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On characterizations of a Prüfer domain

Akira Okabe* and Ryûki Matsuda**

INTRODUCTION

Let D be an integral domain with quotient field K. Then it is easily seen that every invertible fractional ideal of D is finitely generated. An integral domain D is called a *Prüfer domain* if each nonzero finitely generated ideal of D is invertible.

A Prüfer domain may be an example of an integral domain which would have the maximum number of characterizations in all the classes of integral domains which have been already defined in commutative algebra. The number of characterizations of a Prüfer domain is already over eighty now. In this paper, we continue to study a Prüfer domain and we shall give some new characterizations of a Prüfer domain.

In Section 1, we first collect a family of well-known characterizations of a Prüfer domain which is only a part of the known characterizations of a Prüfer domain and we recall some definitions and preliminary results on semistar operations and localizing systemes which will be used in Section 2.

In Section 2, we shall give some new semistar-theoretical characterizations of a Prüfer domain by the use of properties of a semistar operation and a localizing system.

Throughout this paper, D denotes an integral domain with quotient field K. We denote *the integral closure* of an arbitrary integral domain R by R'. The symbol $\subset$ means "proper inclusion". We shall denote the set of all prime ideals (respectively, all maximal ideals) of D by $\mathrm{Spec}(D)$ (respectively, $\mathrm{Max}(D)$).

1. DEFINITIONS AND PRELIMINARY RESULTS

We first collect a family of well-known characterizations of a Prüfer domain.

Proposition 1. Let D be an integral domain. Then the following statements are equivalent:

(1) D is a Prüfer domain;

(2) For each prime ideal P of D, D_P is a valuation domain;

(3) For each maximal ideal M of D, D_M is a valuation domain;

(4) Each nonzero ideal of D that is generated by two elements is invertible;

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*Professor Emeritus, Oyama National College of Technology (aokabe@aw.wakwak.com)
**Professor Emeritus, Ibaraki University (rmazda@adagio.ocn.ne.jp)

(5) Each overring of D is integrally closed;

(6) Each overring of D is flat over D;

(7) D is integrally closed and each overring of D is the intersection of localizations of D;

(8) D is integrally closed and the prime ideals of any overring of D are the extensions of prime ideals of D;

(9) D is integrally closed and for each prime ideal P of D and each overring T of D, there is at most one prime ideal of T lying over P;

(10) D is integrally closed and there is a positive integer $n > 1$ such that $(a, b)^n = (a^n, b^n)$ for all $a, b \in D$;

(11) If $AB = AC$, where A, B, C are nonzero ideals of D and A is a nonzero finitely generated ideal, then $B = C$;

(12) $A(B \bigcap C) = AB \bigcap AC$ for all nonzero ideals A, B, C of D;

(13) $(A + B)(A \bigcap B) = AB$ for all nonzero ideals A, B of D;

(14) If A and C are nonzero ideals of D with C finitely generated and $A \subseteq C$, then there is a nonzero ideal B of D such that $A = BC$;

(15) $(A + B) :_D C = A :_D C + B :_D C$ for all nonzero ideals A, B, C of D with C finitely generated;

(16) $C :_D (A \bigcap B) = C :_D A + C :_D B$ for all nonzero ideals A, B, C of D with A and B finitely generated;

(17) $A \bigcap (B + C) = A \bigcap B + A \bigcap C$ for all nonzero ideals A, B, C of D;

(18) D is integrally closed and there exists a positive integer $n > 1$ such that $a^{n-1}b \in (a^n, b^n)$ for all $a, b \in D$;

(19) $A :_D B + B :_D A = D$ for all nonzero finitely generated ideals A, B of D;

(20) $(x) :_D (y) + (y) :_D (x) = D$ for all $x, y \in D$;

(21) Every ideal of D is a complete ideal;

(22) Every finitely generated ideal of D is a complete ideal;

(23) Every finitely generated ideal of D is an intersection of valuation ideals;

(24) D is integrally closed and K is a P-extension of D;

(25) D is integrally closed and $c_f c_g = c_{fg}$ for all $f, g \in K[X]$, where c_f denotes *the content ideal* of f, i.e., the ideal generated by the coefficients of f.

