Math. J. Ibaraki Univ., 41 (2009), 15–38

Note on localizing systems and Kronecker function rings of semistar operations

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Abstract

We study the results of M. Fontana and J. Huckaba [FHu] on localizing systems and semistar operations, and give a couple of remarks for them. After M. Fontana and K.A. Loper [FL3], we study also Nagata rings, Kronecker function rings, and related semistar operations on semigroups.

This paper consists of §1 and §2. M. Fontana and J. Huckaba [FHu] established a natural bridge between localizing systems and semistar operations. In §1 of this paper, we will study their results, and will give a couple of remarks for them. §1 consists of 4-Parts. Part 1 contains preliminary results, and will review a part of [FHu]. Part 2 concerns with relations between finite type localizing systems and finite type semistar operations. We will give an answer to the problem ([FHu]): Characterize a localizing system $\mathcal{F}$ of D such that $\star_{(\mathcal{F}_f)} = (\star_{\mathcal{F}})_f$. In fact, we treat this problem for all localizing systems, and not for particular ones. Part 3 concerns with $\star$-invertible ideals for semistar operations $\star$. We will study a pseudo-valuation domain D, a quasi-spectral semistar operation $\star$, and $\star$-invertible ideals of D, and we will show that, if I is a $\star$-invertible ideal of D, then I need not be $\bar{\star}$-invertible. The proof of [FHu, Proposition 4.25] seems incomplete. We hear that such an ideal was also given in [FP]. Part 4 concerns with semistar operations which are spectral, quasi-spectral, qq-spectral, and fqq-spectral. We will give a condition for a semistar operation to be spectral. The proof of [FHu, Proposition 4.8] seems incomplete.

M. Fontana and K.A. Loper [FL3] investigated Nagata rings, Kronecker function rings, and related semistar operations. A subsemigroup $\ni 0$ of a torsion-free abelian additive group is said a grading monoid (or, a g-monoid). In §2 of this paper, after [FL3], we will study Nagata rings, Kronecker function rings, and related semistar operations on g-monoids, and will show that almost all statements in [FL3] hold for g-monoids. Since the structure of a g-monoid is simpler than that of a domain, it is expected that the semigroup versions of [FL3] are only straightforward translations from rings to semigroups. However, if or not the semigroup version §2, (1.6) of [FL3, Lemma 2.6] is valid is open. In Appendix, we will give a direct proof for the fact that,

Received 4 June, 2008; revised 15 January, 2009.
2000 *Mathematics Subject Classification.* 13A15.
Key Words and Phrases. localizing system, semistar operation, Kronecker function ring.
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for any integral domain D and for any semistar operation on D, the Kronecker function ring which was defined by M. Fontana, F. Halter-Koch and K.A. Loper ([FL1], [Ha]) is well-defined. Besides, we will show a similar result to [FL2, Theorem 3.5] for our Kronecker function ring $\mathrm{Kr}(D, \star, S)$.

§1 Localizing systems and semistar operations

First, we will review a part of [FHu]. The quotient field of an integral domain D is denoted by $\mathrm{q}(D)$. Let D be an integral domain with $K = \mathrm{q}(D)$. Let $\bar{\mathrm{F}}(D)$ be the set of non-zero D-submodules of K, let $\mathrm{F}(D)$ be the set of non-zero fractional ideals of D, and let $\mathrm{f}(D)$ be the set of non-zero finitely generated D-submodules of K. For every $E, F \in \bar{\mathrm{F}}(D)$, we define $(E : F) = \{x \in K \mid xF \subset E\}$ and $E^{-1} = (D : E)$.

If we set $E^d = E$ (resp., $E^e = K$) for every $E \in \bar{\mathrm{F}}(D)$, then the mapping $E \longmapsto E^d$ (resp., $E \longmapsto E^e$) is a semistar operation, and is called the d-semistar operation (resp., the e-semistar operation) on D. If we set $E^v = (E^{-1})^{-1}$ for every $E \in \bar{\mathrm{F}}(D)$, the mapping $E \longmapsto E^v$ is a semistar operation on D, and is called the v-semistar operation on D. Let T be an overring of D, and let $\star$ be a semistar operation on D. Then there is induced a canonical semistar operation $\alpha(\star)$ on T, and is called the ascent of $\star$ to T.

We say that a semistar operation $\star$ is stable if $(E \cap F)^\star = E^\star \cap F^\star$ for every $E, F \in \bar{\mathrm{F}}(D)$.

A semistar operation $\star$ on D is said of finite type if, for every $E \in \bar{\mathrm{F}}(D)$, $E^\star = \cup\{F^\star \mid F \in \mathrm{f}(D) \text{ with } F \subset E\}$.

For every semistar operation $\star$ on D, a semistar operation $\star_f$ of finite type can be defined in the following way: For every $E \in \bar{\mathrm{F}}(D)$, $E^{\star_f} = \cup\{F^\star \mid F \in \mathrm{f}(D) \text{ with } F \subset E\}$.

We set $v_f = t$. Let $\star_1, \star_2$ be semistar operations on D. If $E^{\star_1} \subset E^{\star_2}$ for every $E \in \bar{\mathrm{F}}(D)$, we set $\star_1 \leq \star_2$.

Let Δ be a non-empty subset of $\mathrm{Spec}(D) - \{(0)\}$. For every $E \in \bar{\mathrm{F}}(D)$, define $E^{\star_\Delta} = \cap\{ED_P \mid P \in \Delta\}$. Then the mapping $E \longmapsto E^{\star_\Delta}$ is a semistar operation on D. A semistar operation $\star$ on D is said spectral, if there is a non-empty subset $\Delta \subset \mathrm{Spec}(D) - \{(0)\}$ such that $\star = \star_\Delta$.

A semistar operation $\star$ on D is said quasi-spectral, if for every non-zero ideal I of D such that $I^\star \not\ni 1$, there is a non-zero prime ideal P with $I \subset P$ such that $P^\star \cap D = P$.

We note that a localizing system $\mathcal{F}$ of D is non-empty and $\mathcal{F} \not\ni (0)$ by definition.

If $\star$ is a semistar operation on D, we consider the following localizing system $\mathcal{F}^\star = \{I \mid I \text{ is an ideal of } D \text{ with } I^\star \ni 1\}$.

If $\star = e$, then $\mathcal{F}^\star = \{I \mid I \text{ is a non-zero ideal of } D\}$.

Let $\star$ be a semistar operation on D, and let $\Pi^\star = \{P \in \mathrm{Spec}(D) - \{(0)\} \mid P^\star \not\ni 1\}$. If the set $\Pi^\star$ is non-empty, we consider the semistar operation $\star_{sp} = \star_{\Pi^\star}$.

If $\mathcal{F}$ is a localizing system of D, we consider the semistar operation $\star_\mathcal{F}$: For every $E \in \bar{\mathrm{F}}(D), E^{\star_\mathcal{F}} = \cup\{(E : I) \mid I \in \mathcal{F}\}$.

If $\mathcal{F} = \{I \mid I \text{ is a non-zero ideal of } D\}$, then $\star_\mathcal{F} = e$.

Let Δ be a non-empty subset of $\mathrm{Spec}(D) - \{(0)\}$. Set $\mathcal{F}(\Delta) = \{I \mid I \text{ is an ideal of } D \text{ with } I \not\subset P \text{ for each } P \in \Delta\}$. Then $\mathcal{F}(\Delta)$ is a localizing system of D. A localizing system $\mathcal{F}$ is said spectral, if there is a non-empty subset $\Delta \subset \mathrm{Spec}(D) - \{(0)\}$ such that $\mathcal{F} = \mathcal{F}(\Delta)$.

A localizing system $\mathcal{F}$ is said finitely spectral if, for every finitely generated ideal $I \notin \mathcal{F}$, there is a prime ideal $P \notin \mathcal{F}$ such that $P \supset I$.

If a localizing system $\mathcal{F}$ is spectral, then $\mathcal{F}$ is finitely spectral.

A localizing system $\mathcal{F}$ of D is said of finite type if, for every $I \in \mathcal{F}$, there is a non-zero finitely generated ideal $J \in \mathcal{F}$ with $J \subset I$.

Given a localizing system $\mathcal{F}$ of D, we consider the following localizing system of finite type $\mathcal{F}_f$:

$\mathcal{F}_f = \{I \in \mathcal{F} \mid$ There is a non-zero finitely generated ideal $J \in \mathcal{F}$ with $J \subset I\}$.

If $\star$ is a semistar operation on D, we consider the semistar operations $\bar{\star} = \star_{\mathcal{F}^\star}$ and $\tilde{\star} = \star_{(\mathcal{F}^\star)_f}$. We have

$E^{\bar{\star}} = \cup\{(E : I) \mid I$ is a non-zero ideal of D with $I^\star \ni 1\}$ for every $E \in \bar{\mathrm{F}}(D)$

and

$E^{\tilde{\star}} = \cup\{(E : I) \mid I$ is a finitely generated non-zero ideal of D with $I^\star \ni 1\}$ for every $E \in \bar{\mathrm{F}}(D)$.

The following (0.1) $\sim$ (0.3) are results in [FHu].

(0.1) Let $\star$ be a semistar operation on D, and let $\mathcal{F}$ be a localizing system of D.

(1) $\mathcal{F}^\star = \mathcal{F}^{\bar{\star}}$.
(2) $\mathcal{F}^{\star_f} = (\mathcal{F}^\star)_f$.
(3) $\bar{\star} \leq \star$.
(4) $\tilde{\star} \leq \star_f$.
(5) If $\star$ is of finite type, then $\star$ is quasi-spectral.
(6) If $\star$ is spectral, then $\star = \bar{\star}$.
(7) If $\star$ is of finite type, then $\bar{\star}$ is of finite type.
(8) $\star$ is stable if and only if $\star = \bar{\star}$.
(9) $\star$ is spectral if and only if $\star$ is quasi-spectral and stable.
(10) $\overline{\star_f} = \tilde{\star}$.
(11) $\mathcal{F} = \mathcal{F}^{\star_{\mathcal{F}}}$.
(12) If $\mathcal{F}$ is of finite type, then $\star_{\mathcal{F}}$ is of finite type.
(13) If $\star$ is of finite type, then $\mathcal{F}^\star$ is of finite type.
(14) $\star_{\mathcal{F}}$ is stable.
(15) If $\star$ is spectral, then $\mathcal{F}^\star$ is spectral.
(16) If $\mathcal{F}$ is spectral, then $\star_{\mathcal{F}}$ is spectral.

(0.2) (1) Let Δ be a non-empty subset of $\mathrm{Spec}(D) - \{(0)\}$. Then $\star_\Delta = \star_{\mathcal{F}(\Delta)}$ and $\mathcal{F}^{\star_\Delta} = \mathcal{F}(\Delta)$.

(2) Let $\star_1, \star_2$ be semistar operations on D such that $\star_1 \leq \star_2$. Then $\overline{\star_1} \leq \overline{\star_2}$.

(0.3) Assume that $\Pi^\star \neq \emptyset$.

(1) $\bar{\star} \leq \star_{sp}$.
(2) If $\star$ is spectral, then $\mathcal{F}^\star = \mathcal{F}(\Pi^\star)$ and $\bar{\star} = \star_{sp}$.
(3) If $\star$ is quasi-spectral, then $\star_{sp} = \bar{\star}$.
(4) $\star$ is spectral if and only if $\star = \star_{sp}$.

Now, we will study the following,

(1.1) ([FHu]) Characterize a localizing system $\mathcal{F}$ such that $\star_{\mathcal{F}_f} = (\star_{\mathcal{F}})_f$.

(1.2) Let $\mathcal{F}$ be a localizing system of D.
(1) For every element $E \in \bar{\mathrm{F}}(D)$, we have
$E^{\star_{\mathcal{F}}} = \cup\{(E : I) \mid I \in \mathcal{F}\}$.
(2) For every element $E \in \mathrm{f}(D)$, we have
$E^{(\star_{\mathcal{F}})_f} = \cup\{(E : I) \mid I \in \mathcal{F}\}$
and
$E^{\star(\mathcal{F}_f)} = \cup\{(E : I) \mid I$ is a finitely generated ideal of D with $I \in \mathcal{F}\}$.

Proof. $E^{\star(\mathcal{F}_f)} = \cup\{(E : J) \mid J \in \mathcal{F}_f\}$
$= \cup\{(E : J) \mid$ There is a finitely generated ideal $I \in \mathcal{F}$ with $I \subset J\}$
$= \cup\{(E : I) \mid I$ is a finitely generated ideal with $I \in \mathcal{F}\}$.

