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Kronecker function rings of semistar operations on semigroups, II

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Abstract

We study the Kronecker function ring of any semistar operation on a grading monoid.

Introduction

We know that various terms in ideal theory are defined analogously for commutative semigroups; those are ideal, integral, divisor, dimension, valuation, star operation, etc. Let G be a torsion-free abelian (additive) group and let S be its subsemigroup containing the zero element. Then S is called a *grading monoid* (or a *g-monoid*). A motivation and an outline of ideas for ideal theory of a grading monoid are as follows: Almost all of ideal theory of a commutative ring R concern properties of ideals of R with respect to the multiplication on R. Abondoning the additon on R we will extract the multiplication on R. Then we have an idea of an algebraic system S of a semigroup which is called a grading monoid.

We already have the Kronecker function ring theory of an e.a.b. semistar operation on a g-monoid ([M2]). In 2001, M. Fontana and K. Loper [FL] outlined a general approach to the theory of Kronecker function rings of an integral domain by semistar operations. In this paper, after them, we will define a Kronecker function ring $Kr(S, *)$ of any semistar operation $*$ on a g-monoid S and will study it. We refer to [G], [GP1, 2] and [M4] for the general theory of a commutative semigroup ring, and [M3] for the general theory of a grading monoid.

1. Preliminary results on semistar operations

Let S be a g-monoid with quotient group G. Let E be a non-empty subset of G such that $S+E \subset E$ with $s+E \subset S$ for some $s \in S$. Then E is called a fractional ideal of S. We denote the set of all fractional ideals of S by $\mathrm{F}(S)$. A non-empty subset E of G is called an S-submodule of G if $S+E \subset E$. We denote the set of all S-submodules

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of G by $\bar{\mathrm{F}}(S)$. The set of all finitely generated members in $\mathrm{F}(S)$ is denoted by $\mathrm{f}(S)$.

Definition (1.1) ([OMS]) A map $* : \bar{\mathrm{F}}(S) \longrightarrow \bar{\mathrm{F}}(S), E \longmapsto E^*$, is called a semistar operation on S if, for all $x \in G$, and for all $E, F \in \bar{\mathrm{F}}(S)$, the following conditions hold:

(1) $(x+E)^* = x+E^*$; (2) $E \subset E^*$; (3) $E \subset F$ implies $E^* \subset F^*$; (4) $(E^*)^* = E^*$.

We denote the set of all the semistar operations on S by $\mathrm{SStar}(S)$.

Let $E, F \in \bar{\mathrm{F}}(S)$. Then we denote the set $\{x \in G \mid x + F \subset E\}$ by $(E : F)$.

Lemma (1.2) Let $*$ be a semistar operation on S, and let $E, F \in \bar{\mathrm{F}}(S)$. Then we have $(E : F)^* \subset (E^* : F^*) = (E^* : F)$.

Proof. Since $(E : F) + F \subset E$, we have $(E : F)^* + F^* \subset E^*$. Hence $(E : F)^* \subset (E^* : F^*)$.

Let $\mathcal{S} = \{S_\lambda \mid \lambda \in \Lambda\}$ be a family of oversemigroups of S. Then the semistar operation $E \longmapsto \cap_\lambda (E + S_\lambda)$ on S is denoted by $*_{\mathcal{S}}$.

A mapping $E \longmapsto E^*$ of $\mathrm{F}(S)$ to $\mathrm{F}(S)$ is called a star operation on S if the following conditions hold for all $x \in G$ and for all $E, F \in \mathrm{F}(S)$ ([M1]):

(1) $(x)^* = (x)$; (2) $(x + E)^* = x + E^*$; (3) $E \subset E^*$; (4) If $E \subset F$, then $E^* \subset F^*$; (5) $(E^*)^* = E^*$.

Let $*$ be a star operation on S. If we set $E^{*_e} = E^*$ for all $E \in \mathrm{F}(S)$, and $E^{*_e} = G$ for all $E \in \bar{\mathrm{F}}(S) - F(S)$, then $*_e$ is a semistar operation on S and is called *the trivial extension of $*$ to a semistar operation.*

Let $*$ be a semistar operation on S. For each $E \in \bar{\mathrm{F}}(S)$, set $E^{*_f} = \cup\{F^* \mid F \in \mathrm{f}(S)$ with $F \subset E\}$. Then $*_f$ is a semistar operation on S, and is called *the finite semistar operation associated to $*$*. A semistar operation $*$ is said to be *of finite type* if $* = *_f$. Since $(*_f)_f = *_f$, $*_f$ is of finite type.

For any subset E of G, the subset $(S : E)$ is also denoted by E^{-1} (We set $\emptyset^{-1} = G$). The mapping $E \longmapsto E^v = (E^{-1})^{-1}$ of $\bar{\mathrm{F}}(S)$ to $\bar{\mathrm{F}}(S)$ is a semistar operation on S and is called *the v-semistar operation* on S.

