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Note on the number of semistar operations, XI

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Abstract

We study star operations and semistar operations on an almost pseudo-valuation domain.

The notion of a star operation is classical, and that of a Kronecker function ring which arises by a star operation is also classical. Let D be an integral domain, let K be its quotient field, and let $\mathrm{F}(D)$ be the set of non-zero fractional ideals of D. A mapping $I \longmapsto I^*$ from $\mathrm{F}(D)$ to $\mathrm{F}(D)$ is called a star operaion on D, if it satisfies the following conditions: (1) $(x)^* = (x)$ for each $x \in K - \{0\}$; (2) $(xI)^* = xI^*$ for each $x \in K - \{0\}$ and $I \in \mathrm{F}(D)$; (3) $I \subset I^*$ for each $I \in \mathrm{F}(D)$; (4) $(I^*)^* = I^*$ for each $I \in \mathrm{F}(D)$; (5) $I \subset J$ implies $I^* \subset J^*$ for each $I, J \in \mathrm{F}(D)$. The Kronecker function ring of D with respect to a star operation $*$ on D was first defined by Prüfer ([P]) and further investigated by Krull ([K]). Let $\mathrm{F}'(D)$ be the set of non-zero D-submodules of K. A mapping $I \longmapsto I^*$ from $\mathrm{F}'(D)$ to $\mathrm{F}'(D)$ is called a semistar operation on D, if it satisfies the following conditions: (1) $(xI)^* = xI^*$ for each $x \in K - \{0\}$ and $I \in \mathrm{F}'(D)$; (2) $I \subset I^*$ for each $I \in \mathrm{F}'(D)$; (3) $(I^*)^* = I^*$ for each $I \in \mathrm{F}'(D)$; (4) $I \subset J$ implies $I^* \subset J^*$ for each $I, J \in \mathrm{F}'(D)$. We confer Fontana-Loper([FL]) and Halter-Koch([HK]) for notions of star operations, semistar operations, and their Kronecker function rings.

Let $\Sigma(D)$ (resp. $\Sigma'(D)$) be the set of star operations (resp. semistar operations) on D. In this paper, we are interested in cardinalities $|\Sigma(D)|$ and $|\Sigma'(D)|$.

Let D be an integrally closed domain. Then D has only a finite number of semistar operations if and only if D is a finite dimensional Prüfer domain with only a finite number of maximal ideals ([M4]).

Let V be a valuation domain with dimension n, let v be a valuation belonging to V, and let Γ be its value group. Let $M = P_n \supsetneqq P_{n-1} \supsetneqq \cdots\cdots P_1 \supsetneqq (0)$ be the prime ideals of V, and let $\{0\} \subsetneqq H_{n-1} \subsetneqq \cdots \subsetneqq H_1 \subsetneqq \Gamma$ be the convex subgroups of Γ. Let m be a positive integer with $n+1 \leq m \leq 2n+1$. Then the following conditions are equivalent: (1) $|\Sigma'(V)| = m$; (2) The maximal ideal of V_{P_i} is principal for exactly $2n+1-m$ of i; (3) Γ/H_i has a minimal positive element for exactly $2n+1-m$ of i ([M1]).

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In [M5], we studied star operations and semistar operations on a pseudo-valuation domain (or a PVD) D. We gave conditions for D to have only a finite number of semistar operations, and showed that conditions for $|\Sigma'(D)| < \infty$ reduce to conditions for fields.

In this paper, we concern with star operations and semistar operations on almost pseudo-valuation domains. We study almost pseudo-valuation domains with simple associated valuation rings, and will prove the following,

Theorem Let D be an almost pseudo-valuation domain, let P be the maximal ideal of D, let $V = (P : P)$, let M be the maximal ideal of V, and let $K = V/M$. Assume that the valuation ring V is discrete with rank one and $D/P = K$. Then we have

(1) If K is a finite field, then $|\Sigma(D)| < \infty$.

(2) If $P = M^2$, then $|\Sigma(D)| = 1$ and $|\Sigma'(D)| = 3$, and if $P = M^3$, then $|\Sigma(D)| = 3$ and $|\Sigma'(D)| = 6$.

(3) If $P = M^4$ and K is an infinite field, then $|\Sigma(D)| = \infty$ and $|\Sigma'(D)| = \infty$.

(4) If $P = M^n$ with $n \geq 5$ and K is an infinite field, then $|\Sigma'(D)| = \infty$.

The paper consists of six sections, Section 1 is a review on almost pseudo-valuation domains, Section 2 is the general case, Section 3 is the case where $P = M^2$ or $P = M^3$, Section 4 is the case where $P = M^4$, Section 5 is the case where $P = M^n$ with $n \geq 5$, and Section 6 is examples.

§1 Review

In this section, we review a result in [M2] on semistar operations on almost pseudo-valuation domains.

Let I be an ideal of a domain D. If $ab \in I$ and $b \notin I$ imply $a^n \in I$ for some $n > 0$ for each elements $a, b \in \mathrm{q}(D)$, then I is called a strongly primary ideal of D, where $\mathrm{q}(D)$ denotes the quotient field of D. If each prime ideal of D is strongly primary, then D is called an almost pseudo-valuation domain (or an APVD). Every PVD is an almost pseudo-valuation domain. We confer Badawi-Houston([BH]) for almost pseudo-valuation domains.

(1.1) Let D be an APVD, let P be a maximal ideal of D, let $V = (P : P)$, and let M be the maximal ideal of V.

(1) $\mathrm{F}'(D) = \mathrm{F}(D) \cup \{\mathrm{q}(D)\}$.

(2) D is a local ring, that is, D has only one maximal ideal.

(3) If D is not a valuation ring, then $V = P^{-1}$.

(4) The set of non-maximal prime ideals of D coincides with the set of non-maximal prime ideals of V, and dim $(V) =$ dim (D).

(5) The integral closure $\bar{D}$ of D is a PVD with maximal ideal M.

(6) Let T be an overring of D. Then either $T \supset V$ or $T \subsetneqq V$.

(7) Let $\Sigma'_1 = \{* \in \Sigma'(D) \mid D^* \supset V\}$. Then there exists a canonical bijection from $\Sigma'(V)$ onto Σ'_1.

