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Note on the number of semistar operations, XIV

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Abstract

We determine conditions for a grading monoid to have only a finite number of semistar operations.

This is a note on the number of semistar operations, and is a continuation of [M3]. The notions of star operations, semistar operations, and their Kronecker function rings of integral domains have been well known. We refer to Fontana-Loper([FL]) and its references for them. Let G be a torsion-free abelian additive group, and let S be an additively closed subset containing 0 of G. Then S is called a grading monoid (or, a g-monoid). We refer to [M1] for notions of star operations, semistar operations, and their Kronecker function rings for g-monoids.

Let $\Sigma(S)$ be the set of star operations on S, and let $\Sigma'(S)$ be the set of semistar operations on S. In §1 of this paper, we are interested in the cardinalities $|\Sigma(S)|$, and $|\Sigma'(S)|$, especially, in when $|\Sigma'(S)| < \infty$? We will determine conditions for $|\Sigma'(S)| < \infty$. §2 is an another note on $|\Sigma'(D)|$ for i-local domains D.

§1 The conditions for $|\Sigma'(S)| < \infty$

Let G be the quotient group of S, and let $\bar{S}$ be the integral closure of S. If S is a group, we have $|\Sigma'(S)| = 1$ trivially. Thus, throughout the section, let S be a g-monoid which is not a group, let M (resp. N) be the maximal ideal of S (resp. $\bar{S}$), let H (resp. L) be the group of units of S (resp. $\bar{S}$). In [M2, Theorem 14] we proved the following fact: Assume that $M = N$. Then we have that $|\Sigma'(S)| < \infty$ if and only if dim $(S) < \infty$, $\bar{S}$ is a valuation semigroup, and L/H is a finite group.

In this section, we will prove the following,

Theorem 1 Assume that $M \neq N$. Then the following conditions are equivalent.
(1) $|\Sigma'(S)| < \infty$.
(2) $\dim(S) < \infty, \bar{S}$ is a valuation semigroup, and $\bar{S} - S$ is a finite set modulo H.

(1.1) (cf. [M2, Proposition 1]) Let V be a valuation semigroup with maximal ideal N. If N is a principal ideal of V, then $|\Sigma(V)| = 1$, and if N is not a principal

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ideal of V, then $|\Sigma(V)| = 2$.

(1.2) Assume that $|\Sigma'(S)| < \infty$. Then there is only a finite number of oversemigroups of S.

Proof. Let T be an oversemigroup of S. Then there arises a semistar operation $I \longmapsto I + T$ on S.

(1.3) Assume that $|\Sigma'(S)| < \infty$. Then L/H is a finite group.

Proof. If L/H is an infinite group, then there is an infinite number of subgroups K of L containing H. Set $T = K \cup N$. Then T is an oversemigroup of S.

Let A be a subset of G. Then $S[A]$ denotes the oversemigroup of S generated by A.

(1.4) Assume that $|\Sigma'(S)| < \infty$. Then $\dim(S) < \infty, \bar{S}$ is a valuation semigroup, and $\bar{S} - S$ is a finite set modulo H.

Proof. Suppose that $\dim(S) = \infty$. Then there is an infinite number of oversemigroups of S. Then $|\Sigma'(S)| = \infty$ by (1.2).

Suppose that $\bar{S}$ is not a valuation semigroup. Then there is an element $x \in G - \bar{S}$ such that $-x \notin \bar{S}$. We have $S[2^n x] \supsetneqq S[2^{n+1}x]$ for each positive integer n. Then $|\Sigma'(S)| = \infty$ by (1.2).

Confering (1.3), let $\alpha_1, \cdots, \alpha_k$ be a complete representative system of L modulo H. Let v be a valuation belongoing to $\bar{S}$. By (1.2), we have $\{S[x] \mid x \in \bar{S} - S\} = \{S[x_1], \cdots, S[x_m]\}$ for some positive integer m. Let $x \in \bar{S} - S$. Then $S[x] = S[x_i]$ for some i. Hence $v(x) = v(x_i)$. Then $x - x_i = \alpha_j + h$ for some j and $h \in H$.

We have seen that (1) implies (2) in Theorem 1.

Thus, in the rest of the section, we assume that $M \neq N$, $\dim(S) < \infty$, $\bar{S}$ is a valuation semigroup, and $\bar{S} - S$ is a finite set modulo H. Set $\bar{S} = V$, let v be a valuation belonging to V, and let Γ be the value group of v.