The finite semistar operation associated to the v-semistar operation is called *the t-semistar operation* on S.

If $\mathcal{W}$ is a family of valuation oversemigroups of S, then $*_{\mathcal{W}}$ is called a *w-semistar operation (associated to $\mathcal{W}$).* If $\mathcal{W}$ is the family of all the valuation oversemigroups of S, then $*_{\mathcal{W}}$ is called *the b-semistar operation* on S.

Let $*_1, *_2$ be semistar operations on S. If $(*_1)_f = (*_2)_f$, then $*_1$ and $*_2$ are said to be equivalent, and is denoted by $*_1 \sim *_2$. By definition, $*_1$ and $*_2$ are equivalent if and only if $E^{*_1} = E^{*_2}$ for each $E \in \mathrm{f}(S)$.

Definition (1.3). A semistar operation $*$ on S is said to be e.a.b. (endlich arithmetisch brauchbar) if, for all $A, B, C \in \mathrm{f}(S), (A + B)^* \subset (A + C)^*$ implies $B^* \subset C^*$, and is said to be a.b. (arithmetisch brauchbar) if, for all $A \in \mathrm{f}(S)$ and for all $B, C \in \bar{\mathrm{F}}(S), (A + B)^* \subset (A + C)^*$ implies $B^* \subset C^*$.

Lemma (1.4). Let $*$ be a semistar operation on S. Then the following conditions are equivalent:

(1) $*$ is e.a.b.

(2) Let $A, B \in \mathrm{f}(S)$ such that $A^* \subset (A+B)^*$. Then $0 \in B^*$.

(3) Let $A, B \in \mathrm{f}(S)$. Then $((A+B)^* : A) \subset B^*$.

(4) Let $A, B, C \in \mathrm{f}(S)$ such that $(A+B)^* = (A+C)^*$. Then $B^* = C^*$.

Proof. (1) $\Longrightarrow$ (2): Then $(A+S)^* \subset (A+B)^*$, hence $S^* \subset B^*$.

(2) $\Longrightarrow$ (3): Let $x \in ((A+B)^* : A)$, then $x + A \subset (A+B)^*$. Then we have $A \subset (A+B-x)^*$. Hence $0 \in (B-x)^*$, and hence $x \in B^*$.

(3) $\Longrightarrow$ (4): Then $A + B \subset (A+C)^*$, hence $B \subset ((A+C)^* : A)$. It follows that $B^* \subset C^*$. Similarly, we have $C^* \subset B^*$.

(4) $\Longrightarrow$ (1): If $(A+B)^* \subset (A+C)^*$, then

$(A+C)^* = ((A+B)^*, (A+C)^*)^* = (A+B, A+C)^* = (A+(B,C))^*$.

Therefore, $C^* = (B,C)^*$, thus $B^* \subset C^*$.

Proposition (1.5) Let T be an oversemigroup of S, and let $*$ be a semistar operation on S. Then we define $\alpha_T(*) : \bar{\mathrm{F}}(T) \longrightarrow \bar{\mathrm{F}}(T)$ by setting:

$E^{\alpha_T(*)} = E^*$ for each $E \in \bar{\mathrm{F}}(T) \subset \bar{\mathrm{F}}(S)$.

(1) $\alpha_T(*)$ is a semistar operation on T.

(2) If $*$ is of finite type on S, then $\alpha_T(*)$ is of finite type on T.

(3) If we set $*' = \alpha_{S^*}(*)$, then $*'|_{\mathrm{F}(S^*)}$, the restriction of $*'$ to $\mathrm{F}(S^*)$, is a star operation on S^*.

(4) If $*$ is an e.a.b. (respectively, a.b.) semistar operation on S, then $*'$ is an e.a.b. (respectively, a.b.) semistar operation on S^*.

Proof. (1), (2) and (3) are easily shown.

(4) Let $E, F, G \in \mathrm{f}(S^*)$ such that $(E+F)^{*'} \subset (E+G)^{*'}$. Note that $E = E_0 + S^*, F = F_0 + S^*, G = G_0 + S^*$, for some $E_0, F_0, G_0 \in \mathrm{f}(S)$. Then,

$(E_0 + F_0)^* = (E_0 + S + F_0 + S)^* = (E+F)^* = (E+F)^{*'}$

$\subset (E+G)^{*'} = (E+G)^* = (E_0 + S^* + G_0 + S^*)^* = (E_0 + G_0)^*$.

Since $*$ is e.a.b, we deduce that $F_0^* \subset G_0^*$, and hence $F^{*'} \subset G^{*'}$. Similar argument shows the a.b. statement.