(8) Let $\Sigma'_2 = \{* \in \Sigma'(D) \mid D^* \subsetneqq V\}$. Then $\Sigma'(D) = \Sigma'_1 \cup \Sigma'_2$.

(9) If $\mid \Sigma'(D) \mid < \infty$, then dim $(D) < \infty$, $V = \bar{D}$, V is a finitely generated D-module, and V/M is a simple extension field of D/P with $[V/M : D/P] < \infty$.

(1.2) Let D be an APVD which is not a PVD, let P be the maximal ideal of D, and let $V = (P : P)$. Assume that $\dim(D) < \infty$, and let $\{T_\lambda \mid \lambda \in \Lambda\}$ be the set of overrings T of D with $T \subsetneqq V$. Let Σ'_1 be the set of semistar operations $*$ on D such that $D^* \supset V$, and let Σ'_2 be the set of semistar operations $*$ on D such that $D^* \subsetneqq V$. Then we have

(1) $\Sigma'(D) = \Sigma'_1 \cup \Sigma'_2$.

(2) $\mid \Sigma'(V) \mid < \infty$.

(3) $\mid \Sigma'_1 \mid = \mid \Sigma'(V) \mid$.

(4) There exists a canonical bijection from the disjoint union $\bigcup_\lambda \Sigma(T_\lambda)$ onto Σ'_2.

§2 The general case

Throughout the paper but the final §6, let D be an APVD which is not a PVD, let P be the maximal ideal of D, let $V = (P : P)$, let M be the maximal ideal of V, let v be a valuation belonging to V, and set $V/M = K$. We assume that v is $\boldsymbol{Z}$-valued and that $K = (D+M)/M$, and let $\{\alpha_i \mid i \in \mathcal{I}\} = \mathcal{K}$ be a system of complete representatives of V modulo M, where $\{0, 1\} \subset \{\alpha_i \mid i \in \mathcal{I}\} \subset D$.

Note: Let V be a $\boldsymbol{Z}$-valued valuation domain of the form $K + M$, where K is a field and M is the maximal ideal of V. Let k be a subfield of K, and let $D = k + M$. Assume that $\dim(D) < \infty$, and K is a simple extension of k with finite degree. Then $|\Sigma(D)|$ need not be finite ([M3]).

There exists $\pi \in V$ such that $M = \pi V$. Then we have $v(\pi) = 1$.

We have $V = P : P = P^{-1}$, where $P : P$ denotes $\{x \in \mathrm{q}(D) \mid xP \subset P\}$ and P^{-1} denotes $D : P$.

Let $I, J \in \mathrm{F}(D)$. If there exists $x \in \mathrm{q}(D)$ such that $xJ = I$, then I and J are said to be similar, and is denoted by $I \sim J$. For each $I \in \mathrm{F}(D)$, set $\{J \in \mathrm{F}(D) \mid J \sim I\} = \mathrm{cl}(I)$.

Let Σ'_1 be the set of semistar operations $*$ on D such that $D^* \supset V$, and let Σ'_2 be the set of semistar operations on D such that $D^* \subsetneqq V$. We can apply (1.1) and (1.2) for D.

(2.1) Let $x \in \mathrm{q}(D) - \{0\}$, and let k be a positive integer with $k > v(x)$. Then x can be expressed uniquely as $x = \alpha_l \pi^l + \alpha_{l+1}\pi^{l+1} + \cdots + \alpha_{k-1}\pi^{k-1} + \pi^k a$, where $l = v(x)$ and each $\alpha_i \in \mathcal{K}$ with $\alpha_l \neq 0$ and $a \in V$.

(2.2) There exists a positive integer $n \geq 2$ uniquely so that $P = M^n$.

Proof. Set $\min\{v(x) \mid x \in P\} = n$. Since $PV = P$, we have $P = M^n$. If $n = 1$, then D is a valuation ring; a contradiction.

For every subset $A \subset \mathrm{q}(D)$, we denote by (A) the D-submodule of $\mathrm{q}(D)$ generated by A. If $P = M^n$, then we have $P = (\pi^n, \pi^{n+1}, \cdots, \pi^{2n-2}, \pi^{2n-1})$, and $V = (1, \pi, \cdots, \pi^{n-1})$.

(2.3) (1) We have $P^v = P$, where P^v denotes $(P^{-1})^{-1}$.
(2) The set Spec(D) of prime ideals of D is $\{P, (0)\}$.

(2) follows from (1.1)(4).

The mapping $I \longmapsto I^v$ from F(D) to F(D) is a star operation on D, and is called the $v-$ *operation. The identity mapping* d from F(D) to F(D) is a star operation on D, and is called the d-operation.

We note that V is a divisorial fractional ideal of D, that is, $V^{\mathrm{v}} = V$. Each star operation on D can be uniquely extended to a semistar operation on D. Also let D' be an overring of D. Then there exists a canonical mapping δ from $\Sigma'(D')$ to $\Sigma'(D)$, and is called the descent mapping. And δ is an injective mapping.

We have $|\Sigma_1'| = |\Sigma'(V)| = 2$.

If $P = M^2$, then we have
$\{I \in \mathrm{F}(D) \mid D \subset I \subset V\} = \{(1), (1, \pi)\}$.

(2.4) Example Assume that $P = M^3$. Set $(1) = I_0, (1, \pi^2) = I_{0,2}, (1, \pi, \pi^2) = I_{0,1,2}$, and $(1, \pi + \alpha\pi^2) = I_{0,1}^{\alpha}$, where $\alpha \in \mathcal{K}$. Then we have
$\{I \in \mathrm{F}(D) \mid D \subset I \subset V\} = \{I_0, I_{0,2}, I_{0,1,2}\} \cup \{I_{0,1}^{\alpha} \mid \alpha \in \mathcal{K}\}$.

Proof. Because, $\{v(x) \mid x \in I - P\}$ is either $\{0\}$ or $\{0, 1\}$ or $\{0, 2\}$ or $\{0, 1, 2\}$.