(1.3) ([FHu, Example 3.5]) There is a domain D and a localizing system $\mathcal{F}$ of D such that $\star_{\mathcal{F}_f} \neq (\star_{\mathcal{F}})_f$.

(1.4) ([FHu, Proposition 3.3]) For every localizing system $\mathcal{F}$, we have $\star_{\mathcal{F}_f} \leq (\star_{\mathcal{F}})_f$.

(1.5) ([M2, Lemma 7]) Let $\mathcal{F}$ be a localizing system of D, and let $\star = \star_{\mathcal{F}}$.
(1) $(\mathcal{F}^{\star})_f = \mathcal{F}^{\star_f}$.
(2) $\star_{\mathcal{F}_f} = \overline{\star_f}$.
(3) $(\star_{\mathcal{F}})_f = (\bar{\star})_f$.

(1.6) ([M2, Proposition 2]) Let $\mathcal{F}$ be a localizing system of D, and let $\star = \star_{\mathcal{F}}$. The following conditions are equivalent.
(1) $\star_{\mathcal{F}_f} = (\star_{\mathcal{F}})_f$.
(2) $\overline{\star_f} = (\bar{\star})_f$.
(3) For every element $E \in \mathrm{f}(D)$, we have
$\cup\{(E : I) \mid I$ is a finitely generated ideal of D with $I^{\star} \ni 1\}$
$= \cup\{(E : I) \mid I$ is an ideal of D with $I^{\star} \ni 1\}$.

(1.7) Let $\mathcal{F}$ be a localizing system of D, and let $\star = \star_{\mathcal{F}}$.
(1) $\star_f = e$ if and only if $\mathcal{F} = \{I \mid I$ is a non-zero ideal of $D\}$.
(2) If $\star_f = e$, then $\star_{\mathcal{F}_f} = (\star_{\mathcal{F}})_f$.
(3) If $\star_f \neq e$, then $\Pi^{(\star_{\mathcal{F}})_f} \neq \emptyset$, hence $((\star_{\mathcal{F}})_f)_{sp}$ is well-defined.

Proof. (1) For, $\mathcal{F} = \mathcal{F}^{\star}$ by (0.1)(11).
(2) Then $\mathcal{F} = \{I \mid I$ is a non-zero ideal of $D\}$ by (1). Hence $\mathcal{F}_f = \mathcal{F}$ and $\star_{\mathcal{F}} = e$.
(3) There is an ideal $I \notin \mathcal{F}$ by (1). Hence the set $\{I \mid I$ is an ideal with $I^{\star_f} \not\ni 1\} = X$ is non-empty. By Zorn's Lemma, X has a maximal member P. Then P is a prime ideal of D, and $P \in \Pi^{\star_f}$.

(1.8) Assume that $\Pi^{\star} \neq \emptyset$. Then we have $(\star_f)_{sp} = \overline{\star_f} \leq \star_f$ and $((\star_f)_{sp})_f =$

$(\star_f)_{sp}$.

Proof. $\star_f$ is quasi-spectral by (0.1)(5). Then $(\star_f)_{sp} = \overline{\star_f}$ by (0.3)(3), and $(\star_f)_{sp}$ is of finite type by (0.1)(7).

By (1.7), if $\mathcal{F} = \{I \mid I$ is a non-zero ideal of $D\}$, then we have $\star_{\mathcal{F}_f} = (\star_{\mathcal{F}})_f$ trivially. And, if otherwise, $((\star_{\mathcal{F}})_f)_{sp}$ is well-defined.

(1.9) Proposition Let $\mathcal{F}$ be a localizing system of D with $\mathcal{F} \subsetneqq \{I \mid I$ is a non-zero ideal of $D\}$, and let $\star = \star_{\mathcal{F}}$. The following conditions are equivalent.

(1) $\star_{\mathcal{F}_f} = (\star_{\mathcal{F}})_f$.

(2) $\overline{\star_f} = (\bar{\star})_f$.

(3) For every element $E \in \mathrm{f}(D)$ and for every ideal I with $I^\star \ni 1$, we have $(E : I) \subset \cup\{(E : J) \mid J$ is a finitely generated ideal with $J^\star \ni 1\}$.

(4) For every ideal $J \in \mathrm{f}(D)$ and for every ideal I with $I^\star \ni 1$, we have $(J : I) \subset \cup\{(J : E) \mid E$ is a finitely generated ideal with $E^\star \ni 1\}$.

(5) For every element $E \in \mathrm{f}(D)$ and for every ideal I with $I^\star \ni 1$ such that $I \subset E$, we have $J \subset E$ for some finitely generated ideal J with $J^\star \ni 1$.

(6) $\star_f$ is stable.

(7) $\star_f$ is spectral.

(8) $(\star_f)_{sp} = \star_f$.

(9) For every element $E \in \mathrm{f}(D)$ and for every ideal $I \in \mathcal{F}$, we have $(E : I) \subset \cup\{(E : J) \mid J$ is a finitely generated ideal with $J \in \mathcal{F}\}$.

(10) For every ideal $J \in \mathrm{f}(D)$ and for every ideal $I \in \mathcal{F}$, we have $(J : I) \subset \cup\{(J : E) \mid E$ is a finitely generated ideal with $E \in \mathcal{F}\}$.

(11) For every element $E \in \mathrm{f}(D)$ and for every ideal $I \in \mathcal{F}$ such that $I \subset E$, we have $J \subset E$ for some finitely generated ideal $J \in \mathcal{F}$.

(12) For every element $E \in \mathrm{f}(D)$ with $E^\star \ni 1$, there is a finitely generated ideal I with $I^\star \ni 1$ such that $I \subset E$.

Proof. (1), (2),(3) are equivalent by (1.6).

(4) $\Longrightarrow$ (3): Let $x \in (E : I)$. There is an element $d \in D - \{0\}$ such that $dE \subset D$. Since $dx \in (dE : I)$, there is a finitely generated ideal J with $J^\star \ni 1$ such that $dx \in (dE : J)$. Then we have $x \in (E : J)$.

(3) $\Longrightarrow$ (4): Trivial.

(5) $\Longrightarrow$ (3): Let $0 \neq x \in (E : I)$. Then we have $I \subset \dfrac{1}{x}E$. Hence there is a finitely generated ideal J with $J^\star \ni 1$ such that $J \subset \dfrac{1}{x}E$. Then $x \in (E : J)$.

(3) $\Longrightarrow$ (5): Since $1 \in (E : I)$, we have $1 \in (E : J)$ for some finitely generated ideal J with $J^\star \ni 1$. Then $J \subset E$.

(6) $\Longrightarrow$ (1): $\mathcal{F}_f = \mathcal{F}^{\star_f}$ by (0.1)(2). Then $\star_{\mathcal{F}_f} = \star_f$ by (0.1)(8).

(1) $\Longrightarrow$ (6): By (0.1)(14).

(7) $\Longrightarrow$ (6): $\star_f = \overline{\star_f}$ by (0.1)(6), and $\star_f$ is stable by (0.1)(8).

(6) $\Longrightarrow$ (7): $\star_f$ is quasi-spectral by (0.1)(5). Then $\star_f$ is spectral by (0.1)(9).

(8) $\Longleftrightarrow$ (7): By (0.3)(4).

(9) $\Longleftrightarrow$ (3), (10) $\Longleftrightarrow$ (4), and, (11) $\Longleftrightarrow$ (5): Because $\mathcal{F} = \mathcal{F}^\star$ by (0.1)(11).

(6) $\Longrightarrow$ (12): Because $1 \in E^\star \cap D^\star = E^{\star_f} \cap D^{\star_f} = (E \cap D)^{\star_f}$.

(12) $\Longrightarrow$ (5): Trivial.

We note that (1) $\Longleftrightarrow$ (6) in (1.9) Proposition was proved in [O, Theorem 6].

Now, we will study a pseudo-valuation domain D, a quasi-spectral semistar operation $\star$ on D, and $\star$-invertible ideals of D.

(2.1) ([FHu, Proposition 4.25]) Let $\star$ be a quasi-spectral semistar operation on D, and let I, J be ideals of D.

(1) $(IJ)^\star = D^\star$ if and only if $(IJ)^{\bar\star} = D^{\bar\star}$.

(2) Assume that $\mathcal{F}^\star = \{D\}$. Then $(IJ)^\star = D^\star$ if and only if $I = J = D$.

Proof. (1) The sufficiency: We have $\bar\star \leq \star$ by (0.1)(3), and hence $(IJ)^\star = D^\star$.

The necessity: Suppose that $(IJ)^{\bar\star} \subsetneqq D^{\bar\star}$. Then we have $IJ \notin \mathcal{F}^{\bar\star}$. Since $\mathcal{F}^\star = \mathcal{F}^{\bar\star}$ by (0.1)(1), we have $IJ \notin \mathcal{F}^\star$. Hence there is a prime ideal P with $P^\star \not\ni 1$ such that $P \supset IJ$. It follows that $(IJ)^\star \subset P^\star \subsetneqq D^\star$; a contradiction.

(2) If $(IJ)^\star = D^\star$, then $IJ \in \mathcal{F}^\star$, hence $IJ = D$.

(2.2) ([FHu, Corollary 4.26]) Let $\star$ be a semistar operation on D, and let I, J be ideals of D.

(1) $(IJ)^{\star_f} = D^{\star_f}$ if and only if $(IJ)^{\tilde\star} = D^{\tilde\star}$.

(2) $(IJ)^t = D^t$ if and only if $(IJ)^{\tilde v} = D^{\tilde v}$.

Proof. $\overline{\star_f} = \tilde\star$ by (0.1)(10). $\star_f$ is quasi-spectral by (0.1)(5). Then we may apply (2.1).

If, for every ideal I with $I^\star \not\ni 1$, there is a prime ideal P with $P^\star \not\ni 1$ such that $P \supset I$, then $\star$ is said a qq-spectral semistar operation (or, a quasi-quasi-spectral semistar operation).

Every quasi-spectral semistar operation is a qq-spectral semistar operation.

(2.3) Let $\star$ be a qq-spectral semistar operation on D, and let I, J be ideals of D.

(1) $(IJ)^\star = D^\star$ if and only if $(IJ)^{\bar\star} = D^{\bar\star}$.

(2) Assume that $\mathcal{F}^\star = \{D\}$. Then $(IJ)^\star = D^\star$ if and only if $I = J = D$.

The proof is similar to that of (2.1).

An element $E \in \bar{\mathrm{F}}(D)$ is said $\star$-invertible if there is an element $F \in \bar{\mathrm{F}}(D)$ such that $(EF)^\star = D^\star$. If E is d-invertible, then E is said invertible.

Set $\mathrm{Inv}^\star(D) = \{E \in \bar{\mathrm{F}}(D) \mid E \text{ is } \star\text{-invertible}\}$, and set $\mathrm{Princ}(D) = \{xD \mid x \in K - \{0\}\}$. $\mathrm{Inv}^\star(D)$ forms a group under a canonical product, and $\mathrm{Prin}(D)$ is a subgroup of $\mathrm{Inv}^\star(D)$. Then the quotient group $\mathrm{Cl}^\star(D) = \dfrac{\mathrm{Inv}^\star(D)}{\mathrm{Princ}(D)}$ is said the $\star$-class group of D.

(2.4) Let $E \in \bar{\mathrm{F}}(D)$. If E is $\bar{\star}$-invertible, then E is $\star$-invertible. If E is $\tilde{\star}$-invertible, then E is $\star_f$-invertible. If E is $\tilde{v}$-invertible, then E is t-invertible.

For, $\bar{\star} \leq \star$ by (0.1)(3), and $\tilde{\star} \leq \star_f$ by (0.1)(4).

(2.5) Let $\star$ be a semistar operation with $D^\star = D$, and let $E \in \bar{\mathrm{F}}(D)$.
(1) E is $\star$-invertible if and only if E is $\bar{\star}$-invertible.
(2) Assume that $\mathcal{F}^\star = \{D\}$. Then E is $\star$-invertible if and only if E is invertible.

Proof. Let $F \in \bar{\mathrm{F}}(D)$ such that $(EF)^\star = D$. Then $EF \in \mathcal{F}^\star$. Since $\mathcal{F}^\star = \mathcal{F}^{\bar{\star}}$ by (0.1)(1), we have $EF \in \mathcal{F}^{\bar{\star}}$. Hence $(EF)^{\bar{\star}} = D$.