Proposition (1.6) Let T be an oversemigroup of S, and let $*$ be a semistar operation on T. We define $\delta_S(*) : \bar{\mathrm{F}}(S) \to \bar{\mathrm{F}}(S)$ by setting:

$E^{\delta_S(*)} = (E+T)^*$ for all $E \in \bar{\mathrm{F}}(S)$.

(1) $\delta_S(*)$ is a semistar operation on S.

(2) If $*$ is an e.a.b. (respectively, a.b.) semistar operation on T, then $\delta_S(*)$ is an e.a.b. (respectively, a.b.) semistar operation on S.

Proof. (1) is straightforward.

(2) Let $E \in \mathrm{f}(S)$ and let $F, G \in \mathrm{f}(S)$ (respectively, $F, G \in \bar{\mathrm{F}}(S)$) such that $(E+F)^{\delta_S(*)} \subset (E+G)^{\delta_S(*)}$. Then, $(E+T+F+T)^* \subset (E+T+G+T)^*$. The conclution follows from the hypothesis on $*$.

Let T be an oversemigroup of S. Then by Propositions (1.5) and (1.6), we have canonical maps:

α: SStar(S) $\longrightarrow$ SStar(T) and δ: SStar(T) $\longrightarrow$ SStar(S).

Proposition (1.7) Let T be an oversemigroup of S. Then

(1) $\alpha(\delta(*)) = *$ for each $* \in$ SStar(T).

(2) The following conditions are equivalent:

(i) δ is bijective.

(ii) α is bijective.

(iii) $S = T$.

Proof. (1) is straightforward.

(2) (i) $\Longrightarrow$ (iii): Since δ is surjective, there is $* \in$ SStar(T) such that $\delta_S(*)$ coincides with the d-semistar operation on S. Therefore, $S = S^d = S^{\delta_S(*)} = (S+T)^* = T^*$. It follows that $S = T^* \supset T$, and hence $S = T$.

2. Background on Kronecker function rings

Let D be an integral domain with quotient field $\mathrm{q}(D) = k$ and let S be a g-monoid with quotient group G. Then the semigroup ring of S over D is denoted by $D[X;S]$. Note that $S \subset D[X;S]$. Let $f = \sum_{i=1}^{n} a_i X^{t_i}$ be a non-zero element of $k[X;G]$, where $a_i \neq 0$ for each i and $t_i \neq t_j$ for each $i \neq j$. Then the fractional ideal $(t_1, \cdots, t_n)$ of S is called the e-content of f, and is denoted by $e_S(f)$ or simply by $e(f)$. The subset $\{t_1, t_2, \cdots, t_n\}$ of G is called the e-support of f, and is denoted by $\mathrm{Supp}_e(f)$ or simply by $\mathrm{Supp}(f)$. We refer to [M4] for semigroup rings.

Proposition (2.1)([M2, Proposition 4]) Let $*$ be an e.a.b. semistar operation on S, let k be a field, and set

$S_*^k = \{f/g \mid f, g \in k[X;S] - \{0\}$ with $e(f)^* \subset e(g)^*\} \cup \{0\}$.

(1) S_*^k is a well-defined extension domain of $k[X;S]$ with $\mathrm{q}(S_*^k) = \mathrm{q}(k[X;S])$ such that $S_*^k \cap G = S^*$.

(2) S_*^k is a Bezout domain.

(3) If $F \in \mathrm{f}(S)$, then $FS_*^k \cap G = F^*$ and $FS_*^k = F^* S_*^k$.

S_*^k is called the Kronecker function ring of S with respect to $*$ and k (or, simply the Kronecker function ring of S with respect to $*$), and is also denoted by $\mathrm{Kr}(S,*,k)$ (or, simply by $\mathrm{Kr}(S,*)$).

Proposition (2.2) Let $*$ be an e.a.b. semistar operation on S.

(1) $\mathrm{Kr}(S,*) = \mathrm{Kr}(S,*_f)$.

(2) Let $\alpha(*)$ be the ascent of $*$ to S^*. Then $\mathrm{Kr}(S,*) = \mathrm{Kr}(S^*,\alpha(*))$.

Proof. (1) is immediate from the definition.

(2) Set $T = S^*$. Suppose that $f, g \in k[X;S] - \{0\}$. Then,

$(e_T(f))^{\alpha(*)} \subset (e_T(g))^{\alpha(*)}$ iff $(e_T(f))^* \subset (e_T(g))^*$ iff $(e_S(f)+T)^* \subset (e_S(g)+T)^*$ iff $(e_S(f)+S)^* \subset (e_S(g)+S)^*$ iff $(e_S(f))^* \subset (e_S(g))^*$ iff $f/g \in \mathrm{Kr}(S,*)$.

