On generalized divisorial semistar operations
on integral domains

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1. INTRODUCTION

Throughout this paper the letter D denotes an integral domain with quotient field K. We shall denote the set of all nonzero D-submodules of K by $\mathbf{K}(D)$ and we shall call each element of $\mathbf{K}(D)$ a *Kaplansky fractional ideal* (for short, *K-fractional ideal*) of D as in [O3]. Let $\mathbf{F}(D)$ be the set of all nonzero *fractional ideals* of D, that is, all elements $E \in \mathbf{K}(D)$ such that there exists a nonzero element $d \in D$ with $dE \subseteq D$. The set of finitely generated K-fractional ideals of D is denoted by $\mathbf{f}(D)$. It is evident that $\mathbf{f}(D) \subseteq \mathbf{F}(D) \subseteq \mathbf{K}(D)$. An *ideal* of D means an *integral ideal* of D and the set of all nonzero integral ideals of D is denoted by $\mathbf{I}(D)$.

If D is a quasi-local domain with maximal ideal M, then we say that (D, M) is a quasi-local domain. In [HHP], a nonzero ideal I of D is called an m-canonical ideal of D if $I : (I : J) = J$ for each nonzero ideal J of D. In [HHP, Proposition 6.2] it was shown that if (D, M) is an integrally closed qausi-local domain, then M is an m-canonical ideal of D if and only if D is a valuation domain. In [BHLP, Proposition 4.1], it was proved that the integrally closed hypothesis in the above result can be eliminated, that is, if (D, M) is a quasi-local domain, then D is a valuation domain if and only if M is an m-canonical ideal of D. Recently, in [B2, Corollary 2.15], it was proved that if a quasi-local integral domain (D, M) admits a proper m-canonical ideal I of D, then the following statements are equivalent:

(1) D is a valuation domain.
(2) I is a divided m-canonical ideal of D.
(3) $cM = I$ for some nonzero element $c \in D$.
(4) $I : M$ is a principal ideal of D.
(5) $I : M$ is an invertible ideal of D.
(6) D is an integrally closed domain and $I : M$ is a finitely generated ideal of D.
(7) $M : M = D$ and $I : M$ is a finitely generated ideal of D.
(8) If $J = I : M$, then J is a finitely generated ideal of D and $J : J = D$.

Let I be a nonzero ideal of D such that $I : I = D$. Then in [HHP, Proposition 3.2], it was proved that the map $J \longmapsto I : (I : J)$ of $\mathbf{F}(D)$ into $\mathbf{F}(D)$ is a star operation

on D. Hence if we denote this star operation by $v(I)$, then I is an m-canonical ideal of D if and only if $v(I)$ is equal to the identity star operation d on D. As easily seen, the star operation $v(D)$ is equal to the divisorial star operation v on D. A star operation $v(I)$ on D is called an *I-divisorial star operation* on D. We shall call a star operation $v(I)$ on D a *generalized divisorial star operation* (for short, *g-divisorial star operation*) on D.

In [P1], Picozza extended the definition of a generalized divisorial star operation to the semistar operation case. In fact, for each $A \in \mathbf{K}(D)$, if we set $E^{v(A)} = A : (A : E)$ for each $E \in \mathbf{K}(D)$, then the map $E \longmapsto E^{v(A)}$ is a semistar operation on D. In [P2, Remark 4.3], it was shown that each semistar operation on a valuation domain V is of the form $v(I)$ for some ideal I of V. In this paper, we shall call a semistar operation $v(A)$ with $A \in \mathbf{K}(D)$ a *generalized divisorial semistar operation* (for short, *g-divisorial semistar operation*) on D.

The purpose of this paper is to continue the investigation of generalized divisorial semistar operations and to give a new characterization of a valuation domain and a new characterization of a strongly discrete valuation domain. In Section 2, we collect some fundamental results on semistar operations. In Section 3, we study generalized divisorial semistar operations. In Section 4, we introduce the notion of an *ideal-divisorial semistar domain* and give a characterization of a valuation domain using this terminology and we also introduce the notion of a *prime-divisorial semistar domain* and give a characterization of a strongly discrete valuation domain using this terminology. Moreover we introduce the notion of an *almost valuation semistar domain* and show that if D is a quasi-local domain, then D is an almost valuation semistar domain if and only if D is a valuation domain.

Throughout this paper, we denote the set of prime ideals (resp. maximal ideals) of D by $\mathrm{Spec}(D)$ (resp. $\mathrm{Max}(D)$) and denote the cardinality of a set X by $|X|$. An integral domain which lies between D and K is called an *overring* of D and an overring R of D is called a *proper overring* of D if $R \neq D$ and $R \neq K$. We denote the set of all overrings of D by $\mathcal{O}(D)$. The symbol $\subset$ always means *proper inclusion*.

2. BACKGROUND ON SEMISTAR OPERATIONS

In [OM], we introduced the notion of a *semistar operation* on D as a generalization of a *star operation*.