(2.6) (cf. [K, Theorem 59]) Assume that D is a quasi-local domain, that is, D has a unique maximal ideal. Then every invertible ideal of D is principal.

Let I be an ideal of D. If, for every element $a, b \in K$, $ab \in I$ and $b \notin I$ imply $a \in I$, then I is called strongly prime. If every prime ideal of D is strongly prime, then D is called a pseud-valuation domain (or, a PVD). We refer to Hedstrom-Houston ([HeHo]) for the notion of a PVD. Thus, every PVD is a quasi-local domain, and if D is a PVD with maximal ideal M, then $V = (M : M)$ is a valuation overring of D with maximal ideal M.

(2.7) Let $\star$ be a quasi-spectral semistar operation on D, and let I be a non-zero ideal of D.
(1) If I is $\star$-invertible, then I need not be $\bar{\star}$-invertible.
(2) If $\mathcal{F}^\star = \{D\}$, and if I is $\star$-invertible, then I need not be invertible.

For a counter example, let D be a PVD which is not a valuation domain, let M be the maximal ideal of D, let $V = (M : M)$, and let $\star$ be the semistar operation $E \longmapsto EV$ on D. Then V is a valuation domain, $M^\star = M, D^\star = V, \star$ is quasi-spectral, $\mathcal{F}^\star = \{D\}$, and $E^{\bar{\star}} = (E : D) = E$ for every $E \in \bar{\mathrm{F}}(D)$. Since D is not a valuation domain, there are elements $a, b \in D - \{0\}$ such that $\dfrac{a}{b} \notin D$ and $\dfrac{b}{a} \notin D$. Then $I = (a, b)$ is not a principal ideal of D. Since IV is a finitely generated ideal of V, we have $IV = xV$ for some element $x \in K - \{0\}$. Then $(Ix^{-1})^\star = V = D^\star$, that is, I is a $\star$-invertible ideal of D. Suppose that I is $\bar{\star}$-invertible. There is an element $E \in \bar{\mathrm{F}}(D)$ such that $(IE)^{\bar{\star}} = D^{\bar{\star}}$, that is, $IE = D$. Then (2.6) implies that I is a principal ideal of D; a contradiction.

(2.8) Let $\star$ be a semistar operation on D with $D^\star = D$.
(1) $\mathrm{Cl}^{\star_f}(D) = \mathrm{Cl}^{\tilde{\star}}(D)$.
(2) $\mathrm{Cl}^{t}(D) = \mathrm{Cl}^{\tilde{v}}(D)$.

Proof. $\overline{\star_f} = \tilde{\star}$ by (0.1)(10). Then we may apply (2.5).

(2.9) Let $\star$ be a semistar operation on D. Then $\mathrm{Cl}^{\star_f}(D) = \mathrm{Cl}^{\tilde{\star}}(D)$ need not be true.

For a counter example, let $D, \star, I$ be those in the counter example of (2.7). Then we have $\star = \star_f$. Let J be an ideal of D with $J^\star \ni 1$. Since $M^\star = M$, we have $J = D$. It follows that $E^{\tilde{\star}} = E$ for every element $E \in \bar{\mathrm{F}}(D)$. The ideal $I = (a, b)$ is $\star_f$-invertible. Suppose that I is $\tilde{\star}$-invertible. There is an element $F \in \bar{\mathrm{F}}(D)$ such that $(IF)^{\tilde{\star}} = D^{\tilde{\star}}$. Then $IF = D$. (2.6) implies a contradiction.

Now, we will study conditions for a semistar operation to be spectral.

(3.1) Proposition Let $\star$ be a semistar operation on D with $\Pi^\star \neq \emptyset$. The following conditions are equivalent.
(1) $\star_{sp} \leq \star$.
(2) $\star$ is qq-spectral.
(3) $E^\star = \cap\{E^\star D_P \mid P \in \Pi^\star\}$ for every element $E \in \bar{\mathrm{F}}(D)$.

Proof. Let $\Pi^\star = \{P_\lambda \mid \lambda \in \Lambda\}$.
(1) $\Longrightarrow$ (2): Let I be an ideal of D such that $I^\star \not\ni 1$. Since $\star_{sp} \leq \star$, we have $I^{\star_{sp}} \not\ni 1$. Hence we have $I \subset P_\lambda$ for some λ, and hence $\star$ is qq-spectral.
(2) $\Longrightarrow$ (3): Suppose that there is an element $z \notin E^\star$ such that $z \in \cap\{E^\star D_{P_\lambda} \mid \lambda \in \Lambda\}$. If we set $J = (E^\star z^{-1}) \cap D$, then $J^\star \not\ni 1$. For every λ, we have $z = \dfrac{x_\lambda}{y_\lambda}$ for some element $x_\lambda \in E^\star$ and for some element $y_\lambda \in D - P_\lambda$. It follows that $J \not\subset P_\lambda$, and that $\star$ is not qq-spectral; a contradiction.
(3) $\Longrightarrow$ (1): $E^{\star_{sp}} = \cap\{ED_{P_\lambda} \mid \lambda \in \Lambda\} \subset \cap\{E^\star D_{P_\lambda} \mid \lambda \in \Lambda\} = E^\star$.

For every semistar operation $\star$, $\star_{sp}$ is spectral by the definition. Hence $\star_{sp} = \overline{\star_{sp}}$ by (0.1)(6).

(3.2) Proposition Assume that $\Pi^\star \neq \emptyset$. The following conditions are equivalent.
(1) $\star$ is qq-spectral.
(2) $\star_{sp} = \bar{\star}$.
(3) $\bar{\star}$ is spectral.
(4) $\mathcal{F}^\star = \mathcal{F}(\Pi^\star)$.
(5) $\mathcal{F}^\star$ is spectral.

Proof. (1) $\Longrightarrow$ (2): $\star_{sp} \leq \star$ by (3.1), hence $\overline{\star_{sp}} \leq \bar{\star}$ by (0.2)(2). On the other hand, $\bar{\star} \leq \star_{sp}$ by (0.3)(1), and $\star_{sp} = \overline{\star_{sp}}$ by (0.1)(6). Then we have $\star_{sp} = \overline{\star_{sp}} \leq \bar{\star} \leq \star_{sp}$.
(2) $\Longrightarrow$ (4): By (0.1)(1) and (0.2)(1), we have
$\mathcal{F}^\star = \mathcal{F}^{\bar{\star}} = \mathcal{F}^{\star_{sp}} = \mathcal{F}^{\star(\Pi^\star)} = \mathcal{F}(\Pi^\star)$.
(4) $\Longrightarrow$ (5): Trivial.
(5) $\Longrightarrow$ (3): There is a non-empty subset $\Delta \subset \mathrm{Spec}(D) - \{(0)\}$ such that $\mathcal{F}^\star = \mathcal{F}(\Delta)$. Then (0.2)(1) implies that
$\bar{\star} = \star_{\mathcal{F}^\star} = \star_{(\mathcal{F}(\Delta))} = \star_\Delta$.
(3) $\Longrightarrow$ (1): $\mathcal{F}^\star = \mathcal{F}^{\bar{\star}}$ by (0.1)(1), and $\mathcal{F}^{\bar{\star}} = \mathcal{F}(\Pi^{\bar{\star}})$ by (0.3)(2), and hence

$\mathcal{F}^\star$ is spectral. Set $\Delta = \Pi^{\bar{\star}}$, and hence $\mathcal{F}^\star = \mathcal{F}(\Delta)$. If I is an ideal with $I^\star \not\ni 1$, then $I \notin \mathcal{F}^\star = \mathcal{F}(\Delta)$. Hence there is a prime ideal $P \in \Delta$ such that $I \subset P$. Since $P \not\ni \mathcal{F}(\Delta) = \mathcal{F}^\star$, we have $P^\star \not\ni 1$.

(3.3) Proposition Assume that $\Pi^\star \neq \emptyset$. If $\star$ is qq-spectral and stable, then $\star$ is spectral.

Proof. $\bar{\star}$ is spectral by (3.2), and $\star = \bar{\star}$ by (0.1)(8). Hence $\star$ is spectral.

(3.4) Assume that $\Pi^\star \neq \emptyset$. If $\star$ is qq-spectral, is $\star$ quasi-spectral?

(3.5) Assume that $\Pi^\star \neq \emptyset$, and that $\star$ is qq-spectral. If $\star$ is of finite fype, or if $\dim(D) < \infty$, then $\star$ is quasi-spectral.

Proof. Let I be an ideal of D with $I^\star \not\ni 1$. Then the set $X = \{P \in \Pi^\star \mid P \supset I\}$ is non-empty. If $\dim(D) < \infty$, obviously X has a maximal member. If $\star$ is of finite type, we may use Zorn's Lemma to find a maximal member in X. Let P be a maximal member in X. Since $P^\star \not\ni 1$, there is a prime ideal Q with $Q^\star \not\ni 1$ such that $Q \supset P^\star \cap D$. By the choice of P, we have $Q = P$. It follows that $P^\star \cap D = P$.

(3.6) ([FHu]) Is there an example of a finitely spectral non-spectral localizing system distinct with $\{I \mid I \text{ is a non-zero ideal of } D\}$?

If, for every finitely generated ideal I of D with $I^\star \not\ni 1$, there is a prime ideal P with $P^\star \not\ni 1$ such that $P \supset I$, then $\star$ is said a fqq-spectral semistar operation (or, a finitely quasi-quasi-spectral semistar operation).

Every qq-spectral semistar operation is a fqq-spectral semistar operation.

(3.7) Let $\star = \star_{\mathcal{F}}$. The following conditions are equivalent.
(1) $\mathcal{F}$ is finitely spectral.
(2) $\star$ is a fqq-spectral semistar operation.

Proof. (1) $\Longrightarrow$ (2): Let I be a finitely generated ideal with $I^\star \not\ni 1$. Since $\mathcal{F} = \mathcal{F}^\star$ by (0.1)(11), we have $I \notin \mathcal{F}$. Hence there is a prime ideal $P \notin \mathcal{F}$ such that $I \subset P$. Since $P \notin \mathcal{F}^\star$, we have $P^\star \not\ni 1$.

The proof of (2) $\Longrightarrow$ (1) is similar.

(3.8) Proposition Let $\mathcal{F}$ be a localizing system of D and let $\star = \star_{\mathcal{F}}$. The following conditions are equivalent.

(1) $\mathcal{F}$ is a finitely spectral non-spectral localizing system distinct with $\{I \mid I$ is a non-zero ideal of $D\}$.

(2) $\star$ is a non-spectral fqq-spectral semistar operation distinct with e.

Proof. (1) $\Longrightarrow$ (2): $\star$ is non-spectral by (0.1)(15). $\star$ is a fqq-spectral semistar operation by (3.7). Clearly, $\star \neq e$. And, $\star$ is stable by (0.1)(14).

The proof of (2) $\Longrightarrow$ (1) is similar.

(3.8) shows that (3.6) is equivalent to the following,

(3.9) Is there a stable non-spectral fqq-spectral semistar operation distinct with e?

(3.10) Let $\star$ be a fqq-spectral semistar operation on D distinct with e. Then we have $\Pi^\star \neq \emptyset$.

Proof. Then we have $D^\star \subsetneqq K$. Hence there is an element $a \in D - \{0\}$ such that $aD^\star \not\ni 1$. Then there is a prime ideal P with $P^\star \not\ni 1$ such that $P \supset aD$.

(3.10) shows that if $\star$ is fqq-spectral distinct with e, then $\star_{sp}$ is well-defined.

(3.11) An example of a domain D, a semistar operation $\star$ on D, a maximal ideal M of D such that $M \subsetneqq M^\star \not\ni 1$.

Example: Let k be a field, let x be an indeterminate over k, and let $T = k[x]$. Let $D = k[x^2, x^4, x^5]$, and let $M = (x^2, x^4, x^5)$ be a maximal ideal of D. Let $\star$ be a semistar operation $E \longmapsto ET$ on D. Then we have $M \not\ni x^3 \in MT = M^\star \not\ni 1$.

(3.12) Assume that, for each prime ideal P in $\Pi^\star$, P is a maximal ideal of some overring T of D. Then, if $\star$ is qq-spectral, then $\star$ is quasi-spectral.