(1.5) (1) Let $I \in \mathrm{F}'(S)$ so that $v(I)$ is not bounded below. Then $I = G$.

(2) $\mathrm{F}'(S) = \mathrm{F}(S) \cup \{G\}$.

(3) Each star operation $*$ on S can be extended uniquely to a semistar operation on S.

(4) $|\Sigma(S)| \leq |\Sigma'(S)|$.

(5) L/H is a finite group.

(6) Let $I \in \mathrm{F}(S)$ so that there does not exist $\inf v(I)$. Then we have $\{x \in G \mid v(x) \geq v(a) \text{ for some } a \in I\} \subset I$.

Proof. (1) Let $x \in G$. There are $x_1, x_2, x_3, \cdots$ in I so that $v(x) > v(x_1) > \cdots$. Then $x - x_i \in N$ for each i, and $x - x_i \not\equiv x - x_j$ modulo H for each $i < j$. It follows that $x - x_n \in S$ for some n, and hence $x \in I$.

(2) follows from (1).
(3) follows from (2).
(4) follows from (3).
(5) follows from the fact that $V - S$ is a finite set modulo H.
(6) Supppose the contrary. There are $x \notin I$ and $a_0 \in I$ such that $v(a_0) \leq v(x)$. There are elements $a_1, a_2, a_3, \cdots$ in I so that $v(a_0) > v(a_1) > v(a_2) > \cdots$. Then $x - a_i \in V - S$ for each i, and $x - a_i \not\equiv x - a_j$ modulo H for each $i < j$; a contradiction.

Let $\alpha_1, \cdots, \alpha_a$ be a complete representative system of L modulo H. And let $z_1, \cdots, z_b$ be a complete representative system of $N - M$ modulo H. We may assume that $v(z_1) \leq \cdots \leq v(z_b)$.

(1.6) Let T be an oversemigroup of S with $T \in \mathrm{F}(S)$, and let $*$ be either a star operation or a semistar operation on S. Then T^* is an oversemigroup of S.

Proof. Let $x, y \in T^*$. Then $x + y \in T^* + T^* \subset (T^* + T^*)^* = (T + T)^* = T^*$.

(1.7) There is min $v(N)$.

Proof. Suppose that $0 < v(s) < v(z_1)$ for some $s \in S$. We have $z_1 - s \equiv z_i$ modulo H for some i. Then $v(z_1) = v(z_i)$, and hence $v(s) = 0$; a contradiction.
Let $x \in N - M$. Then $x \equiv z_i$ modulo H for some i. Hence $v(x) = v(z_i) \geq v(z_1)$.

We may assume that is the rank 1 convex subgroup of Γ. Take an element $\pi \in N$ such that $v(\pi) = 1$.

(1.8) Let T be an oversemigroup of S. Then $T \supset V$ or $T \subset V$.

Proof. Assume that $T \not\subset V$, and take an element $x_0 \in T - V$. Then $-x_0 \in N$. Let $x \in V$. We have $x - kx_0 \in V$ for each $k \geq 0$. If $0 < i < j$, then $x - ix_0 \not\equiv x - jx_0$ modulo H. Therefore $x - mx_0 \in S$ for some m. Then $x \in S[x_0]$, and hence $V \subset T$.

(1.9) There is only a finite number of oversemigroups of S.

Proof. It follows from (1.8), $\dim(V) < \infty$, and the hypothesis that $V - S$ is a finite set modulo H.

(1.10) Let $\dim(S) = 1$. Then $V^v = V$, that is, V is a divisorial fractional ideal of S.

Proof. V^v is an oversemigroup of V by (1.6), and we have $\dim(V) = 1$. Suppose that $V^v \neq V$. Then $V^v = G$. Take an element $x_0 \in M$. Let $1 \leq i \leq b$, and let $0 < j < k$. Then $z_i + jx_0 \not\equiv z_i + kx_0$ modulo H. Hence there is m_i so that $z_i + mx_0 \in S$ for each $m \geq m_i$, that is, $z_i \in (-mx_0)$. Similarly, there is m'_j so that $\alpha_j \in (-mx_0)$ for each $m \geq m'_j$. Let $\max\{m_i, m'_j \mid i, j\} = m_0$. Then $V \subset (-mx_0)$ for each $m \geq m_0$. Since $V^v = \cap\{(x) \mid (x) \supset V\}$, we have $(-mx_0) = G$ for each $m \geq m_0$;

this is clearly impossible.