A map $E \longmapsto E^*$ of $\mathbf{K}(D)$ into $\mathbf{K}(D)$ is called a *semistar operation* on D if the following conditions hold for all $a \in K - \{0\}$ and $E, F \in \mathbf{K}(D)$:

(S_1) $(aE)^* = aE^*$;

(S_2) If $E \subseteq F$, then $E^* \subseteq F^*$; and

(S_3) $E \subseteq E^*$ and $(E^*)^* = E^*$.

We denote the set of all semistar operations on D by $\mathbf{SS}(D)$. For any overring R of D, we denote the set $\{* \in \mathbf{SS}(D) \mid D^* = R\}$ by $\mathbf{SS}(D, R)$.

Here we recall that a map $E \longmapsto E^\star$ of $\mathbf{F}(D)$ into $\mathbf{F}(D)$ is called a *star operation* on D, if the following conditions hold for all $a \in K - \{0\}$ and $E, F \in \mathbf{F}(D)$:

(S_0) $(aD)^\star = aD$;

(S_1) $(aE)^\star = aE^\star$;

(S_2) If $E \subseteq F$, then $E^\star \subseteq F^\star$; and

(S_3) $E \subseteq E^\star$ and $(E^\star)^\star = E^\star$.

If we set $E^d = E$ for all $E \in \mathbf{F}(D)$ then d is a star operation on D and is called *the identity operation* (or simply *the d-operation*). Next, for each $E \in \mathbf{F}(D)$, we set $E^{-1} = D : E = \{x \in K \mid xE \subseteq D\}$ and $E^v = (E^{-1})^{-1}$ for each $E \in \mathbf{F}(D)$, then v is a star operation on D and is called *the v-operation*. We shall denote the set of star operations on D by $\mathbf{S}(D)$.

Proposition 1. Let $*$ be a semistar operation on D and let $E, F \in \mathbf{K}(D)$. Then
(1) $(EF)^* = (E^*F)^* = (EF^*)^* = (E^*F^*)^*$;
(2) $(E + F)^* = (E^* + F)^* = (E + F^*)^* = (E^* + F^*)^*$;
(3) $(E : F)^* \subseteq E^* : F^* = (E^* : F) = (E^* : F)^*$;
(4) $(E \cap F)^* \subseteq E^* \cap F^* = (E^* \cap F^*)^*$, if $E \cap F \neq (0)$.
(5) Let $\{E_\alpha\}$ be a family of K-fractional ideals of D. Then
(a) $(\sum_\alpha E_\alpha)^* = (\sum_\alpha E_\alpha^*)^*$;
(b) $\bigcap_\alpha E_\alpha^* = (\bigcap_\alpha E_\alpha^*)^*$, if $\bigcap_\alpha E_\alpha^* \neq \{0\}$.

Example 2. (1) If we set $E^{\bar{d}} = E$ for each $E \in \mathbf{K}(D)$, then the map $E \mapsto E^{\bar{d}}$ is a semistar operation on D and is called *the identity semistar operation* (or simply *the $\bar{d}$-operation*) on D. If we set $E^{\bar{e}} = K$ for all $E \in \mathbf{K}(D)$, then the map $E \longmapsto E^{\bar{e}}$ is a semistar operation on D and is called *the trivial semistar operation* (or simply *the $\bar{e}$-operation*) on D.

(2) For each $E \in \mathbf{K}(D)$, we set $E^{-1} = \{x \in K \mid xE \subseteq D\}$ and $E^{\bar{v}} = (E^{-1})^{-1} = D : (D : E)$. Then the map $E \longmapsto E^{\bar{v}}$ is a semistar operation on D and is called *the divisorial semistar operation* (or simply *the $\bar{v}$-operation*) on D. If $E \in \mathbf{K}(D) \setminus \mathbf{F}(D)$, then $E^{-1} = (0)$ and so $E^{\bar{v}} = K$.

(3) A semistar operation $*$ on D is said to be *of finite type* (or *of finite character*) if $E^* = \bigcup\{F^* \mid F \subseteq E \text{ and } F \in \mathbf{f}(D)\}$. For each $* \in \mathbf{SS}(D)$ and each $E \in \mathbf{K}(D)$, we set $E^{*_f} = \bigcup\{F^* \mid F \subseteq E \text{ and } F \in \mathbf{f}(D)\}$. Then the map $E \mapsto E^{*_f}$ is a semistar operation of finite type on D and is called *the semistar operation of finite type associated to $*$*. It is easy to see that $*$ is of finite type if and only if $* = *_f$. The semistar operation $\bar{v}_f$ associated to $\bar{v}$ is denoted by $\bar{t}$ and is called *the $\bar{t}$-operation*. It is easily seen that $E^* = E^{*_f}$ for all $E \in \mathbf{f}(D)$. We shall denote the set of all semistar operations of finite type on D by $\mathbf{SS}_f(D)$.