Proof. Let I be an ideal of D with $I^\star \not\ni 1$. Then there is a prime ideal P of D with $P^\star \not\ni 1$ such that $P \supset I$. There is an overring T of D with maximal ideal P. We have $P^\star T \subset (P^\star T)^\star = (PT)^\star = P^\star$, hence $P^\star$ is a T-module. Since $P^\star \not\ni 1$, it follows that $P^\star \cap T = P$, and $P^\star \cap D = P$.

For every element $a, b \in K$, if $ab \in I$ and $b \notin I$ imply $a^n \in I$ for some positive integer n, then I is called strongly primary. If every prime ideal of D is strongly primary, then D is called an almost pseudo-valuation domain (or, an APVD). We refer to Badawi-Houston ([BHo]) for the notion of an APVD. Thus, every APVD is a quasi-local domain. Let M be the maximal ideal of D. Then $V = (M : M)$ is a valuation domain, M is a primary ideal of V, and M is primary to the maximal ideal of V. The set of non-maximal prime ideals of D coincides with the set of non-maximal prime ideals of V.

(3.13) Let D be an APVD. Then every qq-spectral semistar operation $\star$ on D is a quasi-spectral semistar operation.

Proof. Let P be a prime ideal in $\Pi^\star$. Assume that P is not a maximal ideal of D. Then P is a prime ideal of the valuation domain $V = (M : M)$, where M is the maximal ideal of D. It follows that P is the maximal ideal of the valuation domain V_P. Then (3.12) completes the proof.

(3.14) The following conditions are equivalent.

(1) Assume that $\Pi^\star \neq \emptyset$ and that $\star$ is qq-spectral. Then $\star$ is quasi-spectral.

(2) Assume that $\Pi^\star \neq \emptyset$ and that $\star$ is qq-spectral with $D^\star = D$. Then $\star$ is quasi-spectral.

Proof. (2) $\Longrightarrow$ (1): $\star$ induces a canonical semistar operation $\star'$ on $D^\star$. Since $(D^\star)^{\star'} = D^\star$, $\star'$ is quasi-spectral. Let I be a non-zero ideal of D with $I^\star \not\ni 1$. $I^\star$ is an ideal of $D^\star$ with $(I^\star)^{\star'} \not\ni 1$. There is a prime ideal Q of $D^\star$ with $Q^{\star'} = Q$ such that $Q \supset I^\star$. Set $D \cap Q = P$. Then P is a prime ideal of D with $P^\star \cap D = P$ such that $P \supset I$.

§2 Kronecker function rings on semigroups

Throughout the Section, let D be an infinite domain with quotient field K, and let S be a g-monoid $\supsetneqq \{0\}$ with quotient group $\mathrm{q}(S) = G$. We refer to [G2] and [M1] for the general theory of g-monoids. Let $\bar{\mathrm{F}}(S)$ be the set of non-empty subset $E \subset G$ such that $S + E \subset E$, let $\mathrm{F}(S)$ be the set of fractional ideals of S, and let $\mathrm{f}(S)$ be the set of finitely generated fractional ideals of S. Set $E^e = G$ for every $E \in \bar{\mathrm{F}}(S)$. Then the semistar operation $E \longmapsto E^e$ is said the e-semistar operation on S. Set $E^d = E$ for every $E \in \bar{\mathrm{F}}(S)$. Then the semistar operation $E \longmapsto E^d$ is said the d-semistar operation on S.

For every $E, F \in \bar{\mathrm{F}}(S)$, we denote $\{x \in G \mid x + E \subset F\}$ by $(F : E)$. Set $E^{-1} = (S : E) = \{x \in G \mid x + E \subset S\}$, set $\emptyset^{-1} = G$, and set $E^v = (E^{-1})^{-1}$ for every $E \in \bar{\mathrm{F}}(S)$. Then the semistar operation $E \longmapsto E^v$ is said the v-semistar operation on S. Let $\star$ be a semistar operation on S. We define a semistar operation $\star_f$: $E \longmapsto \cup\{F^\star \mid F \in \mathrm{f}(S)$ with $F \subset E\}$. $t = v_f$ is said the t-semistar operation on S.

Let $\star$ be a semistar operation on S, and let T be an oversemigroup of S. There is induced a canonical semistar operation $\alpha_T(\star) = \alpha(\star)$ on T, and is said the ascent of $\star$ to T.

If $\star_1, \star_2$ are semistar operations on S, we say $\star_1 \leq \star_2$ if $E^{\star_1} \subset E^{\star_2}$ for every $E \in \bar{\mathrm{F}}(S)$.

An ideal I of S is said $\star$-ideal if $I^\star = I$. A fractional ideal E of S is said a $\star$-fractional ideal if $E^\star = E$.

A prime ideal P satisfies $P \not\ni 0$ by the definition.

An ideal I of S is said a quasi-$\star$-ideal of S if $I^\star \cap S = I$.

A prime ideal P of S is said a $\star$-prime ideal if $P^\star = P$.

A prime ideal P of S is said a quasi-$\star$-prime ideal if $P^\star \cap S = P$.

An ideal I of S is said a $\star$-maximal ideal if I is maximal in the set $\{I \mid I$ is an ideal with $S \supsetneqq I^\star = I\}$.

An ideal I of S is said a quasi-$\star$-maximal ideal if I is maximal in the set $\{I \mid I$ is an ideal with $S \supsetneqq I = I^\star \cap S\}$.

(1.1) Let $\alpha_{S^\star}(\star) = \alpha(\star)$, and let I be an ideal of S. Then I a quasi-$\star$-ideal of S if and only if $I = E \cap S$, where E is an $\alpha(\star)$-ideal of $S^\star$.

Proof. The sufficiency: $I^\star \cap S = (E \cap S)^\star \cap S \subset E^\star \cap S = E \cap S = I$.

We denote by $\mathrm{Spec}^\star(S)$ the set of $\star$-prime ideals of S, by $\mathrm{Max}^\star(S)$ the set of $\star$-maximal ideals of S, by $\mathrm{QSpec}^\star(S)$ the set of quasi-$\star$-prime ideals of S, by $\mathrm{QMax}^\star(S)$ the set of quasi-$\star$-maximal ideals of S.

We set $\Pi^\star = \{P \in \mathrm{Spec}(S) \mid P^\star \not\ni 0\}$, and set $\Pi^\star_{\max} = \{P \mid P$ is a maximal element in $\Pi^\star\}$.

(1.2) Let $e \neq \star = \star_f$.

(1) If I is an ideal of S with $0 \notin I = I^\star \cap S$, then there is $J \in \mathrm{QMax}^\star(S)$ such that $I \subset J$.

(2) If I is a quasi-$\star$-maximal ideal of S, then I is a quasi-$\star$-prime ideal of S.

(3) If Q is a quasi-$\star$-maximal ideal of S, then there is an $\alpha(\star)$-maximal ideal N of $S^\star$ such that $Q = N \cap S$.

(4) If E is an $\alpha(\star)$-prime ideal of $S^\star$, then $E \cap S$ is a quasi-$\star$-prime ideal of S.

(5) $\mathrm{QSpec}^\star(S) \subset \Pi^\star$ and $\emptyset \neq \Pi^\star_{\max} = \mathrm{QMax}^\star(S)$.

Proof. (1) $\sim$ (4) are straightforward.

(5) There is an element $a \in S$ such that $a + S^\star \subsetneqq S^\star$. Then there is a prime ideal P with $P^\star \not\ni 0$ such that $P \supset (a + S^\star) \cap S$. Hence $\Pi^\star \neq \emptyset$.

Let $e \neq \star = \star_f$. Then we set $\mathcal{M}(\star) = \Pi^\star_{\max}$.

Let $\emptyset \neq \Delta \subset \mathrm{Spec}(S)$. Then we define a semistar operation $\star_\Delta :\ E \longmapsto \cap\{E + S_P \mid P \in \Delta\}$.

(1.3) Let $\emptyset \neq \Delta \subset \mathrm{Spec}(S)$, and set $\star = \star_\Delta$.

(1) $E^\star + S_P = E + S_P$, for every $E \in \bar{\mathrm{F}}(S)$ and $P \in \Delta$.

(2) $(E \cap F)^\star = E^\star \cap F^\star$, for every $E, F \in \bar{\mathrm{F}}(S)$.

(3) $P^\star \cap S = P$ for every $P \in \Delta$.

(4) Let I be an ideal with $I^\star \not\ni 0$, then $I \subset P$ for some $P \in \Delta$.

(5) Assume that $\emptyset \neq \Delta_{\max}$, and that each $P \in \Delta$ is contained in some $Q \in \Delta_{\max}$. Then $\star = \star_{(\Delta_{\max})}$.

The proof is straightforward.

$\star$ is said spectral if $\star = \star_\Delta$ for some $\emptyset \neq \Delta \subset \mathrm{Spec}(S)$.

$\star$ is said quasi-spectral if, for every ideal I with $I^\star \not\ni 0$, there is a prime ideal P with $P^\star \cap S = P$ such that $I \subset P$.

(1.4) Let $\star \neq e$.

(1) $\star$ is spectral if and only if $\star$ is quasi-spectral and stable.

(2) Assume that $\star = \star_f$. Then $\star$ is quasi-spectral and $\mathcal{M}(\star) \neq \emptyset$.

Proof. (1) The sufficiency: Since $\Pi^\star \neq \emptyset$ by the proof of (1.2)(5), [M3, §2,(2.3)] completes the proof.

(2) $\star$ is quasi-spectral by [M3, §1,(3.16)]. And $\mathcal{M}(\star) \neq \emptyset$ by (1.2)(5).

If $\Pi^\star \neq \emptyset$, we set $\star_{sp} = \star_{\Pi^\star}$.

Let $\star \neq e$, and assume that $\star$ is of finite type, then $\Pi^\star \neq \emptyset$ by (1.2)(5), hence $\star_{sp}$ is well defined.

If, for every ideal I with $I^\star \not\ni 0$, there is a prime ideal P with $P^\star \not\ni 0$ such that $P \supset I$, then $\star$ is said a qq-spectral semistar operation.

Every quasi-spectral semistar operation is a qq-spectral semistar operation.

(1.5) Is a qq-semistar operation a quasi-spectral semistar operation?

A canonical semigroup version of Lemma 2.6 in [FL3] is the following.

(1.6) Let $\Pi^\star \neq \emptyset$.
(1) $\star$ is spectral if and only if $\star = \star_{sp}$.
(2) The following statements are equivalent.
(i) $\star_{sp} \leq \star$.
(ii) $\star$ is quasi-spectral.
(iii) $E^\star = \cap\{E^\star + S_P \mid P \in \Pi^\star\}$ for every $E \in \bar{\mathrm{F}}(S)\}$.

(1.7) Let $\Pi^\star \neq \emptyset$.
(1) $\star$ is spectral if and only if $\star = \star_{sp}$.
(2) The following statements are equivalent.
(i) $\star_{sp} \leq \star$.
(ii) $\star$ is qq-spectral.
(iii) $E^\star = \cap\{E^\star + S_P \mid P \in \Pi^\star\}$ for every $E \in \bar{\mathrm{F}}(S)$.

Proof. (1) is [M3, §1,(3.10)], and (2) is [M3, §2,(1.2)].

(1.6)(2) is valid if and only if the answer to (1.5) is yes.

A non-empty subset $\mathcal{F}$ of ideals of S is said a localizing system of S if it satisfies the following conditions:

(1) If $I \in \mathcal{F}$ and J is an ideal of S with $I \subset J$, then $J \in \mathcal{F}$.

(2) If $I \in \mathcal{F}$ and J is an ideal of S such that $(J : x) \cap S \in \mathcal{F}$ for every $x \in I$, then $J \in \mathcal{F}$.

If $\star$ is a semistar operation on S. Then $\mathcal{F}^\star = \{I \mid I$ is an ideal of S with $I^\star \ni 0\}$ is a localizing system of S.

If $\mathcal{F}$ is a localizing system of S, then $\mathcal{F}_f = \{I \mid I$ is an ideal of S which contains a finitely generated ideal $J \in \mathcal{F}\}$ is a localizing system of S.

If $\mathcal{F}$ is a localizing system of S, then the mapping $\star_{\mathcal{F}}$: $E \longmapsto \cup\{(E : I) \mid I \in \mathcal{F}\}$ is a semistar operation on S.

We set the semistar operation $\tilde{\star} = \star_{(\mathcal{F}^\star)_f}$.

(1.8) Assume that $\star \neq e$. Then $\tilde{\star} = (\star_f)_{sp}$.