(1.11) We have $V^v = V$.

Proof. By (1.10), we may assume that $\dim(S) \geq 2$. Let Q be a prime ideal of V with $\mathrm{ht}(Q) = \mathrm{ht}(N) - 1$, and let $P = S \cap Q$, where $\mathrm{ht}(N)$ (resp. $\mathrm{ht}(Q)$) means the height of N (resp. the height of Q). Suppose that $V^v \neq V$. Then $V^v \supset V_Q$. Take an element $x_0 \in M - P$. The similar argumnet to the proof of (1.10) shows that $(-mx_0) \supset V_Q$ for all sufficiently large m. Since $(m+1)x_0 \notin Q$, we have $-(m+1)x_0 \in V_Q \subset (-mx_0)$, and hence $-x_0 \in S$; a contradiction.

(1.12) Let $I \in \mathrm{F}(S)$ so that there does not exist inf $v(I)$. Then $I^v = I$.

Proof. Suppose that $I^v \supsetneqq I$. Take an element $x \in I^v - I$. Then $v(x)$ is a lower bound of $v(I)$ by (1.5)(6). There is a lower bound $v(y)$ of $v(I)$ with $v(x) < v(y)$. Set $I - y = J$. Since $J \subset V$, we have $J^v \subset V$ by (1.11). We have $x - y \in I^v - y = J^v$, and $v(x - y) < 0$. Hence $J^v \not\subset V$; a contradiction.

(1.13) $|\Sigma(S)| < \infty$.

Proof. Let $I \in \mathrm{F}(S)$ with $S \subset I \subset V$. Then I is generated on S by a subset of $\{\alpha_i, z_j \mid i, j\}$. Therefore the set $\{I \in \mathrm{F}(S) \mid S \subset I \subset V\} = X$ is a finite set.

Let $* \in \Sigma(S)$ and let $I \in X$. Set $I^* = g_*(I)$. Then g_* is a mapping from X to X by (1.11), that is, $g_* \in X^X$. Then g is a mapping from $\Sigma(S)$ to X^X.

Let $*, *' \in \Sigma(S), I \in \mathrm{F}(S)$, and assume that $g_* = g_{*'}$. If there does not exist inf $v(I)$, then $I^* = I^{*'} = I$ by (1.12). If there is inf $v(I) = v(x)$, then min $v(I - x) = 0$ by (1.7). Hence $S \subset I - y \subset V$ for some $y \in I$. Since $g_* = g_{*'}$, we have $(I - y)^* = (I - y)^{*'}$, and hence $I^* = I^{*'}$. We have proved that $* = *'$, and hence g is an injection. It follows that $|\Sigma(S)| < \infty$.

(1.14) Let T be an oversemigroup of S with $T \subset V$. Then $|\Sigma(T)| < \infty$.

Proof. Let M' be the maximal ideal of T, and let H' be the group of units of T. We have that $\bar{T} = V$, $\dim(T) = \dim(S) < \infty$, and L/H' is a finite group. If $M' = N$, we have $|\Sigma'(T)| < \infty$ by [M2, Theorem 14], and hence $|\Sigma(T)| < \infty$ by (1.5)(4). If $M' \neq N$, we have $|\Sigma(T)| < \infty$ by (1.13).

(1.15) Let T be an oversemigroup of S. Then $|\Sigma(T)| < \infty$.

Proof. We may assume that $T \not\subset V$ by (1.14). Then $T \supset V$ by (1.8). Then $|\Sigma(T)| \leq 2$ by (1.1).

Confering (1.9), let $\{T_1, \cdots, T_c\}$ be the set of oversemigroups of S. For each $1 \leq i \leq c, * \in \Sigma(T_i)$ and $I \in \mathrm{F}(S)$, set $(I + T_i)^* = I^{\sigma(*)}$ and $G = G^{\sigma(*)}$.

(1.16) (1) If $i \neq j$, then $\Sigma(T_i) \cap \Sigma(T_j) = \emptyset$.