Proof. The equivalence of (1), (2), (4), (11), (12), (13), (14), (15), (16), and (17) is in [11, Theorem 6.6].

The implication (2) $\Rightarrow$ (3) is trivial and the implication (3) $\Rightarrow$ (2) is in [11, Corollary 6.7].

The implications (6) $\Rightarrow$ (3) and (2) $\Rightarrow$ (6) are in [11, Theorem 6.10].

The equivalence of (10) and (18) is in [9, Theorem 24.3]. The equivalence of (1), (19), and (20) is in [9, Theorem 25.2].

The equivalence of (1), (21), (22), and (23) is in [9, Theorem 24.7].

The equivalence of (1), (5), (7), (8), and (9) is in [9, Theorem 26.2]. The equivalence of (1) to (4) is also in [9, Theorem 22.1].

The implication (18) $\Rightarrow$ (24) is trivial. The implication (24) $\Rightarrow$ (3) is in [8, Theorem 2].

The implication (1) $\Rightarrow$ (25) is in [9, Corollary 28.5]. The implication (25) $\Rightarrow$ (18) is in [9, Theorem 28.6].

Let D be an integral domain with quotient field K and let $\mathbf{K}(D)$ be the set of all nonzero D-submodules of K. As in [13], we shall call each element of $\mathbf{K}(D)$ a *Kaplansky fractional ideal* (for short, *K-fractional ideal*) of D.

A map $E \mapsto E^*$ of $\mathbf{K}(D)$ into $\mathbf{K}(D)$ is called a *semistar operation* if the following conditions hold for all $a \in K - \{0\}$ and $E, F \in \mathbf{K}(D)$:
(S_1) $(aE)^* = aE^*$;
(S_2) If $E \subseteq F$, then $E^* \subseteq F^*$; and
(S_3) $E \subseteq E^*$ and $(E^*)^* = E^*$.

The notion of a semistar operation was introduced by Okabe and Matsuda in [11] as a generalization of the notion of a star operation. For more detailed results on semistar operations, the reader is referred to [5], [12], [13], [14], [15], [16], and [17].

A K-fractional ideal E of D is called a *fractional ideal* of D if $dE \subseteq D$ for some nonzero element d of D. We denote the set of all fractional ideals of D by $\mathbf{F}(D)$ and the set of all finitely generated K-fractional ideals of D by $\mathbf{f}(D)$. Moreover we denote the set of all nonzero integral ideals of D by $\mathbf{I}(D)$.

An element $E \in \mathbf{K}(D)$ is called a *$*$-ideal* of D if $E = E^*$. We denote the set of all $*$-ideals of D by $\mathbf{K}^*(D)$. A $*$-ideal P of D is called a *prime $*$-ideal* of D if P is also a prime ideal of D.

We shall now recall the definition of a star operation on D from [9].

A map $E \mapsto E^\star$ of $\mathbf{F}(D)$ into $\mathbf{F}(D)$ is called a *star operation* on D, if the following conditions hold for all $a \in K - \{0\}$ and $E, F \in \mathbf{F}(D)$:
(S_0) $(xD)^\star = xD$ for all $x \in K - \{0\}$;
(S_1) $(aE)^\star = aE^\star$;
(S_2) If $E \subseteq F$, then $E^\star \subseteq F^\star$; and
(S_3) $E \subseteq E^\star$ and $(E^\star)^\star = E^\star$.

The reader can refer to [9, Sections 32 and 34] for basic properties of star operations.

We first recall some representative examples of semistar operations and star operations.

Example 2. (1) If we set $E^{\bar{d}} = E$ for every $E \in \mathbf{K}(D)$, then $\bar{d}$ is a semistar operation on D and is called *the $\bar{d}$-semistar operation* or *the identity semistar operation on D*.