Proposition (2.3) Let $*_1$ and $*_2$ be e.a.b. semistar operations on S.
(1) If $*_1 \leq *_2$, then $\mathrm{Kr}(S,*_1) \subset \mathrm{Kr}(S,*_2)$.
(2) $*_1 \sim *_2$ if and only if $\mathrm{Kr}(S,*_1) = \mathrm{Kr}(S,*_2)$.

Proof. (1) is immediate from the definition.
(2) The sufficiency: For each $F \in \mathrm{f}(S)$, we have
$F^{*_1} = F\mathrm{Kr}(S,*_1) \cap G = F\mathrm{Kr}(S,*_2) \cap G = F^{*_2}$.

Proposition (2.4) Let $\mathcal{S} = \{S_\lambda \mid \lambda \in \Lambda\}$ be a family of oversemigroups of S, and let $*_{\mathcal{S}}$ be the semistar operation associated to $\mathcal{S}$.

(1) If $*_{\{S_\lambda\}}$ is an e.a.b. (respectively, an a.b.) semistar operation for each λ, then $*_{\mathcal{S}}$ is an e.a.b. (respectively, an a.b.) semistar operation on S.

(2) If $*_{\{S_\lambda\}}$ is e.a.b. for each λ, then $\mathrm{Kr}(S,*_{\mathcal{S}}) = \cap_\lambda \mathrm{Kr}(S,*_{\{S_\lambda\}})$.

Proof. (1) Let $E, F, G \in \mathrm{f}(S)$ such that $(E+F)^{*_{\mathcal{S}}} \subset (E+G)^{*_{\mathcal{S}}}$. Then $E+F+S_\lambda \subset E+G+S_\lambda$ for each λ. Since $*_{\{S_\lambda\}}$ is e.a.b., we have $F+S_\lambda \subset G+S_\lambda$. Then $F^{*_{\mathcal{S}}} = \cap_\lambda(F+S_\lambda) \subset \cap_\lambda(G+S_\lambda) = G^{*_{\mathcal{S}}}$.
The proof for the a.b. statement is similar.
(2) For $f, g \in k[X;S] - \{0\}$, $e(f)^{*_{\mathcal{S}}} \subset e(g)^{*_{\mathcal{S}}}$ iff $e(f)^{*_{\{S_\lambda\}}} \subset e(g)^{*_{\{S_\lambda\}}}$ for each λ.

Proposition (2.5) Let $\mathcal{W} = \{V_\lambda \mid \lambda \in \Lambda\}$ be a family of valuation oversemigroups of S, and let W_λ be the trivial valuation extension ring of V_λ to $\mathrm{q}(k[X;S])$. Then the w-semistar operation $*_{\mathcal{W}}$ is a.b. on S, and $\mathrm{Kr}(S,*_{\mathcal{W}}) = \cap_\lambda \mathrm{Kr}(S,*_{\{V_\lambda\}}) = \cap_\lambda W_\lambda$.

Proof. Easy consequence of (2.4) (2).

3. Some semistar operations associated to an semistar operation

Definition (3.1) Let $*$ be a semistar operation on S. An element $x \in G$ is called $*$-integral over S if $x \in (F^* : F^*)$ for some $F \in \mathrm{f}(S)$. The set $S^{[*]} = \cup\{(F^* : F^*) \mid F \in \mathrm{f}(S)\}$ is called *the semistar integral closure of* S *with respect to* $*$ or, simply *the* $*$-*integral closure* of S. If $S = S^{[*]}$, then S is called $*$-*integrally closed*.

Lemma (3.2) $S^{[*]} = \cup\{(F^* : F^*)^{*_f} \mid F \in \mathrm{f}(S)\}$.

Proof. Let $x \in (F^* : F^*)^{*_f}$. There is $H \in \mathrm{f}(S)$ with $H \subset (F^* : F^*)$ such that $x \in H^*$. Since $H^* + F^* \subset F^*$, we have $x + F^* \subset F^*$, and $x \in (F^* : F^*)$. Therefore, $(F^* : F^*)^{*_f} = (F^* : F^*)$.

Proposition (3.3) Let $*$ be a semistar operation on S. We define an operation $[*]$ on S by setting:
$H^{[*]} = \cup\{((F^* : F^*) + H)^{*_f} \mid F \in \mathrm{f}(S)\}$, for each $H \in \mathrm{f}(S)$,
$E^{[*]} = \cup\{H^{[*]} \mid H \in \mathrm{f}(S) \text{ with } H \subset E\}$, for each $E \in \bar{\mathrm{F}}(S)$.

Then the operation $[*]$ is a semistar operation of finite type on S.