(4) Let R be an overring of D. If we set $E^{*(R)} = ER$ for each $E \in \mathbf{K}(D)$, then the map $E \mapsto E^{*(R)}$ is a semistar operation of finite type on D and is called *the semistar operation defined by an overring R*.

(5) Let $\mathcal{B}$ be the set of all valuation overrings of D. If we set $E^{\bar{b}} = \bigcap\{EV_\alpha \mid V_\alpha \in \mathcal{B}\}$ for each $E \in \mathbf{K}(D)$, then the map $E \longmapsto E^{\bar{b}}$ of $\mathbf{K}(D)$ into $\mathbf{K}(D)$ is a semistar operation on D and is called *the $\bar{b}$-operation* on D. By definition, $D^{\bar{b}} = \bigcap\{V_\alpha \mid V_\alpha \in \mathcal{B}\} = \bar{D}$, *the integral closure* of D. Now let $\mathcal{W}$ be a set of valuation overrings of D. If we set $E^{\bar{w}} = \bigcap\{EV_\alpha \mid V_\alpha \in \mathcal{W}\}$ for each $E \in \mathbf{K}(D)$, then the map $E \longmapsto E^{\bar{w}}$ of $\mathbf{K}(D)$ into $\mathbf{K}(D)$ is a semistar operation on D and is called *the $\bar{w}$-operation* on D.

Proposition 3 ([OM, Proposition 17]). Let $\star$ be a star operation on D. Then,

for each $E \in \mathbf{K}(D)$, we set:

$$E^{\star^e} = \begin{cases} E^\star, & \text{for } E \in \mathbf{F}(D) \\ K, & \text{for } E \in \mathbf{K}(D) \backslash \mathbf{F}(D) \end{cases}$$

Then the map $E \longmapsto E^{\star^e}$ is a semistar operation on D. This semistar operation $\star^e$ is called *the trivial semistar extension of a star operation* $\star$.

Here we denote the trivial semistar extension d^e of the d-operation on D by $\bar{f}$. For each overring R of D, we denote the d-operation on R by d_R and denote the trivial semistar extension $(d_R)^e$ of d_R by $\bar{f}_R$. It is easily seen that the map $\bar{v}$ is the trivial semistar extension of v.

Proposition 4 ([OM, Proposition 17 and Corollary 18]). For each star operation $\star$ on D, we set $\chi(\star) = \star^e$. Then
(1) χ is an injective map of $\mathbf{S}(D)$ into $\mathbf{SS}(D)$.
(2) $| \mathbf{S}(D) | \leq | \mathbf{SS}(D) |$.

In [O3], a semistar operation $*$ is said to be *weak* if $D^* = D$ and is said to be *strong* if $D^* \neq D$. We denote the set of weak semistar operations on D by $\mathbf{WS}(D)$. Evidently $\star^e$ is a weak semistar operation for all star operations $\star$.

As in [DF], an integral domain D is called a *conducive domain* if $(D : R) = \{x \in K \mid xR \subseteq D\} \neq (0)$ for each overring R of D other than K. It is evident that each overring of a conducive domain is also conducive (see [DF, Lemma 2.0 (i)]).

Now let us recall that we can define a partial order $\leq$ on $\mathbf{SS}(D)$ in the following way:

$$*_1 \leq *_2 \Longleftrightarrow E^{*_1} \subseteq E^{*_2} \text{ for each } E \in \mathbf{K}(D).$$

For $*_1, *_2 \in \mathbf{SS}(D)$, it is easily seen that $*_1 \leq *_2$ if and only if $(E^{*_1})^{*_2} = E^{*_2}$ for all $E \in \mathbf{K}(D)$ (see [OM, p.6]). It is also easy to see that $\bar{d} \leq \bar{f} \leq \bar{v}$ always holds.

Proposition 5 (cf. [O4, Proposition 7] and [MI, Proposition 2.1]). Let D be an integral domain with quotient field K. Then the following statements are equivalent:
(1) D is a conducive domain;
(2) Every $E \in \mathbf{K}(D)$ such that $D \subseteq E \neq K$ is a fractional ideal of D;
(3) Every $E \in \mathbf{K}(D)$ such that $E \neq K$ is a fractional ideal of D, i.e., $\mathbf{K}(D) = \mathbf{F}(D) \bigcup \{K\}$;
(4) Every valuation overring $V \neq K$ of D is a fractional ideal of D;
(5) Some valuation overring V of D is a fractional ideal of D;
(6) $E^{\bar{v}} \neq K$ for each $E \in \mathbf{K}(D)$ such that $E \neq K$;
(7) For each overring $T \subset K$ of D and for each $* \in \mathbf{SS}(D) \setminus \{\bar{e}\}, T^* \subset K$;
(8) For each valuation overring $V \subset K$ of D and for each $* \in \mathbf{SS}(D) \backslash \{\bar{e}\}, V^* \subset K$;
(9) There is a valuation overring $V \subset K$ of D such that $V^* \subset K$ for each $* \in \mathbf{SS}(D) \setminus \{\bar{e}\}$;
(10) $\bar{d} = \bar{f}$ holds.