(2) If we set $E^{-1} = (D :_K E) = \{x \in K \mid xE \subseteq D\}$ and $E^{\bar{v}} = (E^{-1})^{-1}$ for every $E \in \mathbf{K}(D)$, then $\bar{v}$ is a semistar operation on D and is called *the $\bar{v}$-semistar operation on D*.

(3) If we set $E^{\bar{e}} = K$ for every $E \in \mathbf{K}(D)$, then $\bar{e}$ is a semistar operation on D

and is called *the $\bar{e}$-semistar operation on D* or *the trivial semistar operation on D*.

(4) Let $\{V_\lambda \mid \lambda \in \Lambda\}$ be the set of valuation overrings of D. If we set $E^{\bar{b}} = \bigcap\{EV_\lambda \mid \lambda \in \Lambda\}$ for every $E \in \mathbf{K}(D)$, then $\bar{b}$ is a semistar operation on D and is called *the $\bar{b}$-semistar operation on D*.

(5) If we set $E^d = E$ for all $E \in \mathbf{F}(D)$, then d is a star operation on D and is called *the d-operation* on D.

(6) If we set $E_v = (E^{-1})^{-1}$ for all $E \in \mathbf{F}(D)$, then v is a star operation on D and is called *the v-operation* on D.

We shall denote the set of all semistar operations (respectively, all star operations) on D by $\mathbf{SStar}(D)$ (respectively, $\mathbf{Star}(D)$) as in [5].

As in [13], a semistar operation $*$ is said to be *weak* if $D^* = D$ and is said to be *strong* if $D^* \neq D$. The set of all weak semistar operations on D is denoted by $\mathbf{SStar}_w(D)$ as in [13]. Moreover we denote the set of all strong semistar operations on D by $\mathbf{SStar}_s(D)$.

In [7] the notion of a *localizing (or topologizing) system of ideals* was introduced by P. Gabriel. A set $\mathcal{F}$ of ideals of D is called a *localizing system of ideals* (for short, *localizing system*) if the following conditions are satisfied:

(**LS1**) If $I \in \mathcal{F}$ and $I \subseteq J$ with $J \in \mathbf{I}(D)$, then $J \in \mathcal{F}$.

(**LS2**) If $I \in \mathcal{F}$ and $J \in \mathbf{I}(D)$ such that $J :_D iD \in \mathcal{F}$ for all $i \in I$, then $J \in \mathcal{F}$.

If $\mathcal{F}$ is a localizing system on D and $I, J \in \mathcal{F}$, then $IJ \in \mathcal{F}$ (see, [4, Proposition 5.1.1]) and so $I \bigcap J \in \mathcal{F}$ by (**LS1**). Thus every localizing system is a *generalized multiplicative system.* A localizing system $\mathcal{F}$ on D is said to be *of finite type* if for each $I \in \mathcal{F}$, there exists a finitely generated ideal $J \in \mathcal{F}$ such that $J \subseteq I$.

As in [17], we shall denote the set of localizing systems on D by $\mathbf{LS}(D)$ and the set of localizing systems of finite type on D by $\mathbf{LS}_f(D)$.

Let $\mathcal{F}$ be a localizing system on D. We define

$$D_{\mathcal{F}} = \{x \in K \mid D :_D xD \in \mathcal{F}\}.$$

Then $D_{\mathcal{F}}$ is an overring of D and is called *the ring of fractions with respect to $\mathcal{F}$*. It is easy to see that $D_{\mathcal{F}} = \bigcup\{D :_K I \mid I \in \mathcal{F}\}$ [4, (5. 1b)].

Let $*$ be a semistar operation on D. It is proved in [5, Proposition 2.8] that if we set $\mathcal{F}^* = \{I \in \mathbf{I}(D) \mid I^* = D^*\}$, then $\mathcal{F}^*$ is a localizing system on D.