(2.5) Example Assume that $P = M^4$. For every element $\alpha_i \in \mathcal{K}$, set
$(1) = I_0$,
$(1, \pi + \alpha_1\pi^2 + \alpha_2\pi^3) = I_{0,1}^{\alpha_1,\alpha_2}$,
$(1, \pi^2 + \alpha_1\pi^3) = I_{0,2}^{\alpha_1}$,
$(1, \pi^3) = I_{0,3}$,
$(1, \pi + \alpha_1\pi^3, \pi^2 + \alpha_2\pi^3) = I_{0,1,2}^{\alpha_1,\alpha_2}$,
$(1, \pi + \alpha_1\pi^2, \pi^3) = I_{0,1,3}^{\alpha_1}$,
$(1, \pi^2, \pi^3) = I_{0,2,3}$,
$(1, \pi, \pi^2, \pi^3) = I_{0,1,2,3}$.
Then we have
$\{I \in \mathrm{F}(D) \mid D \subset I \subset V\} = \{I_0, I_{0,1}^{\alpha_1,\alpha_2}, I_{0,2}^{\alpha_1}, I_{0,3}, I_{0,1,2}^{\alpha_1,\alpha_2}, I_{0,1,3}^{\alpha_1}, I_{0,2,3}, I_{0,1,2,3} \mid \text{each } \alpha_i \in \mathcal{K}\}$.

Proof. Let I be a fractional ideal of D such that $D \subset I \subset V$. Let $\tau = \{v(x) | \ x \in I - P\}$, say let $\tau = \{0, 1, 3\}$. Then I contains elements $a, b \in V$ of the form $a = \pi + \alpha_1\pi^2, b = \pi^3$, where $\alpha_1 \in \mathcal{K}$. We have $I \supset (1, a, b)$. Let $I \ni x = \beta_0 + \beta_1\pi + \beta_2\pi^2 + \beta_3\pi^3 + p$, where each $\beta_i \in \mathcal{K}$ and $p \in P$. We have $x = \beta_0 + \beta_1 a + \beta_3 b + \beta'\pi^2 + p'$ for some $\beta' \in \mathcal{K}$ and $p' \in P$. By the choice of τ, we have $\beta' = 0$. Hence $I = (1, a, b)$.

(2.6) Example Assume that $P = M^5$. For every element $\alpha_i \in \mathcal{K}$, set
$(1) = I_0$,
$(1, \pi + \alpha_1\pi^2 + \alpha_2\pi^3 + \alpha_3\pi^4) = I_{0,1}^{\alpha_1,\alpha_2,\alpha_3}$,
$(1, \pi^2 + \alpha_1\pi^3 + \alpha_2\pi^4) = I_{0,2}^{\alpha_1,\alpha_2}$,

$(1, \pi^3 + \alpha_1\pi^4) = I_{0,3}^{\alpha_1}$,
$(1, \pi^4) = I_{0,4}$,
$(1, \pi + \alpha_1\pi^3 + \alpha_2\pi^4, \pi^2 + \alpha_3\pi^3 + \alpha_4\pi^4) = I_{0,1,2}^{\alpha_1,\alpha_2,\alpha_3,\alpha_4}$,
$(1, \pi + \alpha_1\pi^2 + \alpha_2\pi^4, \pi^3 + \alpha_3\pi^4) = I_{0,1,3}^{\alpha_1,\alpha_2,\alpha_3}$,
$(1, \pi + \alpha_1\pi^2 + \alpha_2\pi^3, \pi^4) = I_{0,1,4}^{\alpha_1,\alpha_2}$,
$(1, \pi^2 + \alpha_1\pi^4, \pi^3 + \alpha_2\pi^4) = I_{0,2,3}^{\alpha_1,\alpha_2}$,
$(1, \pi^2 + \alpha_1\pi^3, \pi^4) = I_{0,2,4}^{\alpha_1}$,
$(1, \pi^3, \pi^4) = I_{0,3,4}$,
$(1, \pi + \alpha_1\pi^4, \pi^2 + \alpha_2\pi^4, \pi^3 + \alpha_3\pi^4) = I_{0,1,2,3}^{\alpha_1,\alpha_2,\alpha_3}$,
$(1, \pi + \alpha_1\pi^3, \pi^2 + \alpha_2\pi^3, \pi^4) = I_{0,1,2,4}^{\alpha_1,\alpha_2}$,
$(1, \pi + \alpha_1\pi^2, \pi^3, \pi^4) = I_{0,1,3,4}^{\alpha_1}$,
$(1, \pi^2, \pi^3, \pi^4) = I_{0,2,3,4}$,
$(1, \pi, \pi^2, \pi^3, \pi^4) = I_{0,1,2,3,4}$.
Then we have
$\{I \in \mathrm{F}(D) \mid D \subset I \subset V\} = \{I_0, I_{0,1}^{\alpha_1,\alpha_2,\alpha_3}, I_{0,2}^{\alpha_1,\alpha_2}, I_{0,3}^{\alpha_1}, I_{0,4}, I_{0,1,2}^{\alpha_1,\alpha_2,\alpha_3,\alpha_4}, I_{0,1,3}^{\alpha_1,\alpha_2,\alpha_3}, I_{0,1,4}^{\alpha_1,\alpha_2}, I_{0,2,3}^{\alpha_1,\alpha_2}, I_{0,2,4}^{\alpha_1}, I_{0,3,4}, I_{0,1,2,3}^{\alpha_1,\alpha_2,\alpha_3}, I_{0,1,2,4}^{\alpha_1,\alpha_2}, I_{0,1,3,4}^{\alpha_1}, I_{0,2,3,4}, I_{0,1,2,3,4} \mid \text{each } \alpha_i \in \mathcal{K}\}$.