Proposition 6 ([OM, Lemma 45]). Let R be an overring of D. Then

(1) For each $* \in \mathbf{SS}(R)$, if we define $E^{\delta_D(*)} = (ER)^*$ for all $E \in \mathbf{K}(D)$, then $\delta_D(*) \in \mathbf{SS}(D)$.

(2) If we define $\delta_{R/D} : \mathbf{SS}(R) \to \mathbf{SS}(D)$ by $\delta_{R/D}(*) = \delta_D(*)$, then $\delta_{R/D}$ is an injective map and therefore $| \mathbf{SS}(R) | \leq | \mathbf{SS}(D) |$.

(3) For each $* \in \mathbf{SS}(D)$, if we define $E^{\alpha_R(*)} = E^*$ for all $E \in \mathbf{K}(R)(\subseteq \mathbf{K}(D))$, then $\alpha_R(*) \in \mathbf{SS}(R)$.

(4) If we define $\alpha_{R/D} : \mathbf{SS}(D) \to \mathbf{SS}(R)$ by $\alpha_{R/D}(*) = \alpha_R(*)$, then $\alpha_{R/D} \circ \delta_{R/D}$ is the identity map of $\mathbf{SS}(R)$.

The map $\delta_{R/D}$ (resp. $\alpha_{R/D}$) is called *the descent map* (resp. *the ascent map*).

3. GENERALIZED DIVISORIAL SEMISTAR OPERATIONS

Lemma 7. Let D be an integral domain and let $A \in \mathbf{K}(D)$. If we set $E^{v(A)} = A : (A : E)$ for each $E \in \mathbf{K}(D)$, then $A : E^{v(A)} = (A : E)^{v(A)} = A : E$ for each $E \in \mathbf{K}(D)$.

Proof. By definition, $A : E^{v(A)} \subseteq A : E \subseteq (A : E)^{v(A)} = A : (A : (A : E)) = A : E^{v(A)}$, and therefore the equality holds. $\square$

Proposition 8 (cf. [HHP, Proposition 3.2]). Let D be an integral domain and let $A \in \mathbf{K}(D)$. Then the map $E \mapsto E^{v(A)}$ of $\mathbf{K}(D)$ into $\mathbf{K}(D)$ is a semistar operation on D.

Proof. For each $x \in K \setminus \{0\}$ and each $E \in \mathbf{K}(D)$, we have $(xE)^{v(A)} = A : (A : xE) = A : (x^{-1}(A : E)) = x(A : (A : E)) = xE^{v(A)}$ and therefore the condition (S_1) is satisfied. If $E \subseteq F$ with $E, F \in \mathbf{K}(D)$, then evidently $E^{v(A)} \subseteq F^{v(A)}$ and so the condition (S_2) holds. Lastly, $E \subseteq E^{v(A)}$ is clear for each $E \in \mathbf{K}(D)$ and furthermore $(E^{v(A)})^{v(A)} = E^{v(A)}$ follows from Lemma 7 and thus the condition (S_3) folds for each $E \in \mathbf{K}(D)$. $\square$

For each $A \in \mathbf{K}(D)$, we call a semistar operation $v(A)$ defined in Proposition 8 a *generalized divisorial semistar operation* (for short, a *g-divisorial semistar operation*) on D. In particular, if I is a nonzero integral ideal of D, then $v(I)$ is called an *I-divisorial semistar operation* on D and if P is a nonzero prime ideal of D, then $v(P)$ is called a *P-divisorial semistar operation* on D. For each nonzero integral ideal $I, v(I)$ is generally called an *ideal-divisorial semistar operation* on D and for each nonzero prime ideal $P, v(P)$ is generally called a *prime-divisorial semistar operation* on D.

Proposition 9. Let D be an integral domain. Then the following statements hold:

(1) $v(A)$ is a weak semistar opeation on D if and only if $A : A = D$.

(2) If D is completely integrally closed and A is a fractional ideal of D, then $v(A)$ is a weak semistar operation on D.

(3) If D is integrally closed and A is a finitely generated fractional ideal of D, then $v(A)$ is a weak semistar operation on D.

(4) The semistar operation $v(D)$ is equal to the $\bar{v}$-operation on D.

(5) Let I be a nonzero integral ideal of D. Then $E^{v(I)} = K$ for all $E \in \mathbf{K}(D) \setminus \mathbf{F}(D)$.

(6) If I is an invertible fractional ideal of D, then $v(I) = \bar{v}$.

(7) For each $* \in \mathbf{SS}(D)$, $v(D^*)$ is equal to the $v(*)$-semistar operation on D in [O3].

Proof. (1) By definition, $D^{v(A)} = A : A$ and so our assertion is evident.

(2) and (3) immediately follow from (1), since in each case $A : A = D$ holds.

(4) is evident from definitions of $v(D)$ and the $\bar{v}$-operation.