Proof. (i) To prove $x+E^{[*]}=(x+E)^{[*]}$, it suffices to show that $x+E^{[*]}\subset (x+E)^{[*]}$. Let $a\in x+E^{[*]}$. We have $a-x\in F_1^{[*]}$ for some $F_1\in \mathrm{f}(S)$ with $F_1\subset E$. Then $a-x\in((F_2^*:F_2^*)+F_1)^{*_f}$ for some $F_2\in\mathrm{f}(S)$. Then $a\in((F_2^*:F_2^*)+x+F_1)^{*_f}\subset (x+F_1)^{[*]}\subset(x+E)^{[*]}$.

(ii) Let $x\in E$, and set $(x)=H$. Then we have $x\in(H^*:H^*)+H\subset H^{[*]}\subset E^{[*]}$. Hence $E\subset E^{[*]}$.

(iii) Assume that $E_1\subset E_2$. By the definition, we have $E_1^{[*]}\subset E_2^{[*]}$.

(iv) Let $y\in(E^{[*]})^{[*]}$. We have $y\in F^{[*]}$ for some $F\in\mathrm{f}(S)$ with $F\subset E^{[*]}$. Since F is finitely generated, we have $F\subset H^{[*]}$ for some $H\in\mathrm{f}(S)$ with $H\subset E$.

If $F=(x_1,\cdots,x_n)$, we have, for each i, $x_i\in((F_i^*:F_i^*)+H)^{*_f}$ for some $F_i\in\mathrm{f}(S)$. Then $F\subset((F_1^*:F_1^*)+H,\cdots,(F_n^*:F_n^*)+H)^{*_f}$. Let $G_1=\sum_i F_i$, then $F\subset((G_1^*:G_1^*)+H)^{*_f}$.

On the other hand, $y\in((G_2^*:G_2^*)+F)^{*_f}$ for some $G_2\in\mathrm{f}(S)$. Then,

$y\in((G_2^*:G_2^*)+((G_1^*:G_1^*)+H)^{*_f})^{*_f}=((G_2^*:G_2^*)+(G_1^*:G_1^*)+H)^{*_f}\subset(((G_1+G_2)^*:(G_1+G_2)^*)+H)^{*_f}\subset H^{[*]}\subset E^{[*]}$. Therefore, $(E^{[*]})^{[*]}=E^{[*]}$.

Proposition (3.4) Let $*$ be a semistar operation on S. Then we have

(1) $S^{[*]}$ is an oversemigroup of S.

(2) $S^{[*]}$ is integrally closed.

Proof. (1) Let $a,b\in S^{[*]}$. Then $a\in(F_1^*:F_2^*)$ and $b\in(F_2^*:F_2^*)$ for some $F_1,F_2\in\mathrm{f}(S)$. Then $a+b\in((F_1+F_2)^*:(F_1+F_2)^*)\in S^{[*]}$.

(2) Let $x\in G$ be integral over $S^{[*]}$. We have $nx=a\in S^{[*]}$ for some positive integer n. Since $a\in S^{[*]}$, we have $a\in(F^*:F^*)$ for some $F\in\mathrm{f}(S)$. Set $H=(F,F+x,\cdots,F+(n-1)x)$. Then $x+H\subset H^*$. It follows that $x\in(H^*:H^*)\subset S^{[*]}$.

Definition (3.5) Let $*$ be a semistar operation on S. We define the map $*_a:\bar{\mathrm{F}}(S)\longrightarrow\bar{\mathrm{F}}(S)$ by setting

$F^{*_a}=\cup\{((F+H)^*:H^*)\mid H\in\mathrm{f}(S)\}$, for each $F\in\mathrm{f}(S)$,

$E^{*_a}=\cup\{F^{*_a}\mid F\in\mathrm{f}(S)$ with $F\subset E\}$, for each $E\in\bar{\mathrm{F}}(S)$.

Proposition (3.6) Let $*$ be a semistar operation on S.

(1) $*_a$ is a semistar operation of finite type.

(2) $*_a$ is e.a.b.

(3) $*_f\le[*]\le *_a$.

(4) $[*]=[*_f]=[*]_f$.

(5) $*_a=(*_f)_a=(*_a)_f$.

(6) $*_a=*_f$ if and only if $*_f$ is an e.a.b. semistar operation.

(7) $*_1\le *_2$ implies $(*_1)_a\le(*_2)_a$.

(8) $*_1\le *_2$ implies $[*_1]\le[*_2]$.

(9) $(*_a)_a=*_a$.

(10) $[*]_a=[*_a]=*_a$.

(11) $*_f\le[*]\le[[*]]\le *_a$.

(12) $S^{[*]}=S^{*_a}=S^{[[*]]}$.