Proof. $\tilde{\star} = \star_{(\mathcal{F}^\star)_f} = \star_{\mathcal{F}^{(\star_f)}}$ by [M3, §1,(2.4)]. Since $\tilde{\star}$ is stable and of finite type by [M3, §1,(2.6)], $\tilde{\star}$ is spectral by [M3, §1,(3.16) and §2,(2.3)].

Since $\mathcal{F}^{\star_f} = \mathcal{F}^{\tilde{\star}}$ by [M3, §1, (2.10)], we have $\Pi^{\tilde{\star}} = \Pi^{\star_f} \neq \emptyset$ by (1.2)(5). By [M3, §1, (3.10)], we have $\tilde{\star} = (\tilde{\star})_{sp} = \star_{\Pi^{\tilde{\star}}} = \star_{\Pi^{(\star_f)}} = (\star_f)_{sp}$.

(1.9) Let $\star \neq e$.
(1) $\tilde{\star} = \star_{\mathcal{M}(\star_f)} \leq \star_f$ and $\tilde{\star} \neq e$.
(2) For every $E \in \bar{\mathrm{F}}(S)$,
(a) $E^{\star_f} = \cap\{E^{\star_f} + S_Q \mid Q \in \mathcal{M}(\star_f)\}$,
(b) $E^{\tilde{\star}} = \cap\{E + S_Q \mid Q \in \mathcal{M}(\star_f)\}$.

Proof. (1) Set $\Delta = \Pi^{\star_f}$, then $\Delta_{\max} = \Pi^{\star_f}_{\max} = \mathcal{M}(\star_f)$. By (1.8), we have $\tilde{\star} = (\star_f)_{sp} = \star_\Delta = \star_{(\Delta_{\max})} = \star_{\mathcal{M}(\star_f)}$.
$\tilde{\star} \leq \star_f$ by [M3, §1,(2.8)(3)], and hence $\tilde{\star} \neq e$.
(2) (a) Since $\star_f$ is quasi-spectral, we may use (1.7)(2). Then
$\cap\{E^{\star_f} + S_Q \mid Q \in \mathcal{M}(\star_f)\} = \cap\{E^{\star_f} + S_Q \mid Q \in \Pi^{\star_f}_{\max}\}$
$= \cap\{E^{\star_f} + S_Q \mid Q \in \Pi^{\star_f}\} = E^{\star_f}$.
(b) follows from (1).

Set $D(x) = \{P \in \mathrm{Spec}(S) \mid P \not\ni x\}$ for every $x \in S$. Then $\mathrm{Spec}(S)$ is a topological space with basis $\{D(x) \mid x \in S\}$. A subset $\Delta \subset \mathrm{Spec}(S)$ is said quasi-compact if Δ is contained in a union of open sets $\{G_\lambda \mid \lambda \in \Lambda\}$, then theire is a finie subset $\{\lambda_1, \cdots, \lambda_n\} \subset \Lambda$ such that $\Delta \subset \cup_1^n G_{\lambda_i}$.

(1.10) (a) Let v be the v-semistar operation on S. We have $E^{\tilde{v}} = \cup\{(E : I) \mid I$ is a finitely generated ideal with $I^v \ni 0\}$ for every $E \in \bar{\mathrm{F}}(S)$.
(b) Let $\emptyset \neq \Delta \subset \mathrm{Spec}(S)$. If Δ is quasi-compact, then $\star_\Delta = (\star_\Delta)_f$ and $\mathcal{M}(\star_\Delta) = \Delta_{\max}$.

Proof. (a) follows from [M3, §1, (2.6)(2)].
(b) We have $\mathcal{F}^{\star_\Delta} = \mathcal{F}(\Delta)$ by [M3, §1,(3.4)]. Let $\Delta = \{P_\lambda \mid \lambda \in \Lambda\}$, and let $I \in \mathcal{F}(\Delta)$. There is an element $x_\lambda \in I - P_\lambda$ for every λ. Then $\Delta \subset \cup_\lambda D(x_\lambda)$. Hence, there is a finite subset $\{\lambda_1, \cdots, \lambda_n\} \subset \Lambda$ such that $\Delta \subset D(x_{\lambda_1}) \cup \cdots \cup D(x_{\lambda_n})$. Then $J = (x_{\lambda_1}, \cdots, x_{\lambda_n}) \subset I$, and $J \in \mathcal{F}(\Delta)$, that is, $\mathcal{F}^{\star_\Delta}$ is of finite type. Then $\star_\Delta$ is of finite type by [M3, §1, (1.10)(B)(2) and (2.3)], hence $\star_\Delta = (\star_\Delta)_f$. And $\mathcal{M}(\star_\Delta) = \Pi^{\star_\Delta}_{\max} = \Delta_{\max}$ by definitions.

(1.11) Proposition Let $\star$ be a semistar operation on S. Let $\{I_\lambda \mid \lambda \in \Lambda\}$ be a non-empty set of ideals of S such that if $\lambda_1, \lambda_2 \in \Lambda$, then $I_{\lambda_1} \cup I_{\lambda_2} \subset I_{\lambda_3}$ for some λ_3.
(1) If each I_λ is a $\star_f$-ideal, then $I = \cup\{I_\lambda \mid \lambda \in \Lambda\}$ is a $\star_f$-ideal.
(2) If each I_λ is a $\star_f$-prime ideal, then $I = \cup\{I_\lambda \mid \lambda \in \Lambda\}$ is a $\star_f$-prime ideal.

The proof is straightforward.

(1.12) In (1.11), assume that each $I_\lambda \subsetneqq S$.
(1) If each I_λ is a quasi-$\star_f$-ideal, then I is a quasi-$\star_f$-ideal with $I \subsetneqq S$.
(2) If each I_λ is a quasi-$\star_f$-prime ideal, then I is a quasi-$\star_f$-prime ideal.

The proof is straightforward.

Let D be an infinite domain with quotient field $\mathrm{q}(D) = K$. Let $f = \sum_1^n a_i X^{t_i}$ be a non-zero element of $K[X;G]$, where $a_i \neq 0$ for each i and $t_i \neq t_j$ for each $i \neq j$. Then the fractional ideal $(a_1, \cdots, a_n)$ of D is said the c-content of f, and is denoted by $c_D(f)$ (or, simply by $c(f)$). The subset $\{a_1, \cdots, a_n\}$ of K is denoted by $\mathrm{Coef}(f)$. The fractional ideal $(t_1, \cdots, t_n)$ of S is said the e-content of f, and is denoted by $e_S(f)$ (or, simply by $e(f)$). The subset $\{t_1, \cdots, t_n\}$ of G is denoted by $\mathrm{Exp}(f)$.

We set $N(\star) = \{f \in D[X;S] - \{0\} \mid e(f)^\star \ni 0\}$. Obviously, $N(\star) = N(\star_f)$. We set $D(X;S)_e = D[X;S]_{N(d)}$.

(2.1) Proposition Let $\star \neq e$.

(1) $N(\star)$ is a multiplicatively closed subset of $D[X;S]$.

If $f, g \in D[X;S] - \{0\}$ such that $fg \in N(\star)$, then $f, g \in N(\star)$.

(2) $N(\star) = D[X;S] - \cup\{QD[X;S] \mid Q \in \mathcal{M}(\star_f)\}$.

(3) $\mathrm{Max}(D[X;S]_{N(\star)}) = \{QD[X;S]_{N(\star)} \mid Q \in \mathcal{M}(\star_f)\}$.

(4) $D[X;S]_{N(\star)} = \cap\{D(X;S_Q)_e \mid Q \in \mathcal{M}(\star_f)\}$.

(5) $\mathcal{M}(\star_f) = \{M \cap S \mid M \in \mathrm{Max}(D[X;S]_{N(\star)})\}$.

Proof. (1) follows from Dedekind-Mertens Lemma for S ([GP, 6.2.PROPOSITION]).

(2) Let $f \in D[X;S] - \{0\}$. If $e(f)^\star \not\ni 0$, there is a quasi-$\star_f$-maximal ideal Q such that $e(f) \subset Q$. Then $f \in QD[X;S]$.

(3) It is sufficient to show that each prime ideal H of $D[X;S]$ contained inside $\cup\{QD[X;S] \mid Q \in \mathcal{M}(\star_f)\}$ is contained in $QD[X;S]$ for some $Q \in \mathcal{M}(\star_f)$. Set $\cup\{e(f) \mid f \in H - \{0\}\} = I$. It suffices to show that $I^{\star_f} \not\ni 0$. Suppose the contrary. There are $f_1, \cdots, f_n \in H - \{0\}$ such that $(e(f_1) \cup \cdots \cup e(f_n))^{\star_f} \ni 0$. There are $c_1, \cdots, c_n \in D - \{0\}$ with $c_1 f_1 + \cdots + c_n f_n = g$ such that $\mathrm{Exp}(g) = \mathrm{Exp}(f_1) \cup \cdots \cup \mathrm{Exp}(f_n)$. Hence $e(g)^{\star_f} \ni 0$, and hence $g \in H \cap N(\star)$; a contradiction.

(4) and (5) are consequences of (3).

(2.2) Is (2.1) valid for a finite domain?

We denote $D[X;S]_{N(\star)}$ by $\mathrm{Na}(S, \star, D)$ (or, simply by $\mathrm{Na}(S, \star)$), and we say it the Nagata ring of S with respect to $\star$ and D (or, simply the Nagata ring of S with respect to $\star$). Obviously, $\mathrm{Na}(S, \star) = \mathrm{Na}(S, \star_f)$.

(2.3) Let Q be a prime ideal of S. Then Q is a maximal t-ideal of S if and only if $Q = M \cap S$ for some $M \in \mathrm{Max}(\mathrm{Na}(S, v))$.

The proof follows from (2.1)(5).

(2.4) (1) Let P be a prime ideal of S, and let $\star$ be the semistar operation $E \longmapsto E + S_P$ on S.

(a) $\mathcal{M}(\star_f) = \{P\}$.

(b) $\mathrm{Na}(S, \star) = D(X;S_P)_e$.

(c) $\star = \star_f = \star_{sp} = \tilde{\star}$.

(2) Let $\emptyset \neq \Delta \subset \mathrm{Spec}(S)$, let $\Delta^{\downarrow} = \{H \in \mathrm{Spec}(S) \mid H \subset P$ for some $P \in \Delta\}$, and let $\star = \star_\Delta$. Assume that each $P \in \Delta$ is contained in some $Q \in \Delta_{\max}$.

(a) $\Delta \subset \mathrm{QSpec}^\star(S) \subset \Delta^{\downarrow}$ and $\mathrm{QMax}^\star(S) = \Delta_{\max}$.

Assume that $\Delta_{\max}$ is a quasi-compact subspace of $\mathrm{Spec}(S)$. Then

(b) $\mathrm{Na}(S, \star_\Delta) = \cap\{D(X; S_Q)_e \mid Q \in \Delta_{\max}\} = \cap\{D(X; S_P)_e \mid P \in \Delta\}$,

(c) $\widetilde{(\star_\Delta)} = \star_\Delta$.

Proof. (2) (b) $\star = \star_f$ by [M3, §2,(4.1)]. Since $\mathcal{M}(\star) = \Delta_{\max}$, we have $\mathrm{Max}(\mathrm{Na}(S,\star)) = \{Q\mathrm{Na}(S,\star) \mid Q \in \Delta_{\max}\}$ by (2.1)(3). It follows that

$\mathrm{Na}(S,\star) = \cap\{\mathrm{Na}(S,\star)_{Q\mathrm{Na}(S,\star)} \mid Q \in \Delta_{\max}\} = \cap\{D[X;S]_{QD[X;S]} \mid Q \in \Delta_{\max}\} = \cap\{D(X;S_Q)_e \mid Q \in \Delta_{\max}\} = \cap\{D(X;S_P)_e \mid P \in \Delta\}$.

(c) Since $\star$ is spectral, $\star_{sp} = \star$. Hence $\tilde{\star} = (\star_f)_{sp} = \star_{sp} = \star$.

(2.5) Proposition Let $\star \neq e$, and let $E \in \bar{\mathrm{F}}(S)$.

(1) $E\mathrm{Na}(S,\star) = \cap\{ED(X;S_Q)_e \mid Q \in \mathcal{M}(\star_f)\}$.

(2) $E\mathrm{Na}(S,\star) \cap G = \cap\{E + S_Q \mid Q \in \mathcal{M}(\star_f)\}$.