(5) follows from the fact that $A : E = (0)$ for each $E \in \mathbf{K}(D) \setminus \mathbf{F}(D)$.

(6) is an immediate consequence of [HHP, Lemma 2.1].

(7) follows from [O3, Definition 45]. $\square$

An integral domain D is called an *ideal-divisorial semistar domain* (or simply IDSD) (resp. a *prime-divisorial semistar domain* (or simply PDSD)) if each nontrivial semistar operation on D is of the form $v(I)$ for some nonzero ideal I of D (resp. of the form $v(P)$ for some nonzero prime ideal P of D). In [P2, Remark 4.3], it was proved that each valuation domain is an ideal-divisorial semistar domain. Now note that the trivial semistar operation $\bar{e}$ can be considered as a special prime-divisorial semistar operation on D of the form $v((0))$.

In [D1], a prime ideal P of D is said to be *divided* if $P = PD_P$, or equivalently, P is comparable to every principal ideal of D. An integral domain D is called a *divided domain* if every prime ideal of D is divided. For other characterizations of a divided domain, see [O2, Theorem 2.2]. In [B2], a proper integral ideal I of D is also said to be *divided* if $I \subset (c)$ for every $c \in D \setminus I$. An integral domain D is called a *strongly divided domain* if every proper integral ideal of D is divided. It follows from [O2, Theorem 2.2] that if D is a divided domain, then every radical ideal of D is divided in the sense of [B2]. It is easy to see that every valuation domain is a strongly divided domain.

Lemma 10. Let D be a divided domain. Then $*_{(D_P)} \leq v(P)$ for each $P \in \mathrm{Spec}(D)$.

Proof. Let P be a prime ideal of D. Then $E^{*_{(D_P)}} = ED_P$ for each $E \in \mathbf{K}(D)$. If $t \in P : E$, then $tE \subseteq P$ and so $tE^{*_{(D_P)}} = tED_P \subseteq PD_P = P$. Hence $E^{*_{(D_P)}} \subseteq P : (P : E) = E^{v(P)}$ for each $E \in \mathbf{K}(D)$ which implies that $*_{(D_P)} \leq v(P)$. $\square$

A prime ideal P of D is called *strongly prime* if $x, y \in K$ and $xy \in P$ imply that $x \in P$ or $y \in P$. An integral domain D is called a *pseudo-valuation domain* (for short, PVD) if every prime ideal of D is strongly prime. It was shown in [HH1, Proposition 1.1] that every valuation domain is a PVD.

In [DF, Proposition 2.1], it was proved that every PVD is a conducive domain. Hence it follows that every valuation domain is a conducive domain.

In [HHP], the notion of an *m-canonical ideal* (a *multiplicative canonical ideal*) was

introduced. A nonzero ideal I of D is called an *m-canonical ideal* of D if $J^{v(I)} = J$ for each nonzero ideal J of D. As easily seen, if I is an m-canonical ideal of D, then $I : I = D^{v(I)} = D$.

Lemma 11. Let D be a conducive domain and let I be a nonzero ideal of D. Then I is an m-canonical ideal if and only if $v(I) = \bar{d}$.

Proof. Since D is a conducive domain, I is an m-canonical ideal if and only if $v(I) \mid_{\mathbf{F}(D)}$, the restriction of $v(I)$ to $\mathbf{F}(D)$, is equal to the d-operation on D if and only if $v(I) = \bar{d}$. $\square$

Remark 12. Let D be an integral domain which is not necessarily conducive and let I be a nonzero ideal of D. If $v(I) = \bar{d}$, then I is necessarily an m-canonical ideal of D.

Proposition 13 ([P2, Remark 4.3]). Let V be a valuation domain. Then $v(P) = *_{(V_P)}$ for each $P \in \mathrm{Spec}(V)$.

Proof. Let M be the maximal ideal of V. Then, by [BHLP, Proposition 4.1], M is an m-canonical ideal of V. Then, by Lemma 11, $v(M) = \bar{d} = *_{(V)} = *_{(V_M)}$, since a valuation domain V is a conducive domain.

Next, let P be a non-maximal prime ideal of V. Then V_P is a valuation domain with maximal ideal $PV_P = P$. Hence it follows from Lemma 11 that $v(P) = v(PV_P) = \bar{d}_{V_P} = *_{(V_P)}$. $\square$

Proposition 14. Let T be an overring of D with $T \in \mathbf{F}(D)$. If we set $\bar{v}(T) = \delta_{T/D}(\bar{v}_T)$, then $\bar{v}(T) = v(I)$ for some ideal I of D, where $\bar{v}_T$ is the $\bar{v}$-operation on T.