Proof. Let I be a fractional ideal of D such that $D \subset I \subset V$. Let $\tau = \{v(x) | \ x \in I - P\}$, say let $\tau = \{0, 1, 3\}$. Then I contains elements $a, b \in V$ of the form $a = \pi + \alpha_1\pi^2 + \alpha_2\pi^4, b = \pi^3 + \alpha_3\pi^4$, where $\alpha_1, \alpha_2, \alpha_3 \in \mathcal{K}$. We have $I \supset (1, a, b)$. Let $I \ni x = \beta_0 + \beta_1\pi + \beta_2\pi^2 + \beta_3\pi^3 + \beta_4\pi^4 + p$, where each $\beta_i \in \mathcal{K}$ and $p \in P$. We have $x = \beta_0 + \beta_1 a + \beta_3 b + \beta_1'\pi^2 + \beta_2'\pi^4 + p'$ for some $\beta_i' \in \mathcal{K}$ and $p' \in P$. By the choice of τ, we have $\beta_1' = \beta_2' = 0$. Hence $I = (1, a, b)$.

Each subset τ of $\{0, 1, 2, 3, 4\}$ which contains 0 is called a type associated to D. We have the number 16 of associated types to D. The set of types has a canonical order so that $\{0\}$ is the minimal member and $\{0, 1, 2, 3, 4\}$ is the maximal member: $\{0\}, \{0, 1\}, \{0, 2\}, \{0, 3\}, \{0, 4\}, \{0, 1, 2\}, \{0, 1, 3\}, \{0, 1, 4\}, \{0, 2, 3\}, \{0, 2, 4\}, \{0, 3, 4\}, \{0, 1, 2, 3\}, \{0, 1, 2, 4\}, \{0, 1, 3, 4\}, \{0, 2, 3, 4\}, \{0, 1, 2, 3, 4\}$. Let τ be a type associated to D, say let $\tau = \{0, 1, 2\}$. Then the tuple $< \alpha_1, \alpha_2, \alpha_3, \alpha_4 >$ of elements in $\mathcal{K}$ is called a system of parameters of τ (or, a system of parameters associated to τ), and 4 is called the length of the system of parameters $< \alpha_1, \alpha_2, \alpha_3, \alpha_4 >$. The pair $< 0, 1, 2; \alpha_1, \alpha_2, \alpha_3, \alpha_4 > = \sigma$ of τ and $< \alpha_1, \alpha_2, \alpha_3, \alpha_4 >$ is called a data of τ (or, a data associated to τ). Set $1 = f_1^\sigma, \pi + \alpha_1\pi^3 + \alpha_2\pi^4 = f_2^\sigma, \pi^2 + \alpha_3\pi^3 + \alpha_4\pi^4 = f_3^\sigma$. Then the tuple $< f_1^\sigma, f_2^\sigma, f_3^\sigma >$ is called an associated system of generators to σ. The fractional ideal $(f_1^\sigma, f_2^\sigma, f_3^\sigma) = I$ is called associated to σ, and is denoted by I^σ. The tuple $< f_1^\sigma, f_2^\sigma, f_3^\sigma >$ is also called a canonical system of generators for I associated to σ.

Assume that $P = M^n$ for a positive integer $n \geq 2$. We confer the previous examples. Each subset τ of $\{0, 1, 2, \cdots, n-1\}$ which contains 0 is called a type associated to D. We have the number 2^{n-1} of associated types to D. The set of types has a canonical order so that $\{0\} = \tau_1$ is the minimal member and $\{0, 1, \cdots, n-1\} = \tau_{2^{n-1}}$ is the maximal member. Let $\tau = \{0, k_1, \cdots, k_m\}$ be a type associated to D with $0 < k_1 < \cdots < k_m$. We can define a system of parameters $< \alpha_1, \cdots, \alpha_l >$ of τ. It is a tuple of elements in $\mathcal{K}$. The pair $< 0, k_1, \cdots, k_m; \alpha_1, \cdots, \alpha_l > = \sigma$ of τ and

$< \alpha_1, \cdots, \alpha_l >$ is called a data of τ (or, a data associated to τ). We can define an associated system $< f_1^\sigma, f_2^\sigma, \cdots, f_{m+1}^\sigma >$ of generators to σ. It is a tuple of elements in V. We donote also $f_i^\sigma = f_i(\sigma)$. The fractional ideal $(f_1^\sigma, f_2^\sigma, \cdots, f_{m+1}^\sigma) = I$ is said to be associated to σ, and is denoted by I^σ (or, by $I(\sigma)$). The tuple $< f_1^\sigma, f_2^\sigma, \cdots, f_{m+1}^\sigma >$ is also called a canonical system of generators for I associated to σ.

(2.7) Proposition Assume that $P = M^n$ with $n \geq 2$. Then we have $\{I \in \mathrm{F}(D) \mid D \subset I \subset V\} = \{I(\sigma_1), \cdots, I(\sigma_{2^{n-1}}) \mid$ each σ_i is a data associated to the type τ_i for $1 \leq i \leq 2^{n-1}\}$.

§3 The case where $P = M^2$ or $P = M^3$

(3.1) Proposition Assume that $P = M^2$. Then we have $|\Sigma(D)\ | = 1$ and $|\Sigma'(D)| = 3$.

Proof. Since $\pi^2(1, \pi) = P$, each $I \in \mathrm{F}(D)$ is divisorial. It follows that $|\Sigma(D)| = 1$.

Let T be an overring of D with $V \supsetneqq T \supsetneqq D$, and take $t \in T - D$. There may arise the following two cases: (1) $v(t) = 1$, and (2) $v(t) = 0$.

Case (1): We may assume that $t = \pi + p$ for some $p \in P$. Hence we have $T \ni \pi$, and hence $T = V$; a contradiction.

Case (2): We may assume that $t = 1 + \alpha\pi + p$ for some $\alpha \in \mathcal{K}$ and $p \in P$. Since $t \notin D$, we have $\alpha\pi \in T - D$. Case (1) implies that $T = V$; a contradiction.

We will apply (1.2). Since $|\Sigma_1'|\ = 2$, we have $|\Sigma'(D)|\ = 2 + |\Sigma(D)|\ = 3$.

Throughout the rest of the section, assume that $P = M^3$.

(3.2) We have $\{T \mid T$ is an overring of D with $T \subsetneqq V\} = \{D, D + M^2\}$.