Proof. (1) (i) To prove $(x+E)^{*_a} = x+E^{*_a}$, it suffices to show that $x+E^{*_a} \subset (x+E)^{*_a}$. Let $y \in x+E^{*_a}$. Since $y-x \in E^{*_a}$, there is $F \in \mathrm{f}(S)$ with $F \subset E$ such that $y-x \in F^{*_a}$. Hence $y-x \in ((F+H)^* : H^*)$ for some $H \in \mathrm{f}(S)$. Then $y \in ((x+F+H)^* : H^*) \subset (x+F)^{*_a} \subset (x+E)^{*_a}$. It follows that $x+E^{*_a} \subset (x+E)^{*_a}$.

(ii) Let $x \in E$, and set $(x) = H$. Then $x \in ((H+H)^* : H^*) \subset H^{*_a} \subset E^{*_a}$. Hence $E \subset E^{*_a}$.

(iii) Let $E_1 \subset E_2$. By definition, we have $E_1^{*_a} \subset E_2^{*_a}$.

(iv) Let $y \in (E^{*_a})^{*_a}$. Then $y \in F^{*_a}$ for some $F \in \mathrm{f}(S)$ with $F \subset E^{*_a}$. Since F is finitely generated, we have $F \subset H^{*_a}$ for some $H \in \mathrm{f}(S)$ with $H \subset E$. If $F = (x_1, \cdots, x_n)$, we have, for each i, $x_i + F_i^* \subset (H+F_i)^*$ for some $F_i \in \mathrm{f}(S)$. Let $G_1 = \sum_i F_i$, then $F+G_1 \subset (H+G_1)^*$. On the other hand, $y+G_2 \subset (F+G_2)^*$ for some $G_2 \in \mathrm{f}(S)$. Then $y+G_1+G_2 \subset (F+G_1+G_2)^* \subset (H+G_1+G_2)^*$. Hence $y \in H^{*_a} \subset E^{*_a}$. Hence $(E^{*_a})^{*_a} = E^{*_a}$.

(2) Let $I, J \in \mathrm{f}(S)$, we will show that $((I+J)^{*_a} : I) \subset J^{*_a}$. Let $z \in ((I+J)^{*_a} : I)$, then $z+I \subset (I+J)^{*_a}$. If $I = (x_1, \cdots, x_n)$, we have, for each i, $z+x_i+F_i^* \subset (I+J+F_i)^*$ for some $F_i \in \mathrm{f}(S)$. Let $G = \sum_i F_i$, then $z+I+G \subset (I+J+G)^*$. It follows that $z \in ((J+I+G)^* : (I+G)^*) \subset J^{*_a}$.

(3) Let $x \in E^{*_f}$. We have $x \in H^*$ for some $H \in \mathrm{f}(S)$ with $H \subset E$. Then we have $x \in ((S^* : S^*) + H)^{*_f} \subset H^{[*]} \subset E^{[*]}$.

Hence $E^{*_f} \subset E^{[*]}$, and $*_f \leq [*]$.

Next, let $x \in E^{[*]}$. We have $x \in H^{[*]}$ for some $H \in \mathrm{f}(S)$ with $H \subset E$. There is $F \in \mathrm{f}(S)$ such that $x \in ((F^* : F^*)+H)^{*_f}$. Then $x+F \subset ((F^* : F^*)+H)^{*_f}+F^* \subset (F+H)^*$. Hence $x \in H^{*_a} \subset E^{*_a}$. Therefore $E^{[*]} \subset E^{*_a}$, and $[*] \leq *_a$.

(4), (5), (7) and (8) are obvious from the definitions.

(6) The necessity follows from (2).

The sufficiency: Let $F \in \mathrm{f}(S)$, and let $x \in F^{*_a}$. Then $x+H \subset (F+H)^*$ for some $H \in \mathrm{f}(S)$. Since $*_f$ is e.a.b., we have $x \in F^{*_f}$. Hence $F^{*_a} \subset F^{*_f}$, and $*_a \leq *_f$. The conclusion follows from (3).

(9) Since $*_a = (*_a)_f$ is e.a.b. by (2), we have $(*_a)_a = (*_a)_f$ by (6). Hence $(*_a)_a = *_a$.

(10) By (3),(7), we have $(*_f)_a \leq [*]_a \leq (*_a)_a$ (resp. $(*_a)_f \leq [*_a] \leq (*_a)_a$)). By (5),(9), we have $*_a \leq [*]_a \leq *_a$ (resp., $*_a \leq [*_a] \leq *_a$). Hence $[*]_a = *_a$ (resp. $*_a = [*_a]$).

(11) By (3),(8), we have $[*_f] \leq [[*]] \leq [*_a]$. Then (4),(10) imply the assertion.

(12) We have $S^{*_a} = \cup\{((S+H)^* : H^*) \mid H \in \mathrm{f}(S)\} = \cup\{(H^* : H^*) \mid H \in \mathrm{f}(S)\} = S^{[*]}$. Then (11) completes the proof.

