)^{1/\eta} + \left(M_{\psi^\eta}(|f|^\eta)(x)\right)^{1/\eta}\right), \tag{3.6.1}$$

where M_{ψ^η} is the fractional maximal operator defined by

$$M_{\psi^\eta} f(x) = \sup_{B(a,r)\ni x} \psi(r)^\eta \fint_{B(a,r)} |f(y)|\,dy, \quad x \in \mathbb{R}^n.$$

Next, we note that, for $\theta \in (0,\infty)$,

$$\||g|^\theta\|_{L^{(\Phi,\varphi)}} = \left(\|g\|_{L^{(\Phi((\cdot)^\theta),\varphi))}}\right)^\theta. \tag{3.6.2}$$

Proof of Theorem 3.1.1 (i). First note that T is bounded on $L^{(\Phi,\varphi)}(\mathbb{R}^n)$ as we state just before Theorem 3.1.1. We can take $\eta \in (1,\infty)$ such that $\Phi((\cdot)^{1/\eta}) \in \overline{\nabla}_2$ by Lemma 2.2.6. Then, from (3.1.7) it follows that

$$\psi(r)^\eta \Phi^{-1}(\varphi(r))^\eta \le {C_0}^\eta \Psi^{-1}(\varphi(r))^\eta.$$

By Theorem 3.3.1 with this condition we have the boundedness of M_{ψ^η} from $L^{(\Phi((\cdot)^{1/\eta}),\varphi)}(\mathbb{R}^n)$ to $L^{(\Psi((\cdot)^{1/\eta}),\varphi)}(\mathbb{R}^n)$. Using this boundedness and (3.6.2), we have

$$\begin{aligned}\left\|\left(M_{\psi^\eta}(|Tf|^\eta)\right)^{1/\eta}\right\|_{L^{(\Psi,\varphi)}} &= \left(\|M_{\psi^\eta}(|Tf|^\eta)\|_{L^{(\Psi((\cdot)^{1/\eta}),\varphi)}}\right)^{1/\eta} \\ &\lesssim \left(\||Tf|^\eta\|_{L^{(\Phi((\cdot)^{1/\eta}),\varphi)}}\right)^{1/\eta} \\ &= \|Tf\|_{L^{(\Phi,\varphi)}} \lesssim \|f\|_{L^{(\Phi,\varphi)}},\end{aligned}$$

and

$$\begin{aligned}\|(M_{\psi^\eta}(|f|^\eta))^{1/\eta}\|_{L^{(\Psi,\varphi)}} &= \left(\|M_{\psi^\eta}(|f|^\eta)\|_{L^{(\Psi((\cdot)^{1/\eta}),\varphi)}}\right)^{1/\eta} \\ &\lesssim \left(\||f|^\eta\|_{L^{(\Phi((\cdot)^{1/\eta}),\varphi)}}\right)^{1/\eta} = \|f\|_{L^{(\Phi,\varphi)}}.\end{aligned}$$

Then, using Proposition 3.6.1, we have

$$\|M^\sharp([b,T]f)\|_{L^{(\Psi,\varphi)}} \lesssim \|b\|_{\mathcal{L}_{1,\psi}}\|f\|_{L^{(\Phi,\varphi)}}. \tag{3.6.3}$$

Therefore, once we show that, for $B_r = B(0,r)$,

$$\fint_{B_r} [b,T]f \to 0 \quad \text{as} \quad r \to \infty, \tag{3.6.4}$$

then by Corollary 3.4.3 we have

$$\|[b,T]f\|_{L^{(\Psi,\varphi)}} \lesssim \|b\|_{\mathcal{L}_{1,\psi}}\|f\|_{L^{(\Phi,\varphi)}}, \tag{3.6.5}$$

which is the conclusion.

In the following we show (3.6.4).

Case 1: First we show (3.6.4) for all $f \in L^{(\Phi,\varphi)}(\mathbb{R}^n)$ with compact support. Let $\operatorname{supp} f \subset B_s = B(0,s)$ with $s \geq 1$. Then $f \in L^p(\mathbb{R}^n)$ and $bf \in L^{p_1}(\mathbb{R}^n)$ for some $1 < p_1 < p < \infty$ (see Remark 3.5.1). Since T is bounded on Lebesgue spaces, we see that both $(bTf)\chi_{B_{2s}}$ and $T(bf)\chi_{B_{2s}}$ are in $L^1(\mathbb{R}^n)$ and that

$$⨍_{B_r} (bTf)\chi_{B_{2s}} \to 0, \quad ⨍_{B_r} T(bf)\chi_{B_{2s}} \to 0 \quad \text{as } r \to \infty.$$