Proof. Let T be an overring of D with $V \supsetneqq T \supsetneqq D$, and take $t \in T - D$. The proof of (3.1) shows that, if $T \supsetneqq D + M^2$ then $T = V$. There may arise the following three cases: (1) $v(t) = 2$, (2) $v(t) = 1$, and (3) $v(t) = 0$.

Case (1): We may assume that $t = \pi^2 + p$ for some $p \in P$. Hence we have $T \ni \pi^2$, and $T \supset D + M^2$.

Case (2): We may assume that $t = \pi + \alpha\pi^2 + p$ for some $\alpha \in \mathcal{K}$ and $p \in P$. Since $T \ni t^2$, we have $T \ni \pi^2$, and hence $T \supset D + M^2$.

Case (3): We may assume that $t = 1 + \alpha_1\pi + \alpha_2\pi^2 + p$ for some $\alpha_1, \alpha_2 \in \mathcal{K}$ and $p \in P$. Since $t \notin D$, we have $\alpha_1\pi + \alpha_2\pi^2 \in T - D$. Cases 1 and 2 imply that $T \supset D + M^2$.

(3.3) (1) $I_{0,2}$ and $I_{0,1}^\alpha$ are incomparable for each $\alpha \in \mathcal{K}$.

(2) $I_{0,1}^\alpha \subset I_{0,1}^\beta$ if and only if $\alpha = \beta$.

Proof. (2) Assume that $I_{0,1}^\alpha \subset I_{0,1}^\beta$. Then $\pi + \alpha\pi^2 \in (1, \pi + \beta\pi^2)$ implies $\pi + \alpha\pi^2 = (\pi + \beta\pi^2) + p$ for some $p \in P$. Hence $\alpha = \beta$.

(3.4) (1) Each two in $\{I_0, I_{0,2}, I_{0,1,2}, I_{0,1}^{\alpha}\}$ are not similar for each $\alpha \in \mathcal{K}$.
(2) $I_{0,1}^{\alpha}$ and $I_{0,1}^{\beta}$ are similar for each $\alpha, \beta \in \mathcal{K}$.

Proof. (2) Set $1 + \alpha\pi + \alpha^2\pi^2 = x$. Then we have $x(1, \pi) = (1, \pi + \alpha\pi^2)$.

(3.5) Let $*$ be a star operation on D. Then $(I_{0,2})^*$ is either $I_{0,2}$ or V, and $(I_{0,1}^0)^*$ is either $I_{0,1}^0$ or V.

Proof. Since V is a divisorial fractional ideal of D, we have $(I_{0,2})^* \subset V$ and $(I_{0,1}^0)^* \subset V$.

(3.6) (1) If we set $(I_{0,2})^{*_1} = I_{0,2}$ and $(I_{0,1}^0)^{*_1} = I_{0,1}^0$, then there is determined a unique mapping $*_1$ from $\mathrm{F}(D)$ to $\mathrm{F}(D)$.
(2) If we set $(I_{0,2})^{*_2} = I_{0,2}$ and $(I_{0,1}^0)^{*_2} = V$, then there is determined a unique mapping $*_2$ from $\mathrm{F}(D)$ to $\mathrm{F}(D)$.
(3) If we set $(I_{0,2})^{*_3} = V$ and $(I_{0,1}^0)^{*_3} = I_{0,1}^0$, then there is determined a unique mapping $*_3$ from $\mathrm{F}(D)$ to $\mathrm{F}(D)$.
(4) If we set $(I_{0,2})^{*_4} = V$ and $(I_{0,1}^0)^{*_4} = V$, then there is determined a unique mapping $*_4$ from $\mathrm{F}(D)$ to $\mathrm{F}(D)$.

For, each element $I \in \mathrm{F}(D)$ is similar to one and only one in $\{I_0, I_{0,2}, I_{0,1,2}, I_{0,1}^0\}$.

(3.7) (1) $*_1$ is a star operation on D, and $*_1 = \mathrm{d}$.
(2) $*_2$ is a star operation on D.
(3) $*_3$ is not a star operation on D.
(4) $*_4$ is a star operation on D, and $*_4 = v$.

Proof. (2) For each $x \in \mathrm{q}(D) - \{0\}$, we have $(x)^{*_2} = (x)$.
For each $x \in \mathrm{q}(D) - \{0\}$ and each $I \in \mathrm{F}(D)$, we have $(xI)^{*_2} = xI^{*_2}$.
For each $I \in \mathrm{F}(D)$, we have $I \subset I^{*_2}$.
For each $I \in \mathrm{F}(D)$, we have $(I^{*_2})^{*_2} = I^{*_2}$.
Let $I_1, I_2 \in \mathrm{F}(D)$ with $I_1 \subset I_2$. To prove $I_1^{*_2} \subset I_2^{*_2}$, it is sufficient to show that, if $xI_{0,1}^0 \subset J$ then $xV \subset J$ for each $x \in \mathrm{q}(D) - \{0\}$ and each $J \in \{I_0, I_{0,2}, I_{0,1,2}\}$.
(3) Set $\pi + \pi^2 = x$. Then we have $x(1, \pi^2) \subset (1, \pi + \pi^2)$ and $xV \not\subset (1, \pi + \pi^2)$.

(3.8) Proposition Assume that $P = M^3$. Then we have $|\Sigma(D)| = 3$ and $|\Sigma'(D)| = 6$.

Proof. It follows that $\Sigma(D) = \{d, v, *_2\}$, and that $|\Sigma(D)| = 3$. We can apply (3.1) for $D' = D + M^2$. Then we have
$|\Sigma_2'| = |\Sigma(D)| + |\Sigma(D + M^2)| = 3 + 1 = 4$.
Since $|\Sigma'(V)| = 2$, it follows that
$|\Sigma'(D)| = |\Sigma_1'| + |\Sigma_2'| = 2 + 4 = 6$.

§4 The case where $P = M^4$

In this section, we assume that $P = M^4$.

(4.1) Proposition If K is a finite field, then we have $|\Sigma(D)| < \infty$.