Let T be an overring of D. If we set $\mathcal{F}(T) = \{I \in \mathbf{I}(D) \mid IT = T\}$, then $\mathcal{F}(T)$ is a localizing system on D. In fact, (**LS1**) is evident. Next, let $I \in \mathcal{F}(T)$ and let J be a nonzero ideal of D. Suppose that $J :_D x \in \mathcal{F}(T)$ for each $x \in I$. Then, since $(J :_D x)T = T$, we have $xT = x(J :_D x)T \subseteq JT$ for each $x \in I$ and so $IT \subseteq JT$. Hence $T = IT \subseteq JT \subseteq T$ and therefore $T = JT$ which implies that $J \in \mathcal{F}(T)$. Thus (**LS2**) also holds and hence $\mathcal{F}(T)$ is a localizing system on D. It is easy to see that $\mathcal{F}(T)$ is a localizing system of finite type. A localizing system $\mathcal{F}$ is said to be *of overring type* if $\mathcal{F} = \mathcal{F}(T)$ for some overring T of D.

Lemma 3 ([12, Lemma 45]). Let T be an overring of D. Then

(1) For each $* \in \mathbf{SStar}(T)$, we set $E^{\delta_{T/D}(*)} = (ET)^*$ for each $E \in \mathbf{K}(D)$. Then $\delta_{T/D}(*) \in \mathbf{SStar}(D)$.

(2) For each $* \in \mathbf{SStar}(D)$, we set $E^{\alpha_{T/D}(*)} = E^*$ for each $E \in \mathbf{K}(T)$ ($\subseteq \mathbf{K}(D)$). Then $\alpha_{T/D}(*) \in \mathbf{SStar}(T)$.

(3) $\alpha_{T/D} \circ \delta_{T/D}$ is the identity map of $\mathbf{SStar}(T)$ and so $\delta_{T/D}$ is an injective map.

The map $\delta_{T/D}$ (respectively, $\alpha_{T/D}$) is called *the descent map* (respectively, *the ascent map*).

Let T be an overring of D. If we set $E^{*_{(T)}} = ET$ for each $E \in \mathbf{K}(D)$, then $*_{(T)}$ is a semistar operation on D and is called *the semistar operation on D induced by T*. If $T \neq D$, then $*_{(T)}$ is a strong semistar operation on D. Each semistar operation $*_{(T)}$ for an overring T of D is said to be *of overring type* as in [15]. In this paper an overring T of D is called a *proper overring* in case $T \neq D$ and $T \neq K$. We shall denote the set of all proper overrings of D by $\mathcal{P}(D)$.

Proposition 4. (1) For each overring T of D, $*_{(T)} = \delta_{T/D}(\bar{d}_T)$, where $\bar{d}_T$ is the $\bar{d}$-semistar operation on T.

(2) For overrings $S \subset T$ of D, $*_{(T)} = \delta_{S/D}(\delta_{T/S}(\bar{d}_T))$.

(3) For each overring T of D, $\delta_{T/D}(\bar{e}_T) = \bar{e}$, where $\bar{e}_T$ is the $\bar{e}$-semistar operation on T.

(4) For each $T \in \mathcal{P}(D)$, $\delta_{T/D}(\mathbf{SStar}(T)) \bigcap \mathbf{SStar}_w(D) = \emptyset$. In particular, $\bar{d}$ and $\bar{v}$ are not contained in $\delta_{T/D}(\mathbf{SStar}(T))$.

Proof. (1) For each $E \in \mathbf{K}(D)$, $E^{\delta_{T/D}(\bar{d}_T)} = (ET)^{\bar{d}_T} = ET = E^{*_{(T)}}$ and hence $*_{(T)} = \delta_{T/D}(\bar{d}_T)$.

(2) By definition, $E^{\delta_{S/D}(\delta_{T/S}(\bar{d}_T))} = (ES)^{\delta_{T/S}(\bar{d}_T)} = (ET)^{\bar{d}_T} = ET = E^{*_{(T)}}$ for all $E \in \mathbf{K}(D)$ and so $\delta_{S/D}(\delta_{T/S}(\bar{d}_T)) = *_{(T)}$.