(3) $E^{\tilde{\star}} = E\mathrm{Na}(S,\star) \cap G$.

If $E = E^\star$, then $E = E\mathrm{Na}(S,\star) \cap G$.

(4) Assume that $\star = \star_f$. Then

(i) $\tilde{\star} = \star_{sp}$.

(ii) $S^{\star_{sp}} = \cap\{S_Q \mid Q \in \mathcal{M}(\star)\}$.

(iii) $\star_{sp}$ is of finite type.

Proof. (1) By (2.1), we have

$E\mathrm{Na}(S,\star) = \cap\{(ED[X;S]_{N(\star)})_M \mid M \in \mathrm{Max}(D[X;S]_{N(\star)})\}$

$= \cap\{ED[X;S]_{QD[X;S]} \mid Q \in \mathcal{M}(\star_f)\} = \cap\{ED(X;S_Q)_e \mid Q \in \mathcal{M}(\star_f)\}$.

(2) We have $E\mathrm{Na}(S,\star) \cap G = \cap\{ED(X;S_Q)_e \cap G \mid Q \in \mathcal{M}(\star_f)\}$ by (1). Easily, $ED(X;S_Q)_e \cap G = E + S_Q$.

(3) We have that $E^{\tilde{\star}} = \cap\{E + S_Q \mid Q \in \mathcal{M}(\star_f)\}$ by (1.9)(2). Then $E^{\tilde{\star}} = E\mathrm{Na}(S,\star) \cap G$ by (2).

Assume that $E = E^\star$. Since $\tilde{\star} \leq \star$ by (1.9)(1), $E = E^{\tilde{\star}}$. Hence $E = E\mathrm{Na}(S,\star) \cap G$.

(4) (i) $\tilde{\star} = \star_{\mathcal{M}(\star)}$ by (1.9)(1), and $\star_{sp} = \star_{\Pi^\star} = \star_{\Pi^\star_{\max}} = \star_{\mathcal{M}(\star)}$.

(ii) $S^{\star_{sp}} = S^{\star_{\mathcal{M}(\star)}} = \cap\{S_Q \mid Q \in \mathcal{M}(\star)\}$.

(iii) $\tilde{\star}$ is of finite type by [M3, §1,(2.6)(7)]. Hence $\star_{sp}$ is of finite type by (i).

(2.6) Let $\star \neq e$.

(1) $(\tilde{\star})_f = \tilde{\star} = (\tilde{\star})_{sp} = \tilde{\tilde{\star}}$.

(2) $\mathcal{M}(\star_f) = \mathcal{M}(\tilde{\star})$.

(3) $\mathrm{Na}(S,\star) = \mathrm{Na}(S,\tilde{\star}) = \mathrm{Na}(S^{\tilde{\star}}, \alpha(\tilde{\star}))$.

Proof. (1) $(\tilde{\star})_f = \tilde{\star}$ by [M3, §1,(2.6)(7)]. Hence $\tilde{\star}$ is quasi-spectral by [M3, §1,(3.16)]. $\tilde{\star}$ is stable by [M3, §1,(2.6)(6)]. Hence $\tilde{\star} = (\tilde{\star})_{sp}$ by [M3, §2,(2.3) and §1,(3.10)]. $\tilde{\tilde{\star}} = \tilde{\star}$ by [M3, §1,(2.7)].

(2) Because $\tilde{\star} = \star_{\mathcal{M}(\star_f)}$ by (1.9)(1).

(3) $N(\star) = N(\tilde{\star})$ by (2.1)(2). Hence $\mathrm{Na}(S, \star) = \mathrm{Na}(S, \tilde{\star})$.

The case (C), where $\star = \star_f = \star_\Delta$ and $\emptyset \neq \Delta = \{Q_\lambda \mid \lambda \in \Lambda\} \subset \mathrm{Spec}(S)$ with $Q_\lambda \not\subset Q_{\lambda'}$ for each $\lambda \neq \lambda'$. Then we have $S^\star = \cap_\lambda S_{Q_\lambda}$ and $\mathcal{M}(\star) = \Delta$. If $Q \in \Delta$, then

$(Q + S_Q) \cap S^\star = \cap_\lambda((Q + S_Q) \cap S_{Q_\lambda}) = \cap_\lambda(Q + S_{Q_\lambda}) = Q^\star$, and $Q^\star \cap S = (Q + S_Q) \cap S^\star \cap S = Q$.

Let $M \in \mathcal{M}(\alpha(\star))$. Then $M \subset S^\star$ and $M = M^{\alpha(\star)} = M^\star = \cap_\lambda(M + S_{Q_\lambda})$. Hence $M + S_Q \not\ni 0$ for some $Q \in \Delta$. Then $M \subset (Q + S_Q) \cap S^\star = Q^\star$. By the choice of M, $M = Q^\star$ by (1.2)(3).

It follows that $\mathcal{M}(\alpha(\star)) = \{Q^\star \mid Q \in \Delta\}$.

Since $(S_Q)_{Q+S_Q} \supset (S^\star)_{Q^\star} \supset S_Q$, we have $(S^\star)_{Q^\star} = S_Q$. By (2.1)(4),

$\mathrm{Na}(S^\star, \alpha(\star)) = \cap_\lambda D(X; (S^\star)_{Q_\lambda^\star})_e = \cap_\lambda D(X; S_{Q_\lambda})_e = \mathrm{Na}(S, \star)$.

The general case: Set $\mathcal{M}(\star_f) = \Delta$, then $\tilde{\star} = \star_\Delta$. By the case (C), we have $\mathrm{Na}(S^{\tilde{\star}}, \alpha(\tilde{\star})) = \mathrm{Na}(S, \tilde{\star})$.

(2.7) Let $\star$ be quasi-spectral such that $\Pi^\star \neq \emptyset$. Then $\mathrm{Na}(S, \star) = \mathrm{Na}(S, \star_{sp}) = \mathrm{Na}(S, \tilde{\star})$.

Proof. We have $\tilde{\star} = (\star_f)_{sp} \leq \star_{sp}$ by [M3, §1,(3.8), (4) and (5)]. Hence $\mathrm{Na}(S, \tilde{\star}) \subset \mathrm{Na}(S, \star_{sp})$. Since $\star_{sp} \leq \star$ by (1.7)(2), we have $\mathrm{Na}(S, \star_{sp}) \subset \mathrm{Na}(S, \star)$. The first equality of (2.6)(3) completes the proof.

(2.8) Theorem Assume that $\star \neq e$. We have $\mathrm{Max}(\mathrm{Na}(S, \star)) = \{QD(X; S_Q)_e \cap \mathrm{Na}(S, \star) \mid Q \in \mathcal{M}(\star_f)\}$.

Proof. (2.1), (3) and (4) show that the maximal ideals of $\mathrm{Na}(S, \star)$ are the ideals of the set $\{Q\mathrm{Na}(S, \star) \mid Q \in \mathcal{M}(\star_f)\}$, and $\mathrm{Na}(S, \star) \supsetneqq QD(X; S_Q)_e \cap \mathrm{Na}(S, \star) \supset Q\mathrm{Na}(S, \star)$. The proof is complete.

A valuation oversemigroup V of S is said a $\star$-valuation oversemigroup of S if, for every element $F \in \mathrm{f}(S)$, $F^\star \subset F + V$.

(2.9) Theorem Assume that $\star \neq e$. Let V be a valuation oversemigroup of S. Then V is a $\tilde{\star}$-valuation oversemigroup if and only if V is an oversemigroup of S_P for some $P \in \mathcal{M}(\star_f)$.

Proof. We may assume that $V \subsetneqq G$. The sufficiency: $\tilde{\star} = \star_{\mathcal{M}(\star_f)}$ by (1.9)(1). Set $\mathcal{M}(\star_f) = \{P_\lambda \mid \lambda \in \Lambda\}$, and let $F \in \mathrm{f}(S)$. Then $F^{\tilde{\star}} = \cap\{F + S_{P_\lambda} \mid \lambda \in \Lambda\} \subset F + S_P \subset F + V$.

The necessity: Let M be the maximal ideal of V, let $Q = M \cap S$, and set $\Delta = \mathcal{M}(\star_f)$. Since $\tilde{\star}$ is of finite type, $Q^{\tilde{\star}} = \cup\{F^{\tilde{\star}} \mid F \in \mathrm{f}(S), F \subset Q\}$. And $F^{\tilde{\star}} \subset F + V \subset M$. Hence $Q^{\tilde{\star}} \subset M$.

Suppose that $Q \not\subset P$ for each $P \in \mathcal{M}(\star_f)$. Then $Q^{\tilde{\star}} = Q^{\star_\Delta} = \cap\{Q + S_P \mid P \in \Delta\} \ni 0$; a contradiction.

In the following (3.1) and (3.2), for convenience, we will review [OM, (4.2) and (4.3)] briefly.

(3.1) Let $\star$ be a semistar operation on S. Let $f, g, f', g' \in D[X; S] - \{0\}$ with $\dfrac{f}{g} = \dfrac{f'}{g'}$ such that $(e(f) + e(h))^\star \subset (e(g) + e(h))^\star$ for some $h \in D[X; S] - \{0\}$. Then there is $h' \in D[X; S] - \{0\}$ such that $(e(f') + e(h'))^\star \subset (e(g') + e(h'))^\star$.

Proof. By [GP, 6.2. PROPOSITION], there is a positive integer m such that $(m+1)e(g) + e(f') = me(g) + e(f'g)$ and $(m+1)e(f) + e(g') = me(f) + e(fg')$. It follows that $\{(m+1)e(g) + e(f')\} + me(f) = \{(m+1)e(f) + e(g')\} + me(g)$.

There are elements $s_1, s_2, \cdots, s_n$ of S with $s_i \neq s_j$ for each $i \neq j$ such that $(m+1)(e(g) + e(h)) + m(e(f) + e(h)) = (s_1, s_2, \cdots, s_n)$. If we set $h' = X^{s_1} + X^{s_2} + \cdots + X^{s_n} \in D[X; S] - \{0\}$, we have $e(h') = (m+1)(e(g) + e(h)) + m(e(f) + e(h))$, and therefore

$e(f') + e(h') = \{(m+1)e(g) + e(f') + me(f)\} + (2m+1)e(h)$
$= \{(m+1)e(f) + e(g') + me(g)\} + (2m+1)e(h)$
$= (e(f) + e(h)) + m(e(f) + e(h)) + m(e(g) + e(h)) + e(g')$
$\subset (e(g) + e(h))^\star + m(e(f) + e(h)) + m(e(g) + e(h)) + e(g')$
$\subset (e(g') + e(h'))^\star$.

The set $\{\dfrac{f}{g} \mid f, g \in D[X; S] - \{0\}$ such that $(e(f) + e(h))^\star \subset (e(g) + e(h))^\star$ for some $h \in D[X; S] - \{0\}\} \cup \{0\}$ is denoted by $\mathrm{Kr}(S, \star, D)$ (or, simply by $\mathrm{Kr}(S, \star)$), and is said the Kronecker function ring of S with respect to $\star$ and D (or, simply with respect to $\star$). (3.1) and (3.2) show that $\mathrm{Kr}(S, \star)$ is a well-defined overring of $D[X; S]$.

(3.2) $\mathrm{Kr}(S, \star)$ is an integral domain with quotient field $\mathrm{q}(D[X; S])$.

Proof. Let $\dfrac{f}{g}, \dfrac{f'}{g} \in \mathrm{Kr}(S, \star) - \{0\}$. Then there are $h, h' \in D[X; S] - \{0\}$ such that $(e(f) + e(h))^\star \subset (e(g) + e(h))^\star$ and $(e(f') + e(h'))^\star \subset (e(g) + e(h'))^\star$. There is $j \in D[X; S] - \{0\}$ such that $e(j) = e(h) + e(h')$. Then we have

$(e(f) + e(j))^\star \subset (e(g) + e(j))^\star, (e(f') + e(j))^\star \subset (e(g) + e(j))^\star$.

We may assume that $f + f' \neq 0$. Then it follows that $(e(f + f') + e(j))^\star \subset (e(g) + e(j))^\star$. Hence $\dfrac{f}{g} + \dfrac{f'}{g} \in \mathrm{Kr}(S, \star)$.