(2) There is a canonical mapping σ from $\bigcup_1^c \Sigma(T_i)$ to $\Sigma'(S)$.

Proof. (1) We have $\mathrm{F}(T_i) \neq \mathrm{F}(T_j)$, and $\Sigma(T_i)$ (resp. $\Sigma(T_j)$) is a set of mappings from $\mathrm{F}(T_i)$ to $\mathrm{F}(T_i)$ (resp. from $\mathrm{F}(T_j)$ to $\mathrm{F}(T_j)$).

(2) We see easily that $\sigma(*)$ satisfies the conditions of a semistar operation on S.

(1.17) The mapping σ is a bijection onto $\Sigma'(S)$.

Proof. Let $* \in \Sigma'(S)$. Then $S^* = T_i$ for some T_i. There is a star operation $*' : J \longmapsto J^*$ on T_i. Then we have $\sigma(*') = *$, and hence σ is a surjection.

Let $*_i \in \Sigma(T_i)$ and $*_j \in \Sigma(T_j)$ such that $\sigma(*_i) = \sigma(*_j)$. Then we have $T_i = S^{\sigma(*_i)} = S^{\sigma(*_j)} = T_j$.

(1.18) $|\Sigma'(S)| < \infty$.

Proof. It follows from (1.15), (1.16), and (1.17).

The proof of Theorem 1 is complete.

§2 An another note

In [M4], we determined conditions for $|\Sigma'(D)| < \infty$ for any APVD (or, an almost pseudo-valuation domain) D, and in §1, we determined conditions for $|\Sigma'(S)| < \infty$ for any g-monoid S. Every g-monoid that is not a group has a unique maximal ideal, and every APVD D has the property that D and its integral closure $\bar{D}$ has a unique maximal ideal. We refer to [BH] for APVD's. Thus it is natural to consider the class of domains D such that $\bar{D}$ has a unique maximal ideal. We call such a domain an i-local domain. In §2, we will study $|\Sigma'(D)|$ for i-local domains D.

(2.1) Let D be an i-local domain. Assume that $\bar{D}$ is a valuation domain with maximal ideal M, v be a valuation belonging to $\bar{D}$, and $M^n \subset D$ for some positive integer n. Then either D is a PVD (or, a pseudo-valuation domain), or there is min $v(M)$.

Proof. Suppose the contrary. Let $0 \neq x \in M$. There are elements $x_1, \cdots, x_n \in M$ such that $v(x) > v(x_1) > \cdots > v(x_n) > 0$. Then $x = \dfrac{x}{x_1}\dfrac{x_1}{x_2}\cdots\dfrac{x_{n-1}}{x_n}x_n \in M^n \subset D$. Hence D is a PVD; a contradiction.

Let D be a valuation domain with maximal ideal M, let v be a valuation belonging to D, and let Γ be the balue group of v. If there is min $v(M)$, then we may assume that is the rank one convex subgroup of Γ, and min $v(M) = 1 \in \vec{Z} \subset \Gamma$.

For, the rank one convex subgroup of Γ is isomorphic with the ordered group . Therefore Γ is order isomorphic with an ordered group Γ' the rank one convex subgroup of which is .

(2.2) Let D be an i-local domain with maximal ideal P, let M be the maximal ideal of $\bar{D}$, and assume that $|\Sigma'(D)| < \infty$. Then we have,

(1) $\dim(D) < \infty$.
(2) There is only a finite number of overrings of D.
(3) $\bar{D} = V$ is a valuation domain.
(4) V is a finitely generated D-module.
(5) $V/M = K$ is a simple extension field of $D/P = k$ with $[K : k] < \infty$.
(6) $V, M \in \mathrm{F}(D)$.
(7) $\mathrm{F}'(D) = \mathrm{F}(D) \cup \{\mathrm{q}(D)\}$.

Proof. (1) follows from (2).

(2) If T is an overring of D, then there is a semistar operation $I \longmapsto IT$ on D.

(3) Let $\{V_\lambda \mid \lambda \in \Lambda\}$ be the set of valuation overrings of D. Then we have $\bar{D} = \cap_\lambda V_\lambda$.

(4) $\bar{D}$ is a finitely generated overring of D.

(5) There is only a finite number of intermediate fields between k and K.

(6) There are elements $x_1, \cdots, x_n \in V$ such that $V = \sum_1^n Dx_i$ for some positive integer n.