4. The Kronecker function ring of any semistar operation

Lemma (4.1) (Dedekind-Mertens Lemma for semigroups)([GP1]) Let $f, g \in k[X;S] - \{0\}$. Then there is a positive integer m such that

$(m+1)e(g) + e(f) = me(g) + e(fg)$.

Lemma (4.2) Let $*$ be a semistar operation on S. Let $f, g, f', g' \in k[X;S] - \{0\}$ with $f/g = f'/g'$ such that $(e(f)+e(h))^* \subset (e(g)+e(h))^*$ for some $h \in k[X;S] - \{0\}$.

Then there is $h' \in k[X;S]-\{0\}$ such that $(e(f')+e(h'))^* \subset (e(g')+e(h'))^*$.

Proof. Then we have $fg' = f'g$. By (4.1), there is a positive integer m such that
$(m+1)e(g)+e(f') = me(g)+e(f'g)$,
$(m+1)e(f)+e(g') = me(f)+e(fg')$.
Then it follows that $\{(m+1)e(g)+e(f')\}+me(f) = \{(m+1)e(f)+e(g')\}+me(g)$.
Now, there are a finite set of elements $s_1, s_2, \cdots, s_n$ of S with
$s_i \neq s_j$ for each $i \neq j$ such that $(m+1)(e(g)+e(h))+m(e(f)+e(h)) =$
$(s_1, s_2, \cdots, s_n)$. We set $h' = X^{s_1}+X^{s_2}+\cdots+X^{s_n} \in k[X;S]-\{0\}$.
Then we have $e(h') = (m+1)(e(g)+e(h))+m(e(f)+e(h))$ and therefore
$e(f')+e(h') = \{(m+1)e(g)+e(f')+me(f)\}+(2m+1)e(h)$
$= \{(m+1)e(f)+e(g')+me(g)\}+(2m+1)e(h)$
$= (e(f)+e(h))+m(e(f)+e(h))+m(e(g)+e(h))+e(g')$
$\subset (e(g)+e(h))^*+m(e(f)+e(h))+m(e(g)+e(h))+e(g')$
$\subset (e(g')+e(h'))^*$, as wanted.

Set $\mathrm{Kr}(S,*) = \{f/g \mid f,g \in k[X;S]-\{0\}$ such that $(e(f)+e(h))^* \subset (e(g)+e(h))^*$ for some $h \in k[X;S]-\{0\}\} \cup \{0\}$. (4.2) shows that $\mathrm{Kr}(S,*)$ is a well-defined subset of $\mathrm{q}(k[X;S])$. If $*$ is e.a.b., this coincides with $\mathrm{Kr}(S,*)$ in (2.1).

Proposition (4.3) $\mathrm{Kr}(S,*)$ is an integral domain with quotient field $\mathrm{q}(k[X;S])$.

Proof. Let $f/g, f'/g \in \mathrm{Kr}(S,*)-\{0\}$. Then there are $h, h' \in k[X;S]-\{0\}$ such that
$(e(f)+e(h))^* \subset (e(g)+e(h))^*, (e(f')+e(h'))^* \subset (e(g)+e(h'))^*$. There is $j \in k[X;S]-\{0\}$ such that $e(j) = e(h)+e(h')$. Then we have
$(e(f)+e(j))^* \subset (e(g)+e(j))^*, (e(f')+e(j))^* \subset (e(g)+e(j))^*$.
We may assume that $f+f' \neq 0$. Then it follows that
$(e(f+f')+e(j))^* \subset (e(g)+e(j))^*$. Hence $f/g+f'/g \in \mathrm{Kr}(S,*)$.
Next, we have $(m+2)e(g) = me(g)+e(g^2)$ for some m. There is $j' \in k[X;S]-\{0\}$ such that $e(j') = (m+2)e(g)+2e(j)$. Then we have
$e(ff')+e(j')$
$\subset \{e(f)+e(f')\}+\{(m+2)e(g)+2e(j)\}$
$= \{e(f)+e(j)\}+\{e(f')+e(j)\}+(m+2)e(g)$
$\subset 2(e(g)+e(j))^*+(m+2)e(g)$
$= 2(e(g)+e(j))^*+\{me(g)+e(g^2)\}$
$\subset (e(g^2)+e(j'))^*$.
Therefore $(e(ff')+e(j'))^* \subset (e(g^2)+e(j'))^*$. Hence $(ff')/(gg') \in \mathrm{Kr}(S,*)$.

Proposition (4.4) $\mathrm{Kr}(S,*)$ is a Bezout domain.

Proof. Set $R = \mathrm{Kr}(S,*)$, and let $f \in k[X;S]-\{0\}$ with $\mathrm{Supp}(f) = \{s_1, \cdots, s_n\}$. Then we have $fR = (X^{s_1}, \cdots, X^{s_n})R$.