Next, we have $(m+2)e(g) = me(g) + e(g^2)$ for some m. There is $j' \in D[X; S] - \{0\}$ such that $e(j') = (m+2)e(g) + 2e(k)$. Then we have

$e(ff') + e(j') \subset \{e(f) + e(f')\} + \{(m+2)e(g) + 2e(j)\}$
$= \{e(f) + e(j)\} + \{e(f') + e(j)\} + (m+2)e(g)$
$\subset 2(e(g) + e(j))^\star + (m+2)e(g)$
$= 2(e(g) + e(j))^\star + \{me(g) + e(g^2)\} \subset (e(g^2) + e(j'))^\star$.

Therefore $(e(ff') + e(j'))^\star \subset (e(g^2) + e(j'))^\star$. Hence $\dfrac{ff'}{gg'} \in \mathrm{Kr}(S, \star)$.

We define the mapping $\star_a : \bar{\mathrm{F}}(S) \longrightarrow \bar{\mathrm{F}}(S)$ by setting

$F^{\star_a} = \cup\{((F + H)^\star : H^\star) \mid H \in \mathrm{f}(S)\}$ for every $F \in \mathrm{f}(S)$,
$E^{\star_a} = \cup\{F^{\star_a} \mid F \in \mathrm{f}(S)$ with $F \subset E\}$ for every $E \in \bar{\mathrm{F}}(S)$.

The following (3.3) appears in [OM, (3.6),(4.5) and (4.7)].

(3.3) (1) $\star_a$ is a semistar operation of finite type on S.
(2) $\star_a$ is e.a.b. (that is, endlich arithmetisch brauchbar).
(3) $\star_f = \star_a$ if and only if $\star_f$ is e.a.b.
(4) If $\star_1 \leq \star_2$, then $(\star_1)_a \leq (\star_2)_a$.
(5) If $\star_1 \leq \star_2$, then $\mathrm{Kr}(S, \star_1) \subset \mathrm{Kr}(S, \star_2)$.
(6) Let $\star$ be a semistar operation on S. Then, for every $E \in \bar{\mathrm{F}}(S)$, we have $E^{\star_a} = \cup\{F\mathrm{Kr}(S, \star) \cap G \mid F \in \mathrm{f}(S) \text{ with } F \subset E\}$.

(3.4) Proposition (1) $\star_f \leq \star_a$.
(2) $\mathrm{Kr}(S, \star) = \mathrm{Kr}(S, \star_f) = \mathrm{Kr}(S, \star_a) = \mathrm{Kr}(S^{\star_a}, \alpha(\star_a))$.
(3) $\mathrm{Kr}(S, \star)$ is a Bezout domain.
(4) $\mathrm{Na}(S, \star) \subset \mathrm{Kr}(S, \star)$.
(5) $E^{\star_a} = E\mathrm{Kr}(S, \star) \cap G$ for each $E \in \bar{\mathrm{F}}(S)$.

Proof. The proof follows from [OM, (3.6),(4.4),(4.6) and (4,8)] and (3.3)(6).

(3.5) If $\star$ is a semistar operation on S distinct with e, then $\star_a \neq e$.

Proof. Suppose the contrary. Since $\star_a = e$, we have $S^{\star_a} = G$. Since $\star_a$ is of finite type, $S^\star = G$. Therefore $\star = e$.

(3.6) A valuation oversemigroup V of S is a $\star$-valuation oversemigroup if and only if there is a valuation overring W of $\mathrm{Kr}(S, \star)$ such that $W \cap G = V$.

Proof. Let v be a valuation on G, let $f = \sum_1^n a_i X^{t_i} \in K[X; G]$, where $a_i \neq 0$ for each i and $t_i \neq t_j$ for each $i \neq j$. If we set $v'(f) = \min_i v(t_i)$, we have a valuation v' on $\mathrm{q}(K[X; G])$.

Let V be a $\star$-valuation oversemigroup, let v' be the canonical extension of v to $\mathrm{q}(D[X; S])$, and let V' be the valuation ring of v'. Let $\dfrac{f}{g} \in \mathrm{Kr}(S, \star)$. There is an element $h \in D[X; S] - \{0\}$ such that $(e(f) + e(h))^\star \subset (e(g) + e(h))^\star$. Let $f = \sum_1^n a_i X^{s_i}, g = \sum_1^m b_j X^{t_j}, h = \sum_1^l c_k X^{u_k}$, and let $v(s_{i_0}) = \min_i v(s_i), v(t_{j_0}) = \min_j v(t_j), v(c_{u_0}) = \min_k v(u_k)$. We have

$(e(f) + e(h))^\star + V = e(f) + e(h) + V = e(f) + V + e(h) + V = s_{i_0} + V + e_{u_0} + V = s_{i_0} + u_{k_0} + V$.

Similarly, we have $(e(g) + e(h))^\star + V = t_{j_0} + u_{k_0} + V$. Since $s_{i_0} + u_{k_0} + V \subset t_{j_0} + u_{k_0} + V$, we have $v(s_{i_0}) \geq v(t_{j_0})$. Then $v'(\dfrac{f}{g}) = v'(f) - v'(g) = v(s_{i_0}) - v(t_{j_0}) \geq 0$. Hence $\dfrac{f}{g} \in V'$.

Let W be a valuation overring of $\mathrm{Kr}(S, \star)$, and let $V = W \cap G$. Let $F = (\alpha_1, \cdots, \alpha_n) \in \mathrm{f}(S)$ with $\alpha_i \neq \alpha_j$ for each $i \neq j$, and let $f = X^{\alpha_1} + \cdots + X^{\alpha_n}$.

Let $v(\alpha_{i_0}) = \min_i v(\alpha_i)$. If $z \in F^\star$, we have $(z)^\star \subset e(f)^\star$. Then we have $\dfrac{z}{f} \in \mathrm{Kr}(S,\star) \subset W$, hence $v(z) - v(\alpha_{i_0}) \geq 0$. It follows that $z \in \alpha_{i_0} + V \subset F + V$, hence $F^\star \subset F + V$.

(3.7) Let W be a valuation overring of $\mathrm{Kr}(S,\star)$, and let $V = W \cap G$. Then W is the canonically extended valuation ring of V to $\mathrm{q}(D[X;S])$.

Proof. Let w be a valuation on $\mathrm{q}(D[X;S])$ belonging to W, and set $v(s) = w(X^s)$ for every $s \in S$. Then v is a valuation on G belonging to V. Let v' be the canonical extension of v to $\mathrm{q}(D[X;S])$. If $f = a_0X^{s_0} + \cdots + a_nX^{s_n} \in D[X;S]$ with $a_i \neq 0$ for each i and $s_i \neq s_j$ for each $i \neq j$, and if $v(s_0) = \min_i v(s_i)$, we have $v'(f) = v(s_0)$ and $w(f) \geq \inf_i w(a_iX^{s_i}) = v(s_0)$. Since $\dfrac{X^{s_0}}{f} \in \mathrm{Kr}(S,\star) \subset W, 0 \leq w(\frac{X^{s_0}}{f}) = v(s_0) - w(f)$. Hence $w(f) = v(s_0) = v'(f)$. Therefore $w = v'$.

(3.8) Theorem Assume that $e \neq \star = \star_f$.

(1) Let W be a valuation oversemigroup of $\mathrm{Kr}(S,\star)$ with maximal ideal $N \subsetneqq W$. Set $N_0 = N \cap S$ and $N_1 = N \cap D[X;S]$. Then

(a) $N_1 = N_0D[X;S]$, $N \cap \mathrm{Na}(S,\star) = N_0\mathrm{Na}(S,\star) = N_1\mathrm{Na}(S,\star)$ and $N \cap \mathrm{Na}(S,\star_a) = N_0\mathrm{Na}(S,\star_a) = N_1\ \mathrm{Na}(S,\star_a)$.

(b) N_0 is a quasi-$\star_a$-prime ideal.

(2) If P is a quasi-$\star_a$-prime ideal of S, then there is a quasi-$\star_a$-maximal ideal Q of S and a valuation overring W of $\mathrm{Kr}(S,\star)$ such that $P \subset Q = N \cap S$, where N is the maximal ideal of W.

(3) $\mathcal{M}(\star_a)$ is contained in the canonical image in S of $\mathrm{Max}(\mathrm{Kr}(S,\star))$.

(4) For each $Q \in \mathcal{M}(\star_a)$, there is a $\star$-valuation oversemigroup V of S containing S_Q.

Proof. (1) (a) Let $0 \neq f \in N_1$, and let $\mathrm{Exp}(f) = \{s_1, \cdots, s_n\}$. Then $N \supset f\mathrm{Kr}(S,\star) = (s_1, \cdots, s_n)\mathrm{Kr}(S,\star)$ and $(s_1, \cdots, s_n) \subset N_0$. Hence $f \in N_0D[X;S]$, and hence $N_1 = N_0D[X;S]$.

Let $\dfrac{f}{g} \in N \cap \mathrm{Na}(S,\star)$ with $g \in N(\star)$. Then $f \in gN \subset N$, hence $f \in N_1$. Hence $\dfrac{f}{g} \in N_1\mathrm{Na}(S,\star)$. It follows that $N \cap \mathrm{Na}(S,\star) = N_1\mathrm{Na}(S,\star) = N_0\mathrm{Na}(S,\star)$. Since $\mathrm{Kr}(S,\star) = \mathrm{Kr}(S,\star_a)$, we have $N\cap\ \mathrm{Na}(S,\star_a) = N_0\mathrm{Na}\ (S,\star_a) = N_1\mathrm{Na}(S,\star_a)$.

(b) By (3.4), we have $N_0^{\star_a} = N_0\mathrm{Kr}(S,\star) \cap G \subset N \cap \mathrm{Kr}(S,\star_a) \cap G = N \cap S^{\star_a}$. Hence $N_0^{\star_a} \cap S \subset N \cap S^{\star_a} \cap S = N_0$.

(2) Since $\star_a$ is of finite type, there is a quasi-$\star_a$-maximal ideal Q with $Q \supset P$. $Q^{\star_a} = Q\mathrm{Kr}(S,\star) \cap G$ by (3.4)(5). Hence $Q\mathrm{Kr}(S,\star) \not\ni 1$. Let M be a maximal ideal of $\mathrm{Kr}(S,\star)$ with $M \supset Q\mathrm{Kr}(S,\star)$. $W = \mathrm{Kr}(S,\star)_M$ is a valuation overring of $\mathrm{Kr}(S,\star)$ with maximal ideal $N = MW$. Since Q is a quasi-$\star_a$-maximal ideal, $N \cap S = Q$ by (1)(b).

(3) follows from the proof of (2).

(4) If $Q \in \mathcal{M}(\star_a)$, we can find a valuation overring W of $\mathrm{Kr}(S,\star)$ such that $N \cap S = Q$ by (2), where N is the maximal ideal of W. Set $V = W \cap G$. Then V is a

$\star$-valuation oversemigroup of S containing S_Q by (3.6).

(3.9) Let $\star$ be e.a.b., of finite type and $S = S^\star$ with $\star \neq e$. Let P be a $\star$-maximal ideal of S. Then P is the center of a minimal $\star$-valuation oversemigroup of S.

Proof. $\star_a = \star$ by (3.3)(3). By (3.8)(3), there is a maximal ideal M of $\mathrm{Kr}(S,\star)$ such that $M \cap S = P$. Set $W = \mathrm{Kr}(S,\star)_M$, and let N be the maximal ideal of W. Then $W \cap G = V$ is a $\star$-valuation oversemigroup of S, and P is the center of V in S. Suppose that there is a $\star$-valuation oversemigroup V' with $V' \subset V$, let v' be a valuation on G belonging to V', and let W' be the canonical extension of V', then W' is a valuation overring of $\mathrm{Kr}(S,\star)$. Let $0 \neq \varphi \in W'$. Then $\varphi = \dfrac{\sum a_i X^{\alpha_i}}{\sum b_j \beta_j}$, where each $\alpha_i, \beta_j \in V'$ with $\beta_{j_0} = 0$ for some j_0. It follows that $\varphi \in W$. Hence $W' = W$, and $V' = V$.

(3.10) Assume that $e \neq \star = \star_f$.

(1) $\tilde{\star} \leq \widetilde{(\star_a)} = (\star_a)_{sp} \leq \star_a$ and $\tilde{\star} \leq (\tilde{\star})_a \leq \star_a$.

(2) $\mathrm{Na}(S,\star) = \mathrm{Na}(S,\tilde{\star}) \subset \mathrm{Na}(S,\widetilde{(\star_a)}) = \mathrm{Na}(S,\star_a) \subset \mathrm{Kr}(S,\star_a) = \mathrm{Kr}(S,\star)$.