(7) There is $0 \neq d \in D$ such that $dV \subset D$. Let v be a valuation belonging to V. Let $I \in \mathrm{F}'(D)$ so that $v(I)$ is not bounded below. Let $x \in \mathrm{q}(D)$. There is $a \in I$ such that $v(a) < v(x)$. Then $x \in aV \subset (a/d)D \subset (1/d)I$. Hence $\mathrm{q}(D) \subset (1/d)I$, and hence $I = \mathrm{q}(D)$.

(2.3) Let D be an i-local domain such that $\bar{D} = V$ is a valuation ring, and let M be the maximal ideal of $\bar{D}$. Assume that $M^n \subset D$ for some positive integer n. Then we have,

(1) $\mathrm{F}'(D) = \mathrm{F}(D) \cup \{\mathrm{q}(D)\}$.
(2) Let T be an overring of D. Then either $T \supset V$ or $T \subset V$.
(3) Let $\Sigma'_1 = \{* \in \Sigma'(D) \mid D^* \supset V\}$. Then there is a canonical bijection from $\Sigma'(V)$ onto Σ'_1.
(4) Let $\Sigma'_2 = \{* \in \Sigma'(D) \mid D^* \subsetneqq V\}$. Then $\Sigma'(D) = \Sigma'_1 \cup \Sigma'_2$.
(5) Let $\{T_\lambda \mid \lambda \in \Lambda\}$ be the set of overrings T of D with $T \subsetneqq V$. Then there is a canonical bijection from the disjoint union $\bigcup_\lambda \Sigma(T_\lambda)$ onto Σ'_2.

Proof. (1) Similar to (2.2)(7).

(2) Assume that $T \not\subset V$, and take an element $x \in T - V$. We may assume that $1/x \in M^n$. Let $a \in V$. Then $a(1/x) \in P$, hence $a \in xP \subset T$.

(3) The map $* \longmapsto \delta_D(*)$ gives a bijection from $\Sigma'(V)$ onto Σ'_1.

(4) follows from (1).

(5) Similar to (3).

(2.4) Let D be an i-local domain. Assume that $\bar{D} = V$ is a valuation ring with maximal ideal M, let $\mathcal{K}$ be a complete representative system of V modulo M, v be a valuation belonging to V with value group Γ, assume that is the rank one convex subgroup of Γ, and $v(\pi) = 1 \in \vec{Z}$ for some $\pi \in V$. Let $x \in \mathrm{q}(D) - \{0\}$ with $v(x) \in \vec{Z}$. Let k be a positive integer with $k > v(x)$. Then x can be expressed uniquely as

$x = \alpha_l \pi^l + \alpha_{l+1}\pi^{l+1} + \cdots + \alpha_{k-1}\pi^{k-1} + a\pi^k$, where $l = v(x)$ and each $\alpha_i \in \mathcal{K}$ with $\alpha_l \not\equiv 0 \pmod{M}$ and $a \in V$.

Proof. Since $\frac{x}{\pi^l}$ is a unit of V, we have $\frac{x}{\pi^l} \equiv \alpha_l \pmod{M}$ for a unique $0 \not\equiv \alpha_l \in \mathcal{K}$.

(2.5) Proposition Let D be an i-local domain with maximal ideal P, and assume that $\bar{D} = V$ is a valuation ring with maximal ideal M, v be a valuation belonging to V with the value group Γ. Asume that $D \supset M^3$. Then,

(1) D is either a PVD or, we may assume that is the rank one convex subgroup of Γ.

(2) If $D/P = V/M$, then D is an APVD.

Proof. (1) follows from (2.1).

(2) Suppose the contrary. Then we may apply (2.4), and we may assume that $\mathcal{K} \subset D$. Since D is not an APVD, we may choose $x \in P - M^3$. If $v(x) = 1$, then $x^2 \in P - M^3$ and $x^2 \in M^2$. Hence we may assume that $v(x) = 2$. We have $x = \alpha\pi^2 + a\pi^3$ for $\alpha \in \mathcal{K}$ and $a \in V$. Since $\alpha \in D - P$, we have $\pi^2 \in P$, and hence $M^2 \subset P$. Since D is not an APVD, we may choose $x \in P - M^2$. Then $\pi \in P$, and hence $M = P$; a contradiction.

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