Proof. Since $T \in \mathbf{F}(D)$, $dT = I \subseteq D$ for some $0 \neq d \in D$. Then, for each $E \in \mathbf{K}(D)$, $E^{\bar{v}(T)} = (ET)^{\bar{v}_T} = T : (T : ET) = T : (T : E) = (d^{-1}I : (d^{-1}I : E)) = d^{-1}I : (d^{-1}(I : E)) = dd^{-1}(I : (I : E)) = E^{v(I)}$ and hence $\bar{v}(T) = v(I)$. $\square$

We recall that an integral domain D is called an *h-local domain* if each nonzero prime ideal of D is contained in a unique maximal ideal of D and each nonzero ideal of D is contained in only finitely many maximal ideals of D [MA, p.11].

In [O2], a proper overring of D of the form $I : I$ (resp. $P : P$) for an ideal I (resp. a prime ideal P) of D is called a *conductor overring* (resp. *prime conductor overring*) of D and we also say that an integral domain D has *the* PC-*property* (resp. *the* PPC-*property*) if each proper overring of D is a conductor overring (resp. a prime conductor overring) of D. Any valuation domain has the PPC-property [O1, Theorem 1].

Proposition 15. Let D be an ideal-divisorial semistar domain. Then
(1) D has an m-canonical ideal of D and therefore D is an h-local domain;
(2) D has the PC-property.

Proof. (1) By hypothesis, $\bar{d} = v(I)$ for some nonzero ideal I of D and then I is an m-canonical ideal of D. Hence D is an h-local domain by [HHP, Proposition 2.4].

(2) Let R be a proper overring of D. Then $*_{(R)} = v(I)$ for some nonzero ideal I of D and then $R = DR = D^{*_{(R)}} = D^{v(I)} = I : (I : D) = I : I$. Hence D has the PC-property. $\square$

Proposition 16. If D is a prime-divisorial semistar domain, then D is an h-local domain and D has the PPC-property.

Proof. The proof is the same as that of Proposition 15. $\square$

An integral domain D is called a *divisorial domain* if each nonzero ideal I of D is divisorial, i.e., $I = I^v$. Hence D is a divisorial domain if and only if $d = v$. An integral domain D is said to be *totally divisorial* if each overring of D is divisorial. Note that D is a divisorial domain if and only if $\bar{f} = \bar{v}$, as $\bar{f} = d^e$ and $\bar{v} = v^e$.

Proposition 17 ([HHP, Proposition 3.6]). The following statements are equivalent:
(1) D is a divisorial domain;
(2) D has a principal m-canonical ideal;
(3) D has an invertible m-canonical ideal;
(4) D has a divisorial m-canonical ideal.

Proposition 18 ([MI, Proposition 2.2]). Let D be an integral domain. Then $\bar{v} = \bar{d}$ if and only if D is a conducive divisorial domain.

Proposition 19. Let D be a conducive integral domain. Then the following statements are equivalent:
(1) D is a totally divisorial domain.
(2) $*_{(T)} = \bar{v}(T)$ for each overring T of D.

Proof. (1) $\implies$ (2) Let T be an overring of D and assume that T is a divisorial domain. Then, since T is a conducive divisorial domain, $\bar{d}_T = \bar{v}_T$ by Proposition 18 and then $*_{(T)} = \delta_{T/D}(\bar{d}_T) = \delta_{T/D}(\bar{v}_T) = \bar{v}(T)$.

(1) $\impliedby$ (2) Let T be an overring of D and assume that $*_{(T)} = \bar{v}(T)$. Then $\bar{d}_T = \alpha_{T/D}(\delta_{T/D}(\bar{d}_T)) = \alpha_{T/D}(*_{(T)}) = \alpha_{T/D}(\bar{v}(T)) = \alpha_{T/D}(\delta_{T/D}(\bar{v}_T)) = \bar{v}_T$ and hence T is a divisorial domain. Thus D is a totally divisorial domain. $\square$

4. CHARACTERIZATIONS OF VALUATION DOMAINS

We recall that a valuation domain V is *discrete* if each branched prime ideal of V is not idempotent [G, p.192] and a valuation domain V is *strongly discrete* if each nonzero prime ideal of V is not idempotent[FHP, p.145]. It easily follows from these definitions that each strongly discrete valuation domain is a discrete valuation domain.

Proposition 20. Every strongly discrete valuation domain is a prime-divisorial

semistar domain.

Proof. This follows from [P2, Proposition 4.2 and Remark 4.3]. $\square$

Lemma 21. Let P and Q be prime ideals of an integral domain D such that $D_P = Q : Q$. Then $Q \subseteq P$ and $P : P \subseteq Q : Q = D_P \subseteq D_Q$.

Proof. If $D_P = Q : Q$, then $P(Q : Q) = PD_P$ is the maximal ideal of $Q : Q$ and therefore $Q \subseteq P(Q : Q) = PD_P$. Then $Q \subseteq PD_P \bigcap D = P$ and then, by [O1, Lemma 9 (1)], $P : P \subseteq Q : Q = D_P \subseteq D_Q$. $\square$

Lemma 22. Let P and Q be prime ideals of a divided domain D. If $D_P = Q : Q$, then $P = Q$ and so $D_P = P : P$.