If $x \notin B_{2s}$ and $y \in B(0,s)$, then $|x|/2 \leq |x-y| \leq 3|x|/2$. By (3.1.1) and (3.1.3) we have

$$|Tf(x)| \lesssim \frac{1}{|x|^n}\|f\|_{L^1}, \quad |T(bf)(x)| \lesssim \frac{1}{|x|^n}\|bf\|_{L^1}, \quad x \notin B_{2s}, \tag{3.6.6}$$

which yields

$$b_{B_{2s}} ⨍_{B_r} (Tf)(1-\chi_{B_{2s}}) \to 0, \quad ⨍_{B_r} (T(bf))(1-\chi_{B_{2s}}) \to 0 \quad \text{as } r \to \infty.$$

Next, we show

$$⨍_{B_r} (b - b_{B_{2s}})(Tf)(1-\chi_{B_{2s}}) \to 0 \quad \text{as } r \to \infty. \tag{3.6.7}$$

Then we have (3.6.4) for all $f \in L^{(\Phi,\varphi)}(\mathbb{R}^n)$ with compact support.

Now, since $\Psi \in \Delta_2$, there exists $p \in (1,\infty)$ such that $\Psi^{-1}(u) \lesssim u^{1/p}\,(u \leq 1)$. Let $\nu = \frac{2p}{2p-1}$. Then

$$\left| ⨍_{B_r} (b - b_{B_{2s}})(Tf)(1-\chi_{B_{2s}}) \right| \leq \left(⨍_{B_r} |b - b_{B_{2s}}|^{\nu'} \right)^{1/\nu'} \left(⨍_{B_r} |(Tf)(1-\chi_{B_{2s}})|^{\nu} \right)^{1/\nu}.$$

From Lemma 2.5.3, Remark 2.5.1 and (3.1.7) it follows that

$$\left(⨍_{B_r} |b - b_{B_{2s}}|^{\nu'} \right)^{1/\nu'} \lesssim \int_{2s}^{r} \frac{\psi(t)}{t}\,dt\,\|b\|_{\mathcal{L}_{1,\psi}} \lesssim \psi(r)\log r\,\|b\|_{\mathcal{L}_{1,\psi}} \lesssim \frac{\Psi^{-1}(\varphi(r))}{\Phi^{-1}(\varphi(r))}\log r\,\|b\|_{\mathcal{L}_{1,\psi}}. \tag{3.6.8}$$

From (3.6.6) it follows that

$$\left(\int_{B_r\setminus B_{2s}} |Tf(x)|^\nu \, dx\right)^{1/\nu} \lesssim \left(\int_{B_r\setminus B_{2s}} \left(\frac{1}{|x|^n}\|f\|_{L^1}\right)^\nu \, dx\right)^{1/\nu} \lesssim \|f\|_{L^1}. \tag{3.6.9}$$

By (3.6.8) and (3.6.9) we have

$$\begin{aligned}
&\left|\fint_{B_r} (b-b_{B_{2s}})(Tf)(1-\chi_{B_{2s}})\right| \\
&\lesssim \frac{\Psi^{-1}(\varphi(r))}{\Phi^{-1}(\varphi(r))} \log r\, \|b\|_{\mathcal{L}_{1,\psi}} \frac{1}{r^{n/\nu}} \|f\|_{L^1} = \frac{\log r}{r^{n/\nu}} \frac{\Psi^{-1}(\varphi(r))}{\Phi^{-1}(\varphi(r))} \|b\|_{\mathcal{L}_{1,\psi}} \|f\|_{L^1} \\
&\lesssim \frac{\log r}{r^{n/\nu}} \frac{\varphi(r)^{1/p}}{\varphi(r)} \|b\|_{\mathcal{L}_{1,\psi}} \|f\|_{L^1} = \frac{\log r}{r^{\frac{n}{2p}} (r^n \varphi(r))^{1-\frac{1}{p}}} \|b\|_{\mathcal{L}_{1,\psi}} \|f\|_{L^1} \\
&\to 0 \quad \text{as } r \to \infty.
\end{aligned}$$