(3) $\mathrm{Na}(S,\star) = \mathrm{Na}(S,\tilde{\star}) \subset \mathrm{Na}(S,(\tilde{\star})_a) \subset \mathrm{Kr}(S,(\tilde{\star})_a) = \mathrm{Kr}(S,\tilde{\star}) \subset \mathrm{Kr}(S,\star)$.

(4) For every $E \in \bar{\mathrm{F}}(S)$,

(a) $E^{\widetilde{(\star_a)}} = E\mathrm{Na}(S,\star_a) \cap G \supset E\mathrm{Na}(S,\star) \cap G = E^{\tilde{\star}}$.

(b) $E^{(\tilde{\star})_a} = E\mathrm{Kr}(S,\tilde{\star}) \cap G \subset E\mathrm{Kr}(S,\star) \cap G = E^{\star_a}$.

Proof. (1) Since $\star \leq \star_a$ by (3.4)(1), $\tilde{\star} \leq \widetilde{(\star_a)}$ by [M3, §1,(2.7)(4)]. Since $\star_a$ is of finite type by (3.3)(1), $\widetilde{(\star_a)} = ((\star_a)_f)_{sp} = (\star_a)_{sp}$. Since $\star_a$ is quasi-spectral by [M3, §1,(3.16)], $(\star_a)_{sp} \leq \star_a$ by [M3, §2,(1.2)]. Since $\tilde{\star}$ is of finite type by [M3, §1,(2.6)], $\tilde{\star} \leq (\tilde{\star})_a$ by (3.4)(1). Since $\tilde{\star} \leq \star$ by [M3, §1,(2.6)(3)], $(\tilde{\star})_a \leq \star_a$ by (3.3)(4).

(2) By (2.6)(3), we have $\mathrm{Na}(S,\star) = \mathrm{Na}(S,\tilde{\star})$ and $\mathrm{Na}(S,\star_a) = \mathrm{Na}(S,\widetilde{(\star_a)})$. By (1), $\mathrm{Na}(S,\tilde{\star}) \subset \mathrm{Na}(S,\widetilde{(\star_a)})$.

(3) $\mathrm{Na}(S,\star) = \mathrm{Na}(S,\tilde{\star})$ by (2.6)(3).

Since $\tilde{\star} \leq (\tilde{\star})_a$, $\mathrm{Na}(S,\tilde{\star}) \subset \mathrm{Na}(S,(\tilde{\star})_a)$.

$\mathrm{Na}(S,(\tilde{\star})_a) \subset \mathrm{Kr}(S,(\tilde{\star})_a)$ by (3.4)(4).

$\mathrm{Kr}(S,(\tilde{\star})_a) = \mathrm{Kr}(S,\tilde{\star})$ by (3.4)(2).

Since $\tilde{\star} \leq \star$, $\mathrm{Kr}(S,\tilde{\star}) \subset \mathrm{Kr}(S,\star)$.

(4) (a) Since $\star_f \leq \star_a$, $\mathrm{Na}(S,\star) \subset \mathrm{Na}(S,\star_a)$.

$E^{\widetilde{(\star_a)}} = E\mathrm{Na}(S,\star_a) \cap G$ by (2.5)(3).

(b) From (3.4)(5) and from the fact $\tilde{\star} \leq \star$, we have $E^{(\tilde{\star})_a} = E\mathrm{Kr}(S,\tilde{\star}) \cap G \subset E\mathrm{Kr}(S,\star) \cap G = E^{\star_a}$.

(3.11) Proposition Assume that $\star \neq e$. The following conditions are equivalent.

(1) $\tilde{\star} = \widetilde{(\star_a)}$.

(2) $\mathcal{M}(\star_f) = \mathcal{M}(\star_a)$.

(3) $\mathrm{Na}(S,\star) = \mathrm{Na}(S,\star_a)$.

Proof. (2) $\Longrightarrow$(1): Since $\star_a$ is of finite type, $(\star_a)_f = \star_a$.
By (1.9)(1), $\tilde{\star} = \star_{\mathcal{M}(\star_f)}$ and $\widetilde{(\star_a)} = \star_{\mathcal{M}(\star_a)}$.
(1) $\Longrightarrow$(2): Follows from (2.6)(2).
(2) $\Longrightarrow$ (3): By (2.1)(2), $N(\star) = N(\star_a)$. Hence $\mathrm{Na}(S,\star) = \mathrm{Na}(S,\star_a)$.
(3) $\Longrightarrow$ (2): From (2.1)(5).

(3.12) Assume that $\star \neq e$. The following conditions are equivalent.
(1) $\star_a = (\tilde{\star})_a$.
(2) The set of $\tilde{\star}$-valuation oversemigroups of S coincides with the set of $\star$-valuation oversemigroups of S.
(3) $\mathrm{Kr}(S,\tilde{\star}) = \mathrm{Kr}(S,\star)$.
Moreover, each of the previous conditions implies
(4) $\mathcal{M}(\star_a) = \mathcal{M}((\tilde{\star})_a)$.

Proof. (2) $\Longleftrightarrow$ (3) follows from (3.6).
(1) $\Longrightarrow$ (3): (3.4)(2) implies that $\mathrm{Kr}(S,\tilde{\star}) = \mathrm{Kr}(S,(\tilde{\star})_a) = \mathrm{Kr}(S,\star_a) = \mathrm{Kr}(S,\star)$.
(1) $\Longrightarrow$ (4): Trivial.
(3) $\Longrightarrow$ (1): By (3.4)(5), we have $E^{\star_a} = E\mathrm{Kr}(S,\star)\cap G$ and $E^{(\tilde{\star})_a} = E\mathrm{Kr}(S,\tilde{\star})\cap G$. By (3), we have $E^{\star_a} = E^{(\tilde{\star})_a}$. Hence $\star_a = (\tilde{\star})_a$.

(3.13) Proposition Let $\star_1, \star_2$ be semistar operations on S distinct with e. Then, $\mathrm{Na}(S,\star_1) = \mathrm{Na}(S,\star_2)$ if and only if $\mathcal{M}((\star_1)_f) = \mathcal{M}((\star_2)_f)$.

Proof. The necessity follows from (2.1)(5).
The sufficiency follows from (2.1)(2).

Appendix

Let D be a domain, and let S be a g-monoid $\supsetneqq \{0\}$. Let $D[X;S]$ be the semigroup ring of S over D. If $\vec{Z}_0$ is the non-negative integers, then $D[X;\vec{Z}_0] = D[X]$. After [FL1], we will define the Kronecker function ring $\mathrm{Kr}(D,\star,S)$ of D with respect to $\star$ and S.

(1) (Dedekind-Mertens Lemma)(cf. [GP, 4.3.THEOREM]) Let $f, g \in D[X;S] - \{0\}$. Then there is a positive integer m such that $c(g)^{m+1}c(f) = c(g)^m c(fg)$.

(2) Let $\star$ be a semistar operation on D. Let $f, g, f', g' \in D[X;S] - \{0\}$ with $\dfrac{f}{g} = \dfrac{f'}{g'}$ such that $(c(f)c(h))^\star \subset (c(g)c(h))^\star$ for some $h \in D[X;S] - \{0\}$. Then there is $h' \in D[X;S] - \{0\}$ such that $(c(f')c(h'))^\star \subset (c(g')c(h'))^\star$.

Proof. Then we have $fg' = f'g$. By (1), there is a positive integer m such that
$c(g)^{m+1}c(f') = c(g)^m c(f'g)$, $c(f)^{m+1}c(g') = c(f)^m c(fg')$.
It follows that $\{c(g)^{m+1}c(f')\}c(f)^m = \{c(f)^{m+1}c(g')\}c(g)^m$.
There is $h' \in D[X;S] - \{0\}$ such that $c(h') = (c(g)c(h))^{m+1}(c(f)c(h))^m$.

Then we have
$c(f')c(h') = \{c(g)^{m+1}c(f')c(f)^m\}c(h)^{2m+1}$
$= \{c(f)^{m+1}c(g')c(g)^m\}c(h)^{2m+1} = (c(f)c(h))(c(f)c(h))^m(c(g)c(h))^m c(g')$
$\subset (c(g)c(h))^\star(c(f)c(h))^m(c(g)c(h))^m c(g') \subset (c(g')c(h'))^\star$.
Therefore $(c(f')c(h'))^\star \subset (c(g')c(h'))^\star$.

Set $\mathrm{Kr}(D, \star, S) = \{\frac{f}{g} \mid f, g \in D[X; S] - \{0\}$ such that $(c(f)c(h))^\star \subset (c(g)c(h))^\star$ for some $h \in D[X; S] - \{0\}\} \cup \{0\}$. (2) shows that $\mathrm{Kr}(D, \star, S)$ is a well-defined subset of $\mathrm{q}(D[X; S])$.

(3) $\mathrm{Kr}(D, \star, S)$ is an integral domain with quotient field $\mathrm{q}(D[X; S])$.

Proof. Let $\frac{f}{g}, \frac{f'}{g} \in \mathrm{Kr}(D, \star, S) - \{0\}$. Then there are $h, h' \in D[X; S] - \{0\}$ such that $(c(f)c(h))^\star \subset (c(g)c(h))^\star, (c(f')c(h'))^\star \subset (c(g)c(h'))^\star$.
There is $k \in D[X; S] - \{0\}$ such that $c(k) = c(h)c(h')$. Then we have
$(c(f)c(k))^\star \subset (c(g)c(k))^\star, (c(f')c(k))^\star \subset (c(g)c(k))^\star$.
We may assume that $f+f' \neq 0$. Then it follows that $(c(f+f')c(k))^\star \subset (c(g)c(k))^\star$. Hence $\frac{f}{g} + \frac{f'}{g} \in \mathrm{Kr}(D, \star, S)$.
Next, we have $c(g)^{m+2} = c(g)^m c(g^2)$ for some m. There is $k' \in D[X; S] - \{0\}$ such that $c(k') = c(g)^{m+2}c(k)^2$. Then we have
$c(ff')c(k') \subset \{c(f)c(f')\}\{c(g)^{m+2}c(k)^2\}$
$= \{c(f)c(k)\}\{c(f')c(k)\}c(g)^{m+2} \subset ((c(g)c(k))^2)^\star c(g)^{m+2}$
$= ((c(g)c(k))^2)^\star\{c(g)^m c(g^2)\} \subset (c(g^2)c(k'))^\star$.
Therefore $(c(ff')c(k'))^\star \subset (c(g^2)c(k'))^\star$, and hence $\frac{ff'}{g^2} \in \mathrm{Kr}(D, \star, S)$.

(4) $\mathrm{Kr}(D, \star, S)$ is a Bezout domain.

Proof. Set $R = \mathrm{Kr}(D, \star, S)$, and let $h \in D[X; S] - \{0\}$ with Coef $(f) = \{c_1, \cdots, c_n\}$. Then we have $hR = (c_1, \cdots, c_n)R$.

Let ξ and η be non-zero elements of R. We let $\xi = \frac{f}{g}$ and $\eta = \frac{f'}{g}$ with $f, f', g \in D[X; S] - \{0\}$, and let $\mathrm{Coef}(f) \cup \mathrm{Coef}(f') = \{a_1, \cdots, a_n\}$ with $a_i \neq a_j$ for every $i \neq j$. Then we have, for an element $s \in S - \{0\}$,
$(\xi, \eta)R = (\frac{1}{g})(f, f')R = (\frac{1}{g})(a_1, \cdots, a_n)R$
$= (\frac{1}{g})(a_1 X^s + a_2 X^{2s} + \cdots + a_n X^{ns})R$.
Therefore $(\xi, \eta)R$ is a principal ideal of R.

The above proof is slightly defferent from the corresponding classical one (cf., for instance, [G1, (32.7) THEOREM]).

Let D be a domain with quotient field K, let $\star$ be a semistar operation on D. A valuation overring V of D is said a $\star$-valuation overring if, for every $F \in \mathrm{f}(D)$,

$F^\star \subset FV$. The following similar result to [FL2, Theorem 3.5] is valid, and the proof is similar:

(5) Proposition Let $\star$ be a semistar operation on D, and let V be a valuation overring of D. Then V is a $\star$-valuation overring if and only if there is a valuation overring W of $\mathrm{Kr}(D, \star, S)$ such that $W \cap K = V$.

The author thanks to Professor Ken Nakane for his assistance over a long period of time.

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