Let ξ and η be non-zero elements of R. We set $\xi = f/g$ and $\eta = f'/g$ with $f, f', g \in k[X;S]-\{0\}$, and let $\mathrm{Supp}(f) = \{s_1, \cdots, s_n\}$, let $\mathrm{Supp}(f') = \{t_1, \cdots, t_m\}$ and let $\mathrm{Supp}(f) \cup \mathrm{Supp}(f') = \{u_1, \cdots, u_l\}$ with $u_i \neq u_j$ for each $i \neq j$. Then we have

$(\xi,\eta)R = (\frac{X^{s_1}}{g},\cdots,\frac{X^{s_n}}{g},\eta)R$
$= (\frac{X^{s_1}}{g},\cdots,\frac{X^{s_n}}{g},\frac{X^{t_1}}{g},\cdots,\frac{X^{t_m}}{g})R$
$= (\frac{X^{u_1}}{g},\cdots,\frac{X^{u_l}}{g})R = (\frac{\sum_i X^{u_i}}{g})R.$
Therefore $(\xi,\eta)R$ is a principal ideal of R.

Lemma (4.5) If $*_1 \leq *_2$, then $\mathrm{Kr}(S,*_1) \subset \mathrm{Kr}(S,*_2)$.

Proof. Let $f,g \in k[X;S]-\{0\}$ such that $(e(f)+e(h))^{*_1} \subset (e(g)+e(h))^{*_1}$ for some $h \in k[X;S]-\{0\}$. Then we have $(e(f)+e(h))^{*_2} \subset (e(g)+e(h))^{*_2}$.

Proposition (4.6) Let $*$ be a semistar operation on S. Then we have $\mathrm{Kr}(S,*) = \mathrm{Kr}(S,[*]) = \mathrm{Kr}(S,*_a)$.

Proof. From the definitions, we have $\mathrm{Kr}(S,*_f) = \mathrm{Kr}(S,*)$.

Since $*_f \leq [*] \leq *_a$ by (3.6)(3), we have $\mathrm{Kr}(S,*) \subset \mathrm{Kr}(S,[*]) \subset \mathrm{Kr}(S,*_a)$. Let $\xi \in \mathrm{Kr}(S,*_a)$-$\{0\}$. Then $\xi = f/g$ with $f,g \in k[X,S]-\{0\}$ such that $(e(f)+e(h))^{*_a} \subset (e(g)+e(h))^{*_a}$ for some $h \in k[X;S]-\{0\}$. Let $e(f)+e(h) = (a_1,\cdots,a_n)$. Then, for each i, $a_i+F_i \subset (e(f)+e(g)+F_i)^*$ for some $F_i \in \mathrm{f}(S)$. Set $F = \sum_i F_i$, then $(a_1,\cdots,a_n)+F \subset (e(g)+e(h)+F)^*$. Therefore $(e(f)+e(h)+F)^* \subset (e(g)+e(h)+F)^*$. It follows that $f/g \in \mathrm{Kr}(S,*)$.

Proposition (4.7) Let $*$ be a semistar operation on S. Then, for each $E \in \bar{\mathrm{F}}(S)$, we have $E^{*_a} = \cup\{\mathrm{FKr}(S,*) \cap G \mid F \in \mathrm{f}(S) \text{ with } F \subset E\}$.

Proof. $E^{*_a} = \cup\{F^{*_a} \mid F \in f(S) \text{ with } F \subset E\} = \cup\{F\mathrm{Kr}(S,*_a) \cap G \mid F \in \mathrm{f}(S)$ with $F \subset E\} = \cup\{F\mathrm{Kr}(S,*) \cap G \mid F \in \mathrm{f}(S) \text{ with } F \subset E\}$.

Proposition (4.8) Let $*$ be a semistar operation on S. Set
$T = S^{*_a}$ and $*_T = \alpha_T(*_a) = \alpha(*_a)$.

Then, T is an integrally closed oversemigroup of S, and $*_T$ is an e.a.b. semistar operation on T such that $T^{*_T} = T$ and $\mathrm{Kr}(S,*) = \mathrm{Kr}(T,*_T)$.

Proof. Since $T = S^{[*]}$, T is integrally closed by (3.4)(2). By (3.6)(2), $*_a$ is e.a.b. and, by (1.5)(4), $*_T$ is e.a.b. Since $(E^{*_a})^{*_a} = E^{*_a}$ for each $E \in \bar{\mathrm{F}}(S)$, we have $T^{*_T} = T^{\alpha(*_a)} = T^{*_a} = (S^{*_a})^{*_a} = S^{*_a} = T$. By (2.2) and (4.6), we have $\mathrm{Kr}(S,*) = \mathrm{Kr}(S,*_a) = \mathrm{Kr}(T,*_T)$.

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