(3) This is trivial.

(4) For each $* \in \mathbf{SStar}(T)$, we have $D^{\delta_{T/D}(*)} = (DT)^* = T^* \supseteq T \neq D$ and hence $\delta_{T/D}(*) \notin \mathbf{SStar}_w(D)$. The "in particular" statement is straightforward. □

2. CHARACTERIZATIONS OF A PRÜFER DOMAIN

We first recall some equivalent conditions for an overring T of D to be flat over D.

Proposition 5. Let T be an overring of D. Then the following statements are equivalent:

(1) T is a flat overring of D, i.e., T is a flat D-module.

(2) $T_M = D_{M \cap D}$ for each maximal ideal M of T.

(3) $T = \bigcap\{D_{M \cap D} \mid M \in \mathrm{Max}(T)\}$.

(4) For each prime ideal P of D, either $PT = T$ or $T \subseteq D_P$.

(5) For every $t \in T$, $(D :_D t)T = T$.

(6) There exists a generalized multiplicative system $\mathcal{S}$ of D such that $T = D_{\mathcal{S}}$ and $IT = T$ for every $I \in \mathcal{S}$.

(7) There exists a localizing system $\mathcal{F}$ of finite type on D such that $T = D_{\mathcal{F}}$ and $IT = T$ for every $I \in \mathcal{F}$.

(8) $T = D_{\mathcal{F}(T)}$.

Proof. The equivalence of (1), (2), (3), and (4) is in [11, Propositions 4.10, 4.12, and 4.14]. The equivalence of (1), (6), and (7) is in [4, Remark 5.1.11 (b)]. (1) $\Rightarrow$ (8) is proved in [18, Proposition 1.2 (i)] or in [4, Proposition 5.1.10]. (8) $\Rightarrow$ (7) is trivial. Note that (4) $\Leftrightarrow$ (5), (1) $\Leftrightarrow$ (4), and (1) $\Leftrightarrow$ (2) is proved respectivly in [19, Lemma 1], [19, Theorem 1], and [19, Theorem 2]. $\square$

As in [5], a semistar operation $*$ on D is called a *stable semistar operation* if $(E \bigcap F)^* = E^* \bigcap F^*$ for all $E, F \in \mathbf{K}(D)$.

We first recall when a semistar operation of the form $*_{(T)}$ with overring T of D is a stable semistar operation on D.

Proposition 6 ([15, Remark 37 (1)]). Let T be an overring of D. Then T is flat over D if and only if $*_{(T)}$ is a stable semistar operation.

Proof. ($\Rightarrow$) This follows from [2, Chapter I, §2.6, Proposition 6].
($\Leftarrow$) This follows from [10, Theorem 1]. $\square$

Corollary 7. D is a Prüfer domain if and only if $*_{(T)}$ is a stable semistar operation on D for each overring T of D.

In [11], we defined a partial order $\leq$ on $\mathbf{SStar}(D)$ by $*_1 \leq *_2$ if and only if $E^{*_1} \subseteq E^{*_2}$ for all $E \in \mathbf{K}(D)$.

Proposition 8 ([5, Proposition 2.4]). Let $\mathcal{F}$ be a localizing system on D. If we set $E_{\mathcal{F}} = \bigcup\{E :_K J \mid J \in \mathcal{F}\}$ for each $E \in \mathbf{K}(D)$, then the map $E \mapsto E^{*_{\mathcal{F}}} := E_{\mathcal{F}}$ of $\mathbf{K}(D)$ into $\mathbf{K}(D)$ is a stable semistar operation on D.

A semistar operation $*_{\mathcal{F}}$ is called *the semistar operation associated to a localizing system* $\mathcal{F}$.

Lemma 9. Let T be an overring of D. Then $*_{\mathcal{F}(T)} \leq *_{(T)}$.