Proof. By Lemma 21, $Q \subseteq P$ and $Q : Q \subseteq D_Q$. But, since D is a divided domain, by [O2, Theorem 2.2], $D_Q \subseteq Q : Q$. Hence we get $D_Q = Q : Q = D_P$ and therefore $P = Q$. Thus $D_P = P : P$ as wanted. $\square$

Let D be an integral domain. Then D is called an *almost valuation semistar domain* (or simply AVSD), if $v(P) = *_{(D_P)}$ for all $P \in \mathrm{Spec}(D)$. By Proposition 13, each valuation domain is an AVSD.

Lemma 23. Each almost valuation semistar domain is a divided domain.

Proof. For each $P \in \mathrm{Spec}(D)$, $D_P = D^{*_{(D_P)}} = D^{v(P)} = P : P$ and hence D is a divided domain by [O2, Theorem 2.2]. $\square$

Proposition 24. Let (D, M) be a quasi-local domain. Then D is an AVSD if and only if D is a valuation domain.

Proof. ($\Longrightarrow$) By hypothesis, $v(M) = *_{(D_M)} = *_{(D)} = \bar{d}$ and so, by Remark 12, M is an m-canonical ideal of D. Hence, by [BHLP, Proposition 4.1], D is a valuation domain.
($\Longleftarrow$) This follows from Proposition 13. $\square$

In this paper, a semistar operation $*$ is said to be *of extension type* if $* = *_{(R)}$ for some overring R of D, or equivalently, $* = *_{(D^*)}$, and a domain D is an *extension semistar domain* if each semistar operation on D is of extension type. We denote the set of all semistar operations of extension type on D by $\mathbf{SS}_e(D)$.

Proposition 25. Let D be an integral domain. Then the following statements are equivalent:
(1) D is an extension semistar domain.
(2) D is a conducive and totally divisorial domain.
(3) $\bar{v}_R = \bar{d}_R$ for each overring $R \subset K$ of D.

Proof. (1) $\Longrightarrow$ (2) This follows from [O6, Proposition 51 and Corollaries 61 and 62].

(2) $\Longrightarrow$ (1) This follows from [P2, Proposition 4.9 (ii) $\Rightarrow$ (i)].

(2) $\Longleftrightarrow$ (3) This follows from [DF, Lemma 2.0 (i)] and Proposition 18. $\square$

Corollary 26. If D is an extension semistar domain, then D is an ideal-divisorial semistar domain.

Proof. By Proposition 25, D is a conducive domain and so every overring T of D is a fractional ideal of D. Since D is also totally divisorial by Proposition 25, for each overring T of D, we have $*_{(T)} = \bar{v}(T) = v(I)$ for some ideal I of D by Propositions 14 and 19. Hence D is an ideal-divisorial semistar domain. $\square$

Proposition 27. Assume that D is an extension semistar domain. Then the following statements hold:

(1) Every proper overring of D is also an extension semistar domain.

(2) $| \, \mathbf{SS}(D) \, | = | \, \mathbf{SS}_f(D) \, | = | \, \mathcal{O}(D) \, |$.

(3) D is integrally closed if and only if D is a Prüfer domain.

Proof. (1) follows from [O6, Proposition 51], (2) follows from [O6, Proposition 54 (4)] and (3) follows from [O6, Corollary 63]. $\square$

Proposition 28. Each ideal-divisorial semistar domain is a conducive domain.

Proof. Let D be an ideal-divisorial semistar domain and let T be an overring of D such taht $T \neq K$. Then $*_{(T)} = v(I)$ for some nonzero ideal I of D and then $T = D^{*_{(T)}} = D^{v(I)} = I : I$. Hence I is an ideal of T and so $IT \subseteq I \subseteq D$. Therefore $D : T \neq (0)$, since $I \subseteq D : T$. $\square$

Proposition 29 (cf. [HHP, Lemma 3.1]). Let A and E be K-fractional ideals of D. Then $E^{v(A)} = A : (A : E) = \bigcap \{Au \mid u \in K \text{ and } E \subseteq Au\}$.

Proof. Let $x \in E^{v(A)} = A : (A : E)$ and $E \subseteq Au$. Then, since $u^{-1} \in A : E$, we get $xu^{-1} \in A$, that is, $x \in Au$. Hence $E^{v(A)} \subseteq \bigcap \{Au \mid u \in K \text{ and } E \subseteq Au\}$.

Conversely let $x \in \bigcap \{Au \mid u \in K \text{ and } E \subseteq Au\}$. If $t \in A : E$, then $E \subseteq At^{-1}$ and therefore $x \in At^{-1}$. i.e., $xt \in A$. Thus we get $x \in A : (A : E) = E^{v(A)}$. Hence $\bigcap \{Au \mid u \in K \text{ and } E \subseteq Au\} \subseteq E^{v(A)}$ also holds. $\square$

Proposition 30 (cf. [MI, Corollary 3.4]). Let D be an integral domain. Then the following statements are equivalent:

(1) D is a Prüfer domain.