Proof. Let
$X = \{I_0, I_{0,1}^{\alpha_1,\alpha_2}, I_{0,2}^{\alpha_1}, I_{0,3}, I_{0,1,2}^{\alpha_1,\alpha_2}, I_{0,1,3}^{\alpha_1}, I_{0,2,3}, I_{0,1,2,3} \mid \text{each } \alpha_i \in \mathcal{K}\}$.
Then X is a finite set. Let $*$ be a star operation on D. Since V is a divisorial fractional ideal of D, we have $D \subset I^* \subset V$ for each $I \in X$. If we set $I^* = g_*(I)$, each element $* \in \Sigma(D)$ gives rise to an element $g_* \in X^X$, where X^X is the set of mappings from X to X. And the mapping $g : * \longmapsto g_*$ from $\Sigma(D)$ to X^X is an injection.

(4.2) Let $\alpha_1, \alpha_2, \beta_1, \beta_2 \in \mathcal{K}$.
$I_{0,1}^{\alpha_1,\alpha_2} \subset I_{0,1}^{\beta_1,\beta_2}$ if and only if $\alpha_1 = \beta_1, \alpha_2 = \beta_2$.
$I_{0,2}^{\alpha_1} \subset I_{0,2}^{\beta_1}$ if and only if $\alpha_1 = \beta_1$.
$I_{0,1,2}^{\alpha_1,\alpha_2} \subset I_{0,1,2}^{\beta_1,\beta_2}$ if and only if $\alpha_1 = \beta_1, \alpha_2 = \beta_2$.
$I_{0,1,3}^{\alpha_1} \subset I_{0,1,3}^{\beta_1}$ if and only if $\alpha_1 = \beta_1$.

Proof. For instance, assume that $I_{0,1,2}^{\alpha_1,\alpha_2} \subset I_{0,1,2}^{\beta_1,\beta_2}$. Then we have
$\pi + \alpha_1\pi^3 = (\pi + \beta_1\pi^3) + (\pi^2 + \beta_3\pi^3)p_1 + p_2$
for some $p_i \in P$. Hence $\alpha_1 = \beta_1$. We have $\pi^2 + \alpha_2\pi^3 = (\pi^2 + \beta_2\pi^3) + p_3$ for some $p_3 \in P$. Hence $\alpha_2 = \beta_2$.

(4.3) (1) $I_{0,1}^{\alpha_1,\alpha_2} \sim I_{0,1}^{\beta_1,\beta_2}$ if and only if $\beta_1^2 - \beta_2 \equiv \alpha_1^2 - \alpha_2 \pmod P$.
(2) $I_{0,2}^{\alpha} \sim I_{0,2}^{\beta}$ if and only if $\alpha = \beta$.
(3) $I_{0,1,2}^{\alpha_1,\alpha_2} \sim I_{0,1,2}^{\beta_1,\beta_2}$ for each $\alpha_1, \alpha_2, \beta_1, \beta_2 \in \mathcal{K}$.
(4) $I_{0,1,3}^{\alpha} \sim I_{0,1,3}^{\beta}$ for each $\alpha, \beta \in \mathcal{K}$.

Proof. (1) Assume that $\beta_1^2 - \beta_2 \equiv \alpha_1^2 - \alpha_2 \pmod P$. Set $1 + (\pi + \beta_1\pi^2 + \beta_2\pi^3)(\beta_1 - \alpha_1) = x$. Then we have $xI_{0,1}^{\alpha_1,\alpha_2} = I_{0,1}^{\beta_1,\beta_2}$.
(3) Set $1 - \alpha_2\pi - \alpha_1\pi^2 = x$. Then we have $xI_{0,1,2}^{\alpha_1,\alpha_2} = I_{0,1,2}^{0,0}$.
(4) Set $1 - \alpha\pi = x$. Then we have $xI_{0,1,3}^{\alpha} = I_{0,1,3}^{0}$.

(4.4) Each two in $\{I_0, I_{0,1}^{\alpha_1,\alpha_2}, I_{0,2}^{\alpha_3}, I_{0,3}, I_{0,1,2}^{\alpha_4,\alpha_5}, I_{0,1,3}^{\alpha_6}, I_{0,2,3}, I_{0,1,2,3}\}$ are not similar for each $\alpha_i \in \mathcal{K}$.

Proof. For instance, suppose that there exists $x \in \mathrm{q}(D)$ so that $I_{0,1}^{\alpha_1,\alpha_2} = xI_{0,1,3}^{\alpha_6}$. We may assume that $x = 1 + \beta_1\pi + \beta_2\pi^2 + \beta_3\pi^3 + p$ for some $\beta_i \in \mathcal{K}$ and $p \in P$. Then $x\pi^3 \in I_{0,1}^{\alpha_1,\alpha_2}$ implies a contradiction.

(4.5) Let $x \in \mathrm{q}(D) - \{0\}$.
(1) $xI_{0,1}^{\alpha_1,\alpha_2} \subset I_0$ implies $xV \subset I_0$.
(2) $xI_{0,1,2}^{\alpha_1,\alpha_2} \subset I_0$ implies $xV \subset I_0$.

(3) $xI_{0,1,3}^{\alpha_1} \subset I_0$ implies $xV \subset I_0$.
(4) $xI_{0,1}^{\alpha_{(1)},\alpha_{(2)}} \subset I_{0,1}^{\alpha_1,\alpha_2}$ with $\alpha_{(1)}^2 - \alpha_{(2)} \not\equiv \alpha_1^2 - \alpha_2 \pmod{P}$ implies $xV \subset I_{0,1}^{\alpha_1,\alpha_2}$.
(5) $xI_{0,1}^{\alpha_1,\alpha_2} \subset I_{0,2}^{\beta}$ implies $xV \subset I_{0,2}^{\beta}$.
(6) $xI_{0,1}^{\alpha_1,\alpha_2} \subset I_{0,3}$ implies $xV \subset I_{0,3}$.
(7) $xI_{0,1}^{\alpha_1,\alpha_2} \subset I_{0,2,3}$ implies $xV \subset I_{0,2,3}$.
(8) $xI_{0,1,2}^{\alpha_1,\alpha_2} \subset I_{0,1}^{\beta_1,\beta_2}$ implies $xV \subset I_{0,1}^{\beta_1,\beta_2}$.
(9) $xI_{0,1,3}^{\alpha_1} \subset I_{0,1}^{\beta_1,\beta_2}$ implies $xV \subset I_{0,1}^{\beta_1,\beta_2}$.
(10) $xI_{0,1,2}^{\alpha_1,\alpha_2} \subset I_{0,2}^{\beta}$ implies $xV \subset I_{0,2}^{\beta}$.
(11) $xI_{0,1,3}^{\alpha} \subset I_{0,2}^{\beta}$ implies $xV \subset I_{0,2}^{\beta}$.
(12) $xI_{0,1,2}^{\alpha_1,\alpha_2} \subset I_{0,3}$ implies $xV \subset I_{0,3}$.
(13) $xI_{0,1,3}^{\alpha} \subset I_{0,3}$ implies $xV \subset I_{0,3}$.
(14) $xI_{0,1,2}^{\alpha_1,\alpha_2} \subset I_{0,2,3}$ implies $xV \subset I_{0,2,3}$.
(15) $xI_{0,1,3}^{\alpha} \subset I_{0,2,3}$ implies $xV \subset I_{0,2,3}$.