Therefore, we have (3.6.4) and (3.6.5) for all $f \in L^{(\Phi,\varphi)}(\mathbb{R}^n)$ with compact support.
Case 2: For general $f \in L^{(\Phi,\varphi)}(\mathbb{R}^n)$, using Case 1, we have

$$\|[b,T](f\chi_{B_{2r}})\|_{L^{(\Psi,\varphi)}} \lesssim \|b\|_{\mathcal{L}_{1,\psi}} \|f\chi_{B_{2r}}\|_{L^{(\Phi,\varphi)}} \le \|b\|_{\mathcal{L}_{1,\psi}} \|f\|_{L^{(\Phi,\varphi)}}.$$

Then, by (3.2.5),

$$\begin{aligned}
\fint_{B_r} |[b,T](f\chi_{B_{2r}})| &\le \Psi^{-1}(\varphi(r)) \|[b,T](f\chi_{B_{2r}})\|_{L^{(\Psi,\varphi)}} \\
&\lesssim \Psi^{-1}(\varphi(r)) \|b\|_{\mathcal{L}_{1,\psi}} \|f\|_{L^{(\Phi,\varphi)}}.
\end{aligned}$$

Combining this with Lemma 3.5.3, we have

$$\fint_{B_r} |[b,T]f| \lesssim \Psi^{-1}(\varphi(r)) \|b\|_{\mathcal{L}_{1,\psi}} \|f\|_{L^{(\Phi,\varphi)}},$$

which implies (3.6.4). Therefore, we have (3.6.5) for all $f \in L^{(\Phi,\varphi)}(\mathbb{R}^n)$. The proof is complete. □

Proof of Theorem 3.1.1 (ii). We use the method by Janson [17] as same as the proof of Theorem 2.1.4 (ii). Since $1/K(z)$ is many times infinitely differentiable in an open set, we may choose $z_0 \neq 0$ and $\delta > 0$ such that $1/K(z)$ can be expressed in the neighborhood $|z-z_0| < 2\delta$ as an absolutely convergent Fourier series, $1/K(z) = \sum_j a_j e^{iv_j \cdot z}$. (The exact form of the vectors v_j is irrelevant. For example, if the cube centered at z_0 of side length 4δ is contained in the open set, then we can take $v_j = 2\pi j/(4\delta)$, $j \in \mathbb{Z}^n$.)

Set $z_1 = z_0/\delta$. If $|z - z_1| < 2$, we have the expansion

$$\frac{1}{K(z)} = \frac{\delta^{-n}}{K(\delta z)} = \delta^{-n} \sum_j a_j e^{iv_j \cdot \delta z}, \quad \sum_{j \in \mathbb{Z}^n} |a_j| < \infty$$

Choose now any ball $B = B(x_0, r)$. Set $y_0 = x_0 - rz_1$ and $B' = B(y_0, r)$. Then, if $x \in B$ and $y \in B'$,

$$\left| \frac{x-y}{r} - z_1 \right| \le \left| \frac{x - x_0}{r} \right| + \left| \frac{y - y_0}{r} \right| \le 2.$$

Denote $\operatorname{sgn}(f(x) - f_{B'})$ by $s(x)$. Then

$$\begin{aligned}
&\int_B |b(x) - b_{B'}| \, dx \\
&= \int_B (b(x) - b_{B'}) s(x) \, dx \\
&= \frac{1}{|B'|} \int_B \int_{B'} (b(x) - b(y)) s(x) \, dy \, dx \\
&= \frac{1}{|B'|} \int_{\mathbb{R}^n} \int_{\mathbb{R}^n} (b(x) - b(y)) \frac{r^n K(x-y)}{K(\frac{x-y}{r})} s(x) \chi_B(x) \chi_{B'}(y) \, dy \, dx \\
&= \frac{r^n \delta^{-n}}{|B'|} \int_{\mathbb{R}^n} \int_{\mathbb{R}^n} (b(x) - b(y)) K(x-y) \sum a_j e^{iv_j \cdot \delta \frac{x-y}{r}} s(x) \chi_B(x) \chi_{B'}(y) \, dy \, dx.
\end{aligned}$$

Here, we set $C = \delta^{-n} |B(0,1)|^{-1}$ and

$$g_j(y) = e^{-iv_j \cdot \delta \frac{y}{r}} \chi_{B'}(y), \quad h_j(x) = e^{iv_j \cdot \delta \frac{x}{r}} s(x) \chi_B(x).$$