(2) D is integrally closed and $\mathbf{SS}_f(D) = \mathbf{SS}_e(D)$.

Proof. (1) $\Longrightarrow$ (2) Each Prüfer domain is integrally closed and $\mathbf{SS}_f(D) = \mathbf{SS}_e(D)$ follows from [P2, Lemma 4.4].

(1) $\Longleftarrow$ (2) If we take $* = \bar{t}$, then, by hypothesis, $\bar{t} = *_{(D^{\bar{t}})} = *_{(D)} = \bar{d}$. Hence D

is a Prüfer domain by [G, Proposition 34.12]. □

Proposition 31 ([P2, Proposition 4.2]). Let P be a prime ideal of a valuation domain V.
    (1) If $P \neq P^2$, then $\mathbf{SS}(V, V_P) = \{*_{(V_P)}\}$.
    (2) If $P = P^2$, then $\mathbf{SS}(V, V_P) = \{*_{(V_P)}, \bar{v}(V_P)\}$, where $\bar{v}(V_P) = \delta_{V_P/V}(\bar{v}_{V_P})$.
    (3) $\mathbf{SS}(V) = \{*_{(V_P)} \mid P \in \mathrm{Spec}(V)\} \bigcup \{\bar{v}(V_Q) \mid Q \in \mathrm{Spec}(V) \text{ and } Q = Q^2\}$

Proposition 32 ([B2, Corollary 2.5]). Let (D, M) be a quasi-local domain. Then D is a valuation domain if and only if D admits a divided proper m-canonical ideal.

Theorem 33. Let (D, M) be a quasi-local domain. Then the following conditions sre equivalent:
    (1) D is a valuation domain.
    (2) D is a strongly divided domain and an ideal-divisorial semistar domain.

Proof. (1) $\Longrightarrow$ (2) First, it is easily seen that every proper ideal of a valuation domain is divided and so every valuation domain is strongly divided. Next, it follows from Propositions 13, 14 and 31 that every valuation domain is an ideal-divisorial semistar domain, since a valuation domain is conducive.
    (2) $\Longrightarrow$ (1) Assume that D is an ideal-divisorial semistar domain and a strongly divided domain. First, since D is an ideal-divisorial semistar domain, $\bar{d} = v(I)$ for some proper ideal I of D. Then I is a proper m-canonical ideal of D by Remark 12. Next, since D is a strongly divided domain, I is also a divided ideal of D. Thus D admits a divided proper m-canonical ideal I and so D is a valuation domain by Proposition 32. □

Proposition 34 ([BHLP, Proposition 4.1]). Let (D, M) be a quasi-local domain. Then M is an m-canonical ideal if and only if D is a valuation domain.

Lemma 35. Let (D, M) be a quasi-local domain. If D is a prime-divisorial semistar domain, then D is a valuation domain.

Proof. If D is a prime-divisorial semistar domain, then $\bar{d} = v(P)$ for some prime ideal P of D. Then P is an m-canonical ideal of D by Remark 12 and then P is a maximal ideal of D by [HHP, Lemma 2.2 (i)] and so $P = M$. Hence M is an m-canonical ideal of D and therefore D is a valuation domain by Proposition 34. □

Theorem 36. Let D be an integral domain. Then the following statements are equivalent.
    (1) D is a divided prime-divisorial semistar domain.
    (2) D is a strongly discrete valuation domain.

Proof. (1) $\Longrightarrow$ (2) Let D be a divided prime-divisorial semistar domain and let P be a prime ideal of D. Then, by hypothesis, $*_{(D_P)} = v(Q)$ for some $Q \in \mathrm{Spec}(D)$. Then $D_P = Q : Q$ and therefore $P = Q$ by Lemma 22. Thus $*_{(D_P)} = v(P)$ for all

$P \in \mathrm{Spec}(D)$. Hence D is a quasi-local almost valuation semistar domain and then D is a valuation domain by Proposition 24. Next we shall show that D is strongly discrete. By hypothesis, $\bar{v}(D_P) = v(Q)$ for some $Q \in \mathrm{Spec}(D)$. Then $D_P = (D_P)^{\bar{v}_{D_P}} = D^{\delta_{D_P/D}(\bar{v}_{D_P})} = D^{\bar{v}(D_P)} = D^{v(Q)} = Q : Q$ and so $P = Q$ by Lemma 22. Hence $\bar{v}(D_P) = v(P) = *_{(D_P)}$ for all $P \in \mathrm{Spec}(D)$. Then it follows from Proposition 31 that every prime ideal P of D is not idempotent and therefore D is strongly discrete.

(1) $\Longleftarrow$ (2) It is well-known that every valuation domain is a divided domain. Next, by Proposition 20, every strongly discrete valuation domain is a prime-divisorial semistar domain. $\square$

Remark 37. The fact that each divided prime-divisorial semistar domain is a valuation domain also follows from Lemma 35, since each divided domain is evidently quasi-local.

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