Proof. By definition, $E^{*_{\mathcal{F}(T)}} = \bigcup\{E :_K J \mid J \in \mathcal{F}(T)\}$ for each $E \in \mathbf{K}(D)$. Let $x \in E :_K J$ for some $J \in \mathcal{F}(T)$. Then $xJ \subseteq E$ and therefore $x \in xT = xJT \subseteq ET$. Thus $E :_K J \subseteq ET$ for all $J \in \mathcal{F}(T)$. Hence $E^{*_{\mathcal{F}(T)}} \subseteq ET = E^{*_{(T)}}$ for each $E \in \mathbf{K}(D)$ which implies that $*_{\mathcal{F}(T)} \leq *_{(T)}$. $\square$

Proposition 10 ([12, Proposition 34]). Let $*$ be a semistar operation on D. We set $D^{[*]} = \bigcup\{I^* :_K I^* \mid I \in \mathbf{f}(D)\}$. Then $D^{[*]}$ is an integrally closed overring of D.

Lemma 11. For each overring T of D, we have $T' = D^{[*_{(T)}]}$.

Proof. First we have $T \subseteq D^{[*_{(T)}]} = \bigcup\{I^{*_{(T)}} :_K I^{*_{(T)}} \mid I \in \mathbf{f}(D)\} = \bigcup\{IT :_K IT \mid I \in \mathbf{f}(D)\} \subseteq \{J :_K J \mid J \in \mathbf{f}(T)\} = T'$. Then, since $D^{[*_{(T)}]}$ is integrally closed by Proposition 10, we get $T' \subseteq D^{[*_{(T)}]} \subseteq T'$ and hence $T' = D^{[*_{(T)}]}$. □

Let T be an overing of D. Then we denote the subset $\{M \bigcap D \mid M \in \mathrm{Max}(T)\}$ of $\mathrm{Spec}(D)$ by Δ^T_{max}.

Lemma 12. Let T be an overring of D. Then $\star_{\Delta^T_{\mathrm{max}}} \leq *_{(T)}$.

Proof. We first show that $ET = \bigcap\{ET_M \mid M \in \mathrm{Max}(T)\}$ for each $E \in \mathbf{K}(D)$. $ET \subseteq \bigcap\{ET_M \mid M \in \mathrm{Max}(T)\}$ is trivial. Conversely choose $x \in \bigcap\{ET_M \mid M \in \mathrm{Max}(T)\}$ and set $I = (ET) :_T x$. Then I is an ideal of T and is not contained in any maximal ideal of T. Hence $I = T$ and so $1 \in (ET) :_T x$, that is, $x \in ET$. Therefore $ET = \bigcap\{ET_M \mid M \in \mathrm{Max}(T)\}$ for each $E \in \mathbf{K}(D)$. For each $M \in \mathrm{Max}(T), D_{M\cap D} \subseteq T_M$ is evident and therefore $E^{\star_{\Delta^T_{\mathrm{max}}}} = \bigcap\{ED_{M\cap D} \mid M \in \mathrm{Max}(T)\} \subseteq \bigcap\{ET_M \mid M \in \mathrm{Max}(T)\} = ET = E^{*_{(T)}}$ for all $E \in \mathbf{K}(D)$. Hence $\star_{\Delta^T_{\mathrm{max}}} \leq *_{(T)}$. □

Proposition 13. The following conditions are equivalent.
(1) D is a Prüfer domain;
(2) $\bar{b} = \bar{d}$;
(3) $(\bar{b})_f = \bar{d}$, where $(\bar{b})_f$ is the finite type semistar operation associated to $\bar{b}$;
(4) $T = T^{\bar{b}_T}$ for each overring T of D, where $\bar{b}_T$ is the $\bar{b}$-operation on T.

Proof. (1) ⇒ (2): Let $E \in \mathbf{K}(D)$, and let $x \in K$ with $x \notin E$. Then there exists a maximal ideal M of D such that $E :_D x \subset M$. It follows that $x \notin ED_M$.

(2) ⇒ (1): If $\bar{b} = \bar{d}$, then every ideal of D is a complete ideal and hence D is a Prüfer domain by the implication (21) ⇒ (1) in Proposition 1.