Then

$$\begin{aligned}
&\int_B |b(x) - b_{B'}| \, dx \\
&= C \sum a_j \int_{\mathbb{R}^n} \int_{\mathbb{R}^n} (b(x) - b(y)) K(x-y) g_j(y) h_j(x) \, dy \, dx \\
&= C \sum a_j \int_{\mathbb{R}^n} ([b,T] g_j)(x) h_j(x) \, dx \\
&\le C \sum |a_j| \int_{\mathbb{R}^n} |([b,T] g_j)(x)| |h_j(x)| \, dx \\
&= C \sum |a_j| \int_B |([b,T] g_j)(x)| \, dx \\
&\le C \sum |a_j| |B| \Psi^{-1}(\varphi(r)) \| [b,T] g_j \|_{L^{(\Psi,\varphi)}} \\
&\le C \| [b,T] \|_{L^{(\Phi,\varphi)} \to L^{(\Psi,\varphi)}} |B| \Psi^{-1}(\varphi(r)) \sum |a_j| \| g_j \|_{L^{(\Phi,\varphi)}}.
\end{aligned}$$

By Lemma 3.2.1 we have that $\|g_j\|_{L^{(\Phi,\varphi)}} = \|\chi_{B'}\|_{L^{(\Phi,\varphi)}} \sim \frac{1}{\Phi^{-1}(\varphi(B'))}$. Then

$$\int_B |b(x) - b_{B'}|\, dx \lesssim \|[b,T]\|_{L^{(\Phi,\varphi)} \to L^{(\Psi,\varphi)}} |B| \frac{\Psi^{-1}(\varphi(B))}{\Phi^{-1}(\varphi(B))}.$$

By (3.1.8) we have

$$\frac{1}{\psi(B)} \fint_B |b(x) - b_B|\, dx \le \frac{2}{\psi(B)} \fint_B |b(x) - b_{B'}|\, dx \lesssim \|[b,T]\|_{L^{(\Phi,\varphi)} \to L^{(\Psi,\varphi)}}.$$

That is, $\|b\|_{\mathcal{L}_{1,\psi}} \lesssim \|[b,T]\|_{L^{(\Phi,\varphi)} \to L^{(\Psi,\varphi)}}$ and we have the conclusion. □

3.7 Proof of Theorem 3.1.2

We use the following proposition. We omit its proof because the proof methods are almost the same as [1, Proposition 5.2] and Theorem 2.5.2.

Proposition 3.7.1. *Assume that* $\rho : (0,\infty) \to (0,\infty)$ *satisfies* (1.1.2). *Let* $\rho^*(r)$ *be as in* (2.1.12). *Assume that the condition* (3.1.10) *holds and that* $r \mapsto \rho(r)/r^{n-\epsilon}$ *is almost decreasing for some* $\epsilon > 0$. *Assume also that*

$$\text{(3.7.1)} \quad \int_r^\infty \frac{\rho(t)\Phi^{-1}(\varphi(t))}{t}\, dt < \infty, \qquad \int_r^\infty \frac{\psi(t)}{t} \left(\int_t^\infty \frac{\rho(u)\Phi^{-1}(\varphi(u))}{u}\, du \right) dt < \infty,$$

Then, for any $\eta \in (1,\infty)$, *there exists a positive constant* C *such that, for all* $b \in \mathcal{L}_{1,\psi}(\mathbb{R}^n)$, $f \in L^{(\Phi,\varphi)}(\mathbb{R}^n)$ *and* $x \in \mathbb{R}^n$,

$$M^\sharp([b,I_\rho]f)(x) \le C\|b\|_{\mathcal{L}_{1,\psi}} \left(\left(M_{\psi^\eta}(|I_\rho f|^\eta)(x)\right)^{1/\eta} + \left(M_{(\rho^*\psi)^\eta}(|f|^\eta)(x)\right)^{1/\eta} \right)$$

where $M_{(\rho^*\psi)^\eta}$ *is the fractional maximal operator defined by*

$$M_{(\rho^*\psi)^\eta} f(x) = \sup_{B(a,r)\ni x} (\rho^*(r)\psi(r))^\eta \fint_{B(a,r)} |f(y)|\, dy, \quad x \in \mathbb{R}^n.$$

We note that the condition (3.7.1) is used to prove the well-definedness of $[b, I_\rho]f$.