Proof. (4) We may assume that $v(x) = 0$. Then we may assume that $x = 1 + (\pi+\alpha_1\pi^2+\alpha_2\pi^3)\alpha+p$ for some $\alpha \in \mathcal{K}$ and $p \in P$. Then $x(\pi+\alpha_{(1)}\pi^2+\alpha_{(2)}\pi^3) \in I_{0,1}^{\alpha_1,\alpha_2}$ implies $\alpha_{(1)}^2 - \alpha_{(2)} \equiv \alpha_1^2 - \alpha_2 \pmod{P}$; a contradiction. It follows that $x \in M^4$, and hence $xV \subset I_{0,1}^{\alpha_1,\alpha_2}$.

(4.6) Fix a data $< 0, 1; \alpha_{(1)}, \alpha_{(2)} >$, and set $(I_{0,1}^{\alpha_{(1)},\alpha_{(2)}})^* = V$. For $I_{0,1}^{\alpha_1,\alpha_2}$ with $\alpha_1^2 - \alpha_2 \not\equiv \alpha_{(1)}^2 - \alpha_{(2)} \pmod{P}$, set $(I_{0,1}^{\alpha_1,\alpha_2})^* = I_{0,1}^{\alpha_1,\alpha_2}$. For each $\beta_1, \beta_2 \in \mathcal{K}$, set $(I_{0,2}^{\beta_1})^* = I_{0,2}^{\beta_1}, (I_{0,1,2}^{\beta_1,\beta_2})^* = V, (I_{0,1,3}^{\beta_1})^* = V$, and set $(I_{0,3})^* = I_{0,3}, (I_{0,2,3})^* = I_{0,2,3}$.
Then we have
(1) There is determined a unique mapping $*$ from $\mathrm{F}(D)$ to $\mathrm{F}(D)$.
(2) For each $x \in \mathrm{q}(D) - \{0\}$, we have $(x)^* = (x)$.
(3) For each $x \in \mathrm{q}(D) - \{0\}$ and each $I \in \mathrm{F}(D)$, we have $(xI)^* = xI^*$.
(4) For each $I \in \mathrm{F}(D)$, we have $I \subset I^*$.
(5) For each $I \in \mathrm{F}(D)$, we have $(I^*)^* = I^*$.
(6) For each $I_1, I_2 \in \mathrm{F}(D)$ with $I_1 \subset I_2$, we have $I_1^* \subset I_2^*$.

(6) follows from (4.5).

(4.7) Proposition Assume that $P = M^4$ and K is an infinite field. Then we have $|\Sigma(D)| = \infty$ and $|\Sigma'(D)| = \infty$.

Proof. Let $*_{\alpha_{(1)},\alpha_{(2)}}$ be the star operation on D determined in (4.6). If $I_{0,1}^{\alpha_1,\alpha_2} \not\sim I_{0,1}^{\beta_1,\beta_2}$, we have $*_{\alpha_1,\alpha_2} \neq *_{\beta_1,\beta_2}$. It follows that $|\Sigma(D)| = \infty$.

§5 The case where $P = M^n$ with $n \geq 5$

(5.1) Proposition Assume that $P = M^5$ and K is a finite field. Then we have $|\Sigma(D)| < \infty$.

Proof. Let
$X = \{I_0, I_{0,1}^{\alpha_1,\alpha_2,\alpha_3}, I_{0,2}^{\alpha_1,\alpha_2}, I_{0,3}^{\alpha_1}, I_{0,4}, I_{0,1,2}^{\alpha_1,\alpha_2,\alpha_3,\alpha_4}, I_{0,1,3}^{\alpha_1,\alpha_2,\alpha_3}, I_{0,1,4}^{\alpha_1,\alpha_2}, I_{0,2,3}^{\alpha_1,\alpha_2}, I_{0,2,4}^{\alpha_1}, I_{0,3,4}, I_{0,1,2,3}^{\alpha_1,\alpha_2,\alpha_3}, I_{0,1,2,4}^{\alpha_1,\alpha_2}, I_{0,1,3,4}^{\alpha_1}, I_{0,2,3,4}, I_{0,1,2,3,4} \mid \text{each } \alpha_i \in \mathcal{K}\}$.
The similar argument to the proof of (4.1) completes the proof.