(1) ⇔ (3) follows from the equivalence (1) ⇔ (22) in Proposition 1.

(1) ⇔ (4) follows from the equivalence (1) ⇔ (5) in Proposition 1. □

Let $*$ be a semistar operation on D. Then $*$ is said to be *arithmetisch brauchbar* (abbreviated a.b.) if for all $E \in \mathbf{f}(D)$ and for all $F, G \in \mathbf{K}(D), (EF)^* \subseteq (EG)^*$ implies $F^* \subseteq G^*$ and $*$ is said to be *endlich arithmetisch brauchbar* (abbreviated e.a.b.) if for all $E, F, G \in \mathbf{f}(D), (EF)^* \subseteq (EG)^*$ implies $F^* \subseteq G^*$.

We get the following characterizations of a Prüfer domain using the a.b. property or the e.a.b. property of semistar operations:

Proposition 14. Let D be an integral domain. Then the following statements are equivalent:

(1) D is a Prüfer domain;

(2) Each semistar operation on D is a.b.;

(3) Each semistar operation on D is e.a.b.;

(4) $\bar{d}$ is a.b.;

(5) $\bar{d}$ is e.a.b..

Proof. The implications (2) $\Rightarrow$ (3) $\Rightarrow$ (5) and (2) $\Rightarrow$ (4) $\Rightarrow$ (5) are clear. The implication (5) $\Rightarrow$ (1) is shown in [9, Theorem 24.3].

(1) $\Rightarrow$ (2) : Let $*$ be a semistar operation on D. Assume that $(EF)^* \subseteq (EG)^*$ for $E \in \mathbf{f}(D)$ and for $F, G \in \mathbf{K}(D)$. Then $(E^{-1}(EF)^*)^* \subseteq (E^{-1}(EG)^*)^*$ and hence $F^* = ((EE^{-1})F^*)^* \subseteq ((EE^{-1})G^*)^* = G^*$ which implies that $*$ is a.b.. $\square$

Let $*$ be a semistar operation on D. If we set $\bar{*} = *_{\mathcal{F}^*}$, then it is easily seen that $\bar{*}$ is stable by Proposition 8. Moreover, it is proved in [5, Theorem 2.10 (B)] that $*$ is stable if and only if $* = \bar{*}$.

We shall now give some new characterizations of a Prüfer domain.

Theorem 15. Let D be an integral domain. Then the following conditions are equivalent:

(1) D is a Prüfer domain;

(2) $*_{\mathcal{F}(T)} = *_{(T)}$ for each overring T of D;

(3) $T = D^{[*_{(T)}]}$ for each overring T of D.

(4) $\star_{\Delta^T_{\max}} = *_{(T)}$.

Proof. (1) $\Rightarrow$ (2) : First, note that $\mathcal{F}^{*_{(T)}} = \mathcal{F}(T)$ for each overring T of D. In fact, $\mathcal{F}^{*_{(T)}} = \{I \in \mathbf{I}(D) \mid I^{*_{(T)}} = D^{*_{(T)}}\} = \{I \in \mathbf{I}(D) \mid IT = DT = T\} = \mathcal{F}(T)$. Then $*_{\mathcal{F}(T)} = *_{\mathcal{F}^{*_{(T)}}} = \overline{*_{(T)}}$. But, since T is flat over D, $*_{(T)}$ is stable by Proposition 6 and therefore $\overline{*_{(T)}} = *_{(T)}$ by [5, Theorem 2.10 (B)]. Hence $*_{\mathcal{F}(T)} = *_{(T)}$.

(2) $\Rightarrow$ (1) : Assume that $*_{\mathcal{F}(T)} = *_{(T)}$ for each overring T of D. Then, by Proposition 8, $*_{(T)}$ is stable and so T is flat over D by Proposition 6. Thus every overring T is flat over D and therefore D is a Prüfer domain by Proposition 1.