Proof of Theorem 3.1.2 (i). We may assume that $\Phi, \Psi \in \Delta_2 \cap \nabla_2$ and $\Theta \in \nabla_2$. We can choose $\eta \in (1,\infty)$ such that $\Phi((\cdot)^{1/\eta})$, $\Psi((\cdot)^{1/\eta})$ and $\Theta((\cdot)^{1/\eta})$ are in ∇_2 by Lemma 2.2.6. Then from (3.1.12) it follows that

$$\psi(r)^\eta \Theta^{-1}(\varphi(r))^\eta \le C_1^\eta \Psi^{-1}(\varphi(r))^\eta.$$

Hence, by Theorem 3.3.1 we see that M_{ψ^η} is bounded from $L^{(\Theta((\cdot)^{1/\eta}),\varphi)}(\mathbb{R}^n)$ to $L^{(\Psi((\cdot)^{1/\eta}),\varphi)}(\mathbb{R}^n)$. Moreover, as we mentioned just before Theorem 3.1.2, I_ρ is bounded from $L^{(\Phi,\varphi)}(\mathbb{R}^n)$ to $L^{(\Theta,\varphi)}(\mathbb{R}^n)$ by (3.1.11). Then, using (3.6.2), we have

$$\begin{aligned}\left\| \left(M_{\psi^\eta}(|I_\rho f|^\eta)\right)^{1/\eta} \right\|_{L^{(\Psi,\varphi)}} &= \left(\|M_{\psi^\eta}(|I_\rho f|^\eta)\|_{L^{(\Psi((\cdot)^{1/\eta}),\varphi)}} \right)^{1/\eta} \\ &\lesssim \left(\||I_\rho f|^\eta\|_{L^{(\Theta((\cdot)^{1/\eta}),\varphi)}} \right)^{1/\eta} = \|I_\rho f\|_{L^{(\Theta,\varphi)}} \lesssim \|f\|_{L^{(\Phi,\varphi)}}.\end{aligned}$$

From (3.1.11) and (3.1.12) it follows that

$$(\rho^*(r)\psi(r))^\eta \left(\Phi^{-1}(\varphi(r))\right)^\eta \le (C_0C_1)^\eta \left(\Psi^{-1}(\varphi(r))\right)^\eta .$$

By using Theorem 3.3.1, we have the boundedness of $M_{(\rho^*\psi)^\eta}$ from $L^{(\Phi((\cdot)^{1/\eta}),\varphi)}(\mathbb{R}_n)$ to $L^{(\Psi((\cdot)^{1/\eta}),\varphi)}$. That is,

$$\begin{aligned}\left\| \left(M_{(\rho^*\psi)^\eta}(|f|^\eta)\right)^{1/\eta} \right\|_{L^{(\Psi,\varphi)}} &= \left(\left\|M_{(\rho^*\psi)^\eta}(|f|^\eta)\right\|_{L^{(\Psi((\cdot)^{1/\eta}),\varphi)}} \right)^{1/\eta} \\ &\lesssim \left(\||f|^\eta\|_{L^{(\Phi((\cdot)^{1/\eta}),\varphi)}} \right)^{1/\eta} = \|f\|_{L^{(\Phi,\varphi)}}.\end{aligned}$$

Therefore, if we show that, for $B_r = B(0,r)$,

$$\fint_{B_r} [b, I_\rho]f \to 0 \quad \text{as} \quad r \to \infty, \tag{3.7.2}$$

then we have

$$\|[b, I_\rho]f\|_{L^{(\Psi,\varphi)}} \lesssim \|b\|_{\mathcal{L}_{1,\psi}} \|f\|_{L^{(\Phi,\varphi)}}, \tag{3.7.3}$$

by Corollary 3.4.3.

In the following we show (3.7.2).

Case 1: First we show (3.7.2) for all $f \in L^{(\Phi,\varphi)}(\mathbb{R}^n)$ with compact support. Let $\operatorname{supp} f \subset B_s = B(0,s)$ with $s \ge 1$. Then $f \in L^p(\mathbb{R}^n)$ and $bf \in L^{p_1}(\mathbb{R}^n)$ for some $1 < p_1 < p < \infty$ (see Remark 3.5.2). Since $\frac{\rho(|y|)}{|y|^n}$ is locally integrable with respect to y, we see that $(bI_\rho f)\chi_{B_{2s}}$ and $I_\rho(bf)\chi_{B_{2s}}$ are in $L^1(\mathbb{R}^n)$ and that

$$\fint_{B_r} (bI_\rho f)\chi_{B_{2s}} \to 0, \quad \fint_{B_r} I_\rho(bf)\chi_{B_{2s}} \to 0 \quad \text{as } r \to \infty.$$

If $x \notin B_{2s}$ and $y \in B(0,s)$, then $|y| < |x-y|$ and $|x|/2 \le |x-y| \le 3|x|/2$,

$$\rho(|x-y|) \le \sup_{|x|/2 \le t \le 3|x|/2} \rho(t). \tag{3.7.4}$$

Then we have

$$\frac{\rho(|x-y|)}{|x-y|^n} \lesssim \frac{\sup_{|x|/2\le t\le 3|x|/2}\rho(t)}{|x|^n} \sim \sup_{|x|/2\le t\le 3|x|/2}\frac{\rho(t)}{t^n},$$

and

$$|I_\rho f(x)| \lesssim \sup_{|x|/2\le t\le 3|x|/2}\frac{\rho(t)}{t^n}\|f\|_{L^1}, \quad |I_\rho(bf)(x)| \lesssim \sup_{|x|/2\le t\le 3|x|/2}\frac{\rho(t)}{t^n}\|bf\|_{L^1}.$$

From the almost decreasingness of $t\mapsto \rho(t)/t^{n-\epsilon}$ for some $\epsilon\in(0,n)$, it follows that $\frac{\rho(t)}{t^n}\to 0$ as $t\to\infty$, which yields

$$b_{B_{2s}}\fint_{B_r}(I_\rho f)(1-\chi_{B_{2s}})\to 0, \quad \fint_{B_r}(I_\rho(bf))(1-\chi_{B_{2s}})\to 0 \quad \text{as } r\to\infty.$$

Next, we show

$$\fint_{B_r}(b-b_{B_{2s}})(I_\rho f)(1-\chi_{B_{2s}})\to 0 \quad \text{as } r\to\infty. \tag{3.7.5}$$

Then we have (3.7.2) for all $f\in L^{(\Phi,\varphi)}(\mathbb{R}^n)$ with compact support.

Now, since $\Psi\in\Delta_2$, there exists $p\in(1,\infty)$ such that $\Psi^{-1}(u)\lesssim u^{1/p}$ $(u\le 1)$. Let $\nu=\frac{2p}{2p-1}$, then

$$\left|\fint_{B_r}(b-b_{B_{2s}})(I_\rho f)(1-\chi_{B_{2s}})\right| \le \left(\fint_{B_r}|b-b_{B_{2s}}|^{\nu'}\right)^{1/\nu'}\left(\fint_{B_r}|(I_\rho f)(1-\chi_{B_{2s}})|^{\nu}\right)^{1/\nu}.$$

From Lemma 2.5.3, Remark 2.5.1 and (3.1.12) it follows that

$$\left(\fint_{B_r}|b-b_{B_{2s}}|^{\nu'}\right)^{1/\nu'} \lesssim \int_{2s}^{r}\frac{\psi(t)}{t}\,dt\,\|b\|_{\mathcal{L}_{1,\psi}} \lesssim \psi(r)\log r\,\|b\|_{\mathcal{L}_{1,\psi}} \lesssim \frac{\Psi^{-1}(\varphi(r))}{\Theta^{-1}(\varphi(r))}\log r\,\|b\|_{\mathcal{L}_{1,\psi}}. \tag{3.7.6}$$

For $j=0,1,2,\dots$, from (3.7.4) and (1.1.3) it follows that

$$\begin{aligned}&\left(\int_{2^{j+2}B_s\setminus 2^{j+1}B_s}|I_\rho f(x)|^\nu\,dx\right)^{1/\nu}\\ &\lesssim \left(\int_{2^{j+2}B_s\setminus 2^{j+1}B_s}\left(\frac{\sup_{|x|/2\le t\le 3|x|/2}\rho(t)}{|x|^n}\|f\|_{L^1}\right)^{\nu}dx\right)^{1/\nu}\\ &\lesssim (2^js)^{(-n\nu+n)/\nu}\sup_{2^js\le t\le 3\cdot 2^{j+1}s}\rho(t)\,\|f\|_{L^1} \lesssim \int_{2^jK_1s}^{3\cdot 2^jK_2s}\frac{\rho(t)}{t}\,dt\,\|f\|_{L^1},\end{aligned}$$

since $s \geq 1$. Take the integer j_0 such that $r \leq 2^{j_0+2}s < 2r$. Then, by (3.1.11),

$$(3.7.7)\quad \left(\fint_{B_r} |(I_\rho f)(1-\chi_{B_{2s}})|^\nu\right)^{1/\nu} \leq \frac{1}{r^{n/\nu}} \sum_{j=0}^{j_0} \left(\int_{2^{j+2}B_s \setminus 2^{j+1}B_s} |I_\rho f|^\nu\right)^{1/\nu}$$
$$\lesssim \frac{1}{r^{n/\nu}} \int_0^{3K_2 r/2} \frac{\rho(t)}{t}\, dt\, \|f\|_{L^1} \lesssim \frac{1}{r^{n/\nu}} \frac{\Theta^{-1}(\varphi(r))}{\Phi^{-1}(\varphi(r))} \|f\|_{L^1}.$$

By (3.7.6) and (3.7.7), we have

$$\begin{aligned}
&\left|\fint_{B_r} (b-b_{B_{2s}})(I_\rho f)(1-\chi_{B_{2s}})\right| \\
&\lesssim \frac{\Psi^{-1}(\varphi(r))}{\Theta^{-1}(\varphi(r))} \log r \frac{1}{r^{n/\nu}} \frac{\Theta^{-1}(\varphi(r))}{\Phi^{-1}(\varphi(r))} \|b\|_{\mathcal{L}_{1,\psi}} \|f\|_{L^1} \\
&= \frac{\log r}{r^{n/\nu}} \frac{\Psi^{-1}(\varphi(r))}{\Phi^{-1}(\varphi(r))} \|b\|_{\mathcal{L}_{1,\psi}} \|f\|_{L^1} \\
&\lesssim \frac{\log r}{r^{n/\nu}} \frac{\varphi(r)^{1/p}}{\varphi(r)} \|b\|_{\mathcal{L}_{1,\psi}} \|f\|_{L^1} = \frac{\log r}{r^{\frac{n}{2p}} (r^n \varphi(r))^{1-\frac{1}{p}}} \|b\|_{\mathcal{L}_{1,\psi}} \|f\|_{L^1} \\
&\to 0 \quad \text{as } r \to \infty.
\end{aligned}$$

Therefore, we have (3.7.2) and (3.7.3) for all $f \in L^{(\Phi,\varphi)}(\mathbb{R}^n)$ with compact support.
Case 2: For general $f \in L^{(\Phi,\varphi)}(\mathbb{R}^n)$, using Case 1, we have

$$\|[b, I_\rho](f\chi_{B_{2r}})\|_{L^{(\Psi,\varphi)}} \lesssim \|b\|_{\mathcal{L}_{1,\psi}} \|f\chi_{B_{2r}}\|_{L^{(\Phi,\varphi)}} \leq \|b\|_{\mathcal{L}_{1,\psi}} \|f\|_{L^{(\Phi,\varphi)}}.$$

Then, by (3.2.5),

$$\begin{aligned}
\fint_{B_r} [b, I_\rho](f\chi_{B_{2r}}) &\leq \Psi^{-1}(\varphi(r)) \|[b, I_\rho](f\chi_{B_{2r}})\|_{L^{(\Psi,\varphi)}} \\
&\lesssim \Psi^{-1}(\varphi(r)) \|b\|_{\mathcal{L}_{1,\psi}} \|f\|_{L^{(\Phi,\varphi)}}.
\end{aligned}$$

Combining this with Lemma 3.5.6, we have

$$\fint_{B_r} [b, I_\rho] f \lesssim \Psi^{-1}(\varphi(r)) \|b\|_{\mathcal{L}_{1,\psi}} \|f\|_{L^{(\Phi,\varphi)}},$$

which implies (3.7.2). Therefore, we have (3.7.3) for all $f \in L^{(\Phi,\varphi)}(\mathbb{R}^n)$. The proof is complete. □

Proof of Theorem 3.1.2 (ii)**.** In a similar way to the proof of Theorem 3.1.1 (ii), we can conclude that $\|b\|_{\mathcal{L}_{1,\psi}} \lesssim \|[b, I_\rho]\|_{L^{(\Phi,\varphi)} \to L^{(\Psi,\varphi)}}$, by calculating $|z|^{n-\alpha}$ instead of $1/K(z)$. □

3.8 Proofs of Theorems 3.1.3 and 3.1.4

Finally, we prove Theorems 3.1.3 and 3.1.4.

Proof of Theorem 3.1.3. Let $B_r = B(0,r)$. By Theorem 3.4.1 we have that, for every $b \in \mathcal{L}^{(\Phi_0,\psi)}(\mathbb{R}^n)$, b_{B_r} converges as $r \to \infty$ and $\|b - \lim_{r\to\infty} b_{B_r}\|_{L^{(\Phi_0,\psi)}} \sim \|b\|_{\mathcal{L}^{(\Phi_0,\psi)}}$. Let $b_0 = b - \lim_{r\to\infty} b_{B_r}$. Then $\|b_0\|_{L^{(\Phi_0,\psi)}} \sim \|b\|_{\mathcal{L}^{(\Phi_0,\psi)}}$ and $[b,T]f = b_0Tf - T(b_0f)$. Using the boundedness of T on $L^{(\Phi,\varphi)}(\mathbb{R}^n)$ and on $L^{(\Psi,\theta)}(\mathbb{R}^n)$ and generalized Hölder's inequality (Lemma 3.2.4) with the assumption (3.1.14), we have

$$\begin{aligned}\|[b,T]f\|_{L^{(\Psi,\theta)}} &\le \|b_0Tf\|_{L^{(\Psi,\theta)}} + \|T(b_0f)\|_{L^{(\Psi,\theta)}} \\ &\lesssim \|b_0\|_{L^{(\Phi_0,\psi)}}\|Tf\|_{L^{(\Phi,\varphi)}} + \|b_0f\|_{L^{(\Psi,\theta)}} \\ &\lesssim \|b_0\|_{L^{(\Phi_0,\psi)}}\|f\|_{L^{(\Phi,\varphi)}} \sim \|b\|_{\mathcal{L}^{(\Phi_0,\varphi)}}\|f\|_{L^{(\Phi,\varphi)}}.\end{aligned}$$

This is the conclusion. □

Proof of Theorem 3.1.4. We use the same method as the proof of Theorem 3.1.3. For $b \in \mathcal{L}^{(\Phi_0,\varphi)}(\mathbb{R}^n)$, let $b_0 = b - \lim_{r\to\infty} b_{B_r}$. Then $\|b_0\|_{L^{(\Phi_0,\varphi)}} \sim \|b\|_{\mathcal{L}^{(\Phi_0,\varphi)}}$ and $[b,I_\rho]f = b_0I_\rho f - I_\rho(b_0f)$. As we mentioned just before Theorem 3.1.2 I_ρ is bounded from $L^{(\Phi,\varphi)}(\mathbb{R}^n)$ to $L^{(\Theta,\varphi)}(\mathbb{R}^n)$ by the assumption (3.1.11). Moreover, we see that I_ρ is bounded from $L^{(\Psi_0,\varphi)}(\mathbb{R}^n)$ to $L^{(\Psi,\varphi)}(\mathbb{R}^n)$, since

$$\begin{aligned}&\int_0^r \frac{\rho(t)}{t}\,dt\,\Psi_0^{-1}(\varphi(r)) + \int_r^\infty \frac{\rho(t)\Psi_0^{-1}(\varphi(t))}{t}\,dt \\ &\sim \int_0^r \frac{\rho(t)}{t}\,dt\,\Phi^{-1}(\varphi(r))\Phi_0^{-1}(\varphi(r)) + \int_r^\infty \frac{\rho(t)\Phi^{-1}(\varphi(t))\Phi_0^{-1}(\varphi(t))}{t}\,dt \\ &\lesssim \left(\int_0^r \frac{\rho(t)}{t}\,dt\,\Phi^{-1}(\varphi(r)) + \int_r^\infty \frac{\rho(t)\Phi^{-1}(\varphi(t))}{t}\,dt\right)\Phi_0^{-1}(\varphi(r)) \\ &\lesssim \Theta^{-1}(\varphi(r))\Phi_0^{-1}(\varphi(r)) \lesssim \Psi^{-1}(\varphi(r)).\end{aligned}$$

In the above we use the almost decreasingness of $r \mapsto \Phi_0^{-1}(\varphi(r))$. Then, using these boundedness of I_ρ and generalized Hölder's inequality (Lemma 3.2.4), we have

$$\begin{aligned}\|[b,I_\rho]f\|_{L^{(\Psi,\varphi)}} &\le \|b_0I_\rho f\|_{L^{(\Psi,\varphi)}} + \|I_\rho(b_0f)\|_{L^{(\Psi,\varphi)}} \\ &\lesssim \|b_0\|_{L^{(\Phi_0,\varphi)}}\|I_\rho f\|_{L^{(\Theta,\varphi)}} + \|b_0f\|_{L^{(\Psi_0,\varphi)}} \\ &\lesssim \|b_0\|_{L^{(\Phi_0,\varphi)}}\|f\|_{L^{(\Phi,\varphi)}} \sim \|b\|_{\mathcal{L}^{(\Phi_0,\varphi)}}\|f\|_{L^{(\Phi,\varphi)}}.\end{aligned}$$

This is the conclusion. □

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