(5.2) Assume that $P = M^5$. Let $\alpha_i, \beta_j \in \mathcal{K}$ for each i, j.
$I_{0,1}^{\alpha_1,\alpha_2,\alpha_3} \subset I_{0,1}^{\beta_1,\beta_2,\beta_3}$ if and only if $\alpha_i = \beta_i$ for each i.
$I_{0,2}^{\alpha_1,\alpha_2} \subset I_{0,2}^{\beta_1,\beta_2}$ if and only if $\alpha_i = \beta_i$ for each i.
$I_{0,3}^{\alpha_1} \subset I_{0,3}^{\beta_1}$ if and only if $\alpha_1 = \beta_1$.
$I_{0,1,2}^{\alpha_1,\alpha_2,\alpha_3,\alpha_4} \subset I_{0,1,2}^{\beta_1,\beta_2,\beta_3,\beta_4}$ if and only if $\alpha_i = \beta_i$ for each i.
$I_{0,1,3}^{\alpha_1,\alpha_2,\alpha_3} \subset I_{0,1,3}^{\beta_1,\beta_2,\beta_3}$ if and only if $\alpha_i = \beta_i$ for each i.
$I_{0,1,4}^{\alpha_1,\alpha_2} \subset I_{0,1,4}^{\beta_1,\beta_2}$ if and only if $\alpha_i = \beta_i$ for each i.
$I_{0,2,3}^{\alpha_1,\alpha_2} \subset I_{0,2,3}^{\beta_1,\beta_2}$ if and only if $\alpha_i = \beta_i$ for each i.
$I_{0,2,4}^{\alpha_1} \subset I_{0,2,4}^{\beta_1}$ if and only if $\alpha_1 = \beta_1$.
$I_{0,1,2,3}^{\alpha_1,\alpha_2,\alpha_3} \subset I_{0,1,2,3}^{\beta_1,\beta_2,\beta_3}$ if and only if $\alpha_i = \beta_i$ for each i.
$I_{0,1,2,4}^{\alpha_1,\alpha_2} \subset I_{0,1,2,4}^{\beta_1,\beta_2}$ if and only if $\alpha_i = \beta_i$ for each i.
$I_{0,1,3,4}^{\alpha_1} \subset I_{0,1,3,4}^{\beta_1}$ if and only if $\alpha_1 = \beta_1$.

Proof. For instance, assume that $I_{0,1,2}^{\alpha_1,\alpha_2,\alpha_3,\alpha_4} \subset I_{0,1,2}^{\beta_1,\beta_2,\beta_3,\beta_4}$. Then we have
$\pi + \alpha_1\pi^3 + \alpha_2\pi^4 = (\pi + \beta_1\pi^3 + \beta_2\pi^4) + (\pi^2 + \beta_3\pi^3 + \beta_4\pi^4)p_1 + p_2$
for some $p_i \in P$. Hence $\alpha_1 = \beta_1$ and $\alpha_2 = \beta_2$. We have
$\pi^2 + \alpha_3\pi^3 + \alpha_4\pi^4 = (\pi^2 + \beta_3\pi^3 + \beta_4\pi^4) + p_3$
for some $p_3 \in P$. Hence $\alpha_3 = \beta_3$ and $\alpha_4 = \beta_4$.

(5.3) Assume that $P = M^5$. Then each two in $\{I_0, I_{0,1}^{\alpha_1,\alpha_2,\alpha_3}, I_{0,2}^{\alpha_4,\alpha_5}, I_{0,3}^{\alpha_6}, I_{0,4}, I_{0,1,2}^{\alpha_7,\alpha_8,\alpha_9,\alpha_{10}}, I_{0,1,3}^{\alpha_{11},\alpha_{12},\alpha_{13}}, I_{0,1,4}^{\alpha_{14},\alpha_{15}}, I_{0,2,3}^{\alpha_{16},\alpha_{17}}, I_{0,2,4}^{\alpha_{18}}, I_{0,3,4}, I_{0,1,2,3}^{\alpha_{19},\alpha_{20},\alpha_{21}}, I_{0,1,2,4}^{\alpha_{22},\alpha_{23}}, I_{0,1,3,4}^{\alpha_{24}}, I_{0,2,3,4}, I_{0,1,2,3,4}\}$ are not similar for each $\alpha_i \in \mathcal{K}$.

Proof. Because the each two have distinct types.

(5.4) Proposition Assume that $P = M^n$ with $n \geq 6$ and K is a finite field. Then we have $|\Sigma(D)| < \infty$.

The proof is similar to that of (5.1).

(5.5) Proposition Assume that $P = M^n$ with $n \geq 5$ and K is an infinite field. Then we have $|\Sigma'(D)| = \infty$.

Proof. Set $D + M^4 = D'$. Then D' is an APVD with maximal ideal M^4. D' is not a PVD. We have $|\Sigma(D')| = \infty$ by (4.7). Hence $|\Sigma'(D')| = \infty$. Since the descent map δ from $\Sigma'(D')$ to $\Sigma'(D)$ is an injection, we have $|\Sigma'(D)| = \infty$.

The proof of our Theorem is complete.

§6 Examples

In this final section, D does not necessarily denote an APVD which is not a PVD. We will apply our Theorem to some APVD's.

(6.1) Proposition Let V be a rank one discrete valuation domain of the form $K+M$, where K is a field and M is the maximal ideal of V, and let $D=K+M^n$ for a positive integer $n\geq 2$. Then we have

(1) If K is a finite field, then $|\Sigma(D)|<\infty$.

(2) If $n=2$, then $|\Sigma(D)|=1$ and $|\Sigma'(D)|=3$, and if $n=3$, then $|\Sigma(D)|=3$ and $|\Sigma'(D)|=6$.

(3) If $n=4$ and K is an infinite field, then $|\Sigma(D)|=\infty$ and $|\Sigma'(D)|=\infty$.

(4) If $n\geq 5$ and K is an infinite field, then $|\Sigma'(D)|=\infty$.

Proof. Then D is an APVD which is not a PVD, $P=M^n$ is the maximal ideal of D, $V=(P:P)$, and $(D+M)/M=V/M$.

(6.2) Let $V=K[[X]]$ be the formal power series ring of a variable X over a field K, let M be the maximal ideal of V, and let $D=K+M^n$ for a positive integer $n\geq 2$. Then we have the same (1) $\sim$ (4) of (6.1).

(6.3) Let V be a rank one discrete valuation domain of the form $K+M$, where K is a field and M is the maximal ideal of V, and let $D=k+M^n$ for a positive integer $n\geq 2$ and for a subfield k of K. Then, if $n\geq 4$ and K is an infinite field, then $|\Sigma'(D)|=\infty$.

Proof. Set $K+M^n=D'$. Then $|\Sigma'(D')|=\infty$ by (6.1). Since the descent map δ from $\Sigma'(D')$ to $\Sigma'(D)$ is an injection, we have $|\Sigma'(D)|=\infty$.

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