(1) $\Leftrightarrow$ (3) : This follows from Proposition 1 ((1) $\Leftrightarrow$ (5)), Proposition 10 and Lemma 11.

(4) $\Rightarrow$ (1) : Let T be an arbitrary overring of D. By hypothesis, $D^{\star_{(T)}} = D^{\Delta^T_{\max}}$ and therefore $T = \bigcap\{D_{M \cap D} \mid M \in \mathrm{Max}(T)\}$. Then, by (1) $\Leftrightarrow$ (3) in Proposition 5, T is flat over D. Hence, by (1) $\Leftrightarrow$ (6) in Proposition 1, D is a Prüfer domain.

(1) $\Rightarrow$ (4) : Assume that D is a Prüfer domain. Then, by Propositions 1 and 5, $D_{M \cap D} = T_M$ for each $M \in \mathrm{Max}(T)$ and hence $E^{\star_{\Delta^T_{\max}}} = \bigcap\{ED_{M \cap D} \mid M \in \mathrm{Max}(T)\} = \bigcap\{ET_M \mid M \in \mathrm{Max}(T)\} = ET = E^{*_{(T)}}$ for all $E \in \mathbf{K}(D)$. Hence $\star_{\Delta^T_{\max}} = *_{(T)}$. $\square$

References

[1] D. D. Anderson and D. F. Anderson, Examples of star operations on integral domains, Communications in Algebra, 18(1990), 1621-1643.

[2] N. Bourbaki, Commutative Algebra, Chapters 1-7, Springer-Verlag, 1985.

[3] D. E. Dobbs and R. Fedder, Conducive integral domains, J. Algebra, 86(1984), 494-510.

[4] M. Fontana, J. A. Huckaba and I. J. Papick, Prüfer Domains, Marcel Dekker, New York, 1997.

[5] M. Fontana and J. A. Huckaba, Localizing systems and semistar operations, in "Non-Noetherian Commutative Ring Theory" (S. T. Chapman and S. Glaz (Eds.)), Kluwer Academic Publishers, (2000), Chapter 8, 169-197.

[6] M. Fontana and K. Alan Loper, Nagata rings, Kronecker function rings, and related semistar operations, Communications in Algebra, 31 (2003), 4775-4805.

[7] P. Gabriel, Des Catégories abéliennes, Bull. Soc. Math. France, 90(1962), 323-448.

[8] R. Gilmer and J. F. Hoffmann, A characterization of Prüfer domains in terms of polynomials, Pacific Journal of Mathematics, 60(1975), 81-85.

[9] R. Gilmer, Multiplicative Ideal Theory, Queen's Papers in Pure and Applied Mathematics, Vol. 90, Kingston, Ontario, Canada, 1992.

[10] C. U. Jensen, A remark on flat and projective modules, Canadian J. Math., 18(1966), 943-949.

[11] M. D. Larsen and P. J. McCarthy, Multiplicative theory of ideals, Academic Press, New York and London, 1971.

[12] A. Okabe and R. Matsuda, Semistar operations on integral domains, Math. J. Toyama Univ., 17(1994), 1-21.

[13] A. Okabe, Some results on semistar operations, JP Jour. of Algebra, Number Theory and Applications, 3(2003), 187-210.

[14] A. Okabe, Semistar operations on conducive domains, Math. J. Ibaraki Univ., 35(2003), 11-19.

[15] A. Okabe, On generalized Dedekind domains, Math. J. Ibaraki Univ., 36(2004), 5-23.

[16] A. Okabe, On amply strong semistar domains, Math. J. Ibaraki Univ., 36(2004), 45-55.

[17] A. Okabe, A note on semistar operations and localizing systems, JP Jour. Algebra, Number Theory and Applications, 6(1) (2006), 149-174.

[18] N. Popesch, On a class of Prüfer domains, Rev. Roumanie Math. Pures Appl., 29(1984), 777-786.

[19] F. Richman, Generalized quotient rings, Proc. Amer. Math. Soc., 16(1965